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Existence results for a coupled system of Caputo type fractional integro-differential equations with multi-point and sub-strip boundary conditions

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Abstract

This paper is concerned with the existence and uniqueness of solutions for a coupled system of Liouville–Caputo type fractional integro-differential equations with multi-point and sub-strip boundary conditions. The fractional integro-differential equations involve Caputo derivative operators of different orders and finitely many Riemann–Liouville fractional integral and non-integral type nonlinearities. The boundary conditions at the terminal position $t = 1$ involve sub-strips and multi-point contributions. The Banach fixed point theorem and the Leray–Schauder alternative are used to establish our results. The obtained results are illustrated with the aid of examples.

MSC: 34A08; 34B10; 34B15

Keywords: Caputo derivative; Riemann–Liouville integral; Coupled system; Multi-point boundary conditions; Existence; Fixed point theorem

1 Introduction

Fractional calculus has evolved as an important area of investigation owing to its extensive applications in natural and social sciences. Examples include bio-engineering [1], ecology [2], financial economics [3], chaos and fractional dynamics [4], etc. The tools of fractional calculus have improved the mathematical modeling of many real-world problems [5–7]. It has been mainly due to the nonlocal nature of fractional-order differential and integral operators. Fractional-order mathematical models often consist of coupled systems of fractional-order differential and integro-differential equations. For theoretical treatment of such systems, we refer the reader to the papers [8–19] and the references cited therein. In [20], a fractional-order nonlinear mixed coupled system with coupled integro-differential boundary conditions was studied. In a recent article [21], the authors investigated the existence of solutions for the systems of Caputo and Riemann–Liouville type mixed-order coupled fractional differential equations and inclusions equipped with with coupled integral fractional boundary conditions.

In the present study, we investigate a new class of nonlinear coupled systems of Liouville–Caputo type fractional integro-differential equations

$$\begin{cases} {}^cD^q x(t) + \sum_{i=1}^k I^{p_i} g_i(t,x(t),y(t)) = f_1(t,x(t),y(t)), & 1 < q \le 2, t \in [0,1], \\ {}^cD^\delta y(t) + \sum_{j=1}^l I^{\upsilon_j} h_j(t,x(t),y(t)) = f_2(t,x(t),y(t)), & 1 < \delta \le 2, t \in [0,1], \end{cases} \tag{1}$$

subject to the boundary conditions

$$\begin{cases} x(0) = a_1, \qquad y(0) = b_1, \\ \alpha_1 x(1) + \beta_1 x'(1) = \gamma_1 \int_0^\zeta y(s)\,ds + \sum_{m=1}^\omega \mu_m y(\eta_m), \\ \alpha_2 y(1) + \beta_2 y'(1) = \gamma_2 \int_0^\zeta x(s)\,ds + \sum_{m=1}^\omega \xi_m x(\eta_m), \end{cases} \tag{2}$$

where ${}^cD^q$, ${}^cD^\delta$ respectively denote the Caputo fractional derivative operators of order $q,\delta \in (1,2]$ and $f_1, f_2, g_i, h_j : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $(i = 1,\dots,k)$ $(j = 1,\dots,l)$ are continuous functions, $a_1, b_1, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \mu_m, \xi_m \in \mathbb{R}$ and $\zeta, \eta_m \in (0,1)$, $m = 1,2,\dots,\omega$.

Existence and uniqueness results for the given problem are established via Banach fixed point theorem and Leray–Schauder alternative. The main results are presented in Sect. 3. In Sect. 2, some basic definitions from fractional calculus are recalled and an auxiliary result concerning the linear version of problem (1)–(2), which is essential to define the solution of problem (1)–(2), is proved. Examples illustrating the obtained results are constructed in Sect. 4.

2 Preliminaries

Let us recall some basic definitions on fractional calculus [22].

Definition 1 For a function $g \in AC^n[a,b]$, the Caputo derivative of fractional order $q \in (n-1,n]$, $n \in \mathbb{N}$, existing almost everywhere on $[a,b]$, is defined as follows:

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} g^{(n)}(s)\,ds, \quad t \in [a,b].$$

Definition 2 The Riemann–Liouville fractional integral of order $q > 0$ for $g \in L_1[a,b]$, existing almost everywhere on $[a,b]$, is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_a^t \frac{g(s)}{(t-s)^{1-q}}\,ds, \quad t \in [a,b].$$

Lemma 1 *For $m-1 < q \le m$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{m-1} t^{m-1},$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\dots,m-1$.

In view of Lemma 1, it follows that

$$I^{q\,c}D^q x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{m-1} t^{m-1} \tag{3}$$

for some $c_i \in \mathbb{R}$, $i = 0,1,2,\dots,m-1$.

To study nonlinear problem (1), we first consider the associated linear problem and obtain its solution.

Lemma 2 *Let $\Lambda_1 \neq 0$. For $\widehat{f_1}, \widehat{f_2} \in C([0,1],\mathbb{R})$ the integral solution of the linear system of fractional differential equations*

$$\begin{cases} {}^cD^q x(t) = \widehat{f_1}(t), & 1 < q \leq 2, t \in [0,1], \\ {}^cD^\delta y(t) = \widehat{f_2}(t), & 1 < \delta \leq 2, t \in [0,1], \end{cases} \tag{4}$$

supplemented with the boundary conditions (2) is given by

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \widehat{f_1}(s)\,ds + a_1 - \frac{t}{\Lambda_1}\Bigg[\sigma_1 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \widehat{f_1}(s)\,ds \\ & + \sigma_2 \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \widehat{f_1}(s)\,ds - \sigma_3 \int_0^\zeta \int_0^s \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} \widehat{f_2}(u)\,du\,ds \\ & - \sigma_4 \int_0^{\eta_m} \frac{(\eta_m-s)^{\delta-1}}{\Gamma(\delta)} \widehat{f_2}(s)\,ds + \sigma_5 \int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)} \widehat{f_2}(s)\,ds \\ & + \sigma_6 \int_0^1 \frac{(1-s)^{\delta-2}}{\Gamma(\delta-1)} \widehat{f_2}(s)\,ds - \sigma_7 \int_0^\zeta \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \widehat{f_1}(u)\,du\,ds \\ & - \sigma_8 \int_0^{\eta_m} \frac{(\eta_m-s)^{q-1}}{\Gamma(q)} \widehat{f_1}(s)\,ds + \Lambda_2\Bigg] \end{aligned} \tag{5}$$

and

$$\begin{aligned} y(t) = {} & \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} \widehat{f_2}(s)\,ds + b_1 - \frac{t}{\Lambda_1}\Bigg[\sigma_9 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \widehat{f_1}(s)\,ds \\ & + \sigma_{10} \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \widehat{f_1}(s)\,ds - \sigma_{11} \int_0^\zeta \int_0^s \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} \widehat{f_2}(u)\,du\,ds \\ & - \sigma_{12} \int_0^{\eta_m} \frac{(\eta_m-s)^{\delta-1}}{\Gamma(\delta)} \widehat{f_2}(s)\,ds + \sigma_{13} \int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)} \widehat{f_2}(s)\,ds \\ & + \sigma_{14} \int_0^1 \frac{(1-s)^{\delta-2}}{\Gamma(\delta-1)} \widehat{f_2}(s)\,ds - \sigma_{15} \int_0^\zeta \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \widehat{f_1}(u)\,du\,ds \\ & - \sigma_{16} \int_0^{\eta_m} \frac{(\eta_m-s)^{q-1}}{\Gamma(q)} \widehat{f_1}(s)\,ds + \Lambda_3\Bigg], \end{aligned} \tag{6}$$

where

$$\begin{aligned} \Lambda_1 &= (\alpha_1+\beta_1)(\alpha_2+\beta_2) - \left(\gamma_1\frac{\zeta^2}{2} + \sum_{m=1}^{\omega}\mu_m\eta_m\right)\left(\gamma_2\frac{\zeta^2}{2} + \sum_{m=1}^{\omega}\xi_m\eta_m\right), \\ \Lambda_2 &= (\alpha_2+\beta_2)\left[a_1\alpha_1 - b_1\gamma_1\zeta - b_1\sum_{m=1}^{\omega}\mu_m\right] \\ &\quad + \left(\gamma_1\frac{\zeta^2}{2} + \sum_{m=1}^{\omega}\mu_m\eta_m\right)\left[b_1\alpha_2 - a_1\gamma_2\zeta - a_1\sum_{m=1}^{\omega}\xi_m\right], \end{aligned}$$

$$\Lambda_3 = (\alpha_1+\beta_1)\left[b_1\alpha_2 - a_1\gamma_2\zeta - a_1\sum_{m=1}^{\omega}\xi_m\right] + \left(\gamma_2\frac{\zeta^2}{2}+\sum_{m=1}^{\omega}\xi_m\eta_m\right)\left[a_1\alpha_1 - b_1\gamma_1\zeta - b_1\sum_{m=1}^{\omega}\mu_m\right],$$

$$\begin{cases} \sigma_1 = \alpha_1(\alpha_2+\beta_2), \quad \sigma_7 = \gamma_2(\gamma_1\frac{\zeta^2}{2}+\sum_{m=1}^{\omega}\mu_m\eta_m), \quad \sigma_{13} = \alpha_2(\alpha_1+\beta_1), \\ \sigma_2 = \beta_1(\alpha_2+\beta_2), \quad \sigma_8 = (\gamma_1\frac{\zeta^2}{2}+\sum_{m=1}^{\omega}\mu_m\eta_m)\sum_{m=1}^{\omega}\xi_m, \\ \sigma_{14} = \beta_2(\alpha_1+\beta_1), \quad \sigma_3 = \gamma_1(\alpha_2+\beta_2), \quad \sigma_9 = \alpha_1(\gamma_2\frac{\zeta^2}{2}+\sum_{m=1}^{\omega}\xi_m\eta_m), \\ \sigma_{15} = \gamma_2(\alpha_1+\beta_1), \quad \sigma_4 = (\alpha_2+\beta_2)\sum_{m=1}^{\omega}\mu_m, \\ \sigma_{10} = \beta_1(\gamma_2\frac{\zeta^2}{2}+\sum_{m=1}^{\omega}\xi_m\eta_m), \quad \sigma_{16} = (\alpha_1+\beta_1)\sum_{m=1}^{\omega}\xi_m, \\ \sigma_5 = \alpha_2(\gamma_1\frac{\zeta^2}{2}+\sum_{m=1}^{\omega}\mu_m\eta_m), \quad \sigma_{11} = \gamma_1(\gamma_2\frac{\zeta^2}{2}+\sum_{m=1}^{\omega}\xi_m\eta_m), \\ \sigma_6 = \beta_2(\gamma_1\frac{\zeta^2}{2}+\sum_{m=1}^{\omega}\mu_m\eta_m), \quad \sigma_{12} = (\gamma_2\frac{\zeta^2}{2}+\sum_{m=1}^{\omega}\xi_m\eta_m)\sum_{m=1}^{\omega}\mu_m. \end{cases} \tag{7}$$

Proof Applying the integral operators I^q and I^δ respectively on the first and second equations of (4) and then using (3), we get

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\widehat{f_1}(s)\,ds - c_0 - c_1 t, \tag{8}$$

$$y(t) = \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\widehat{f_2}(s)\,ds - d_0 - d_1 t, \tag{9}$$

where c_0, c_1, d_0, d_1 are arbitrary constants. Using the conditions $x(0) = a_1$ and $y(0) = b_1$ respectively in (8) and (9), we find that $c_0 = -a_1$, $d_0 = -b_1$; consequently, we have

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\widehat{f_1}(s)\,ds + a_1 - c_1 t, \tag{10}$$

$$y(t) = \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\widehat{f_2}(s)\,ds + b_1 - d_1 t. \tag{11}$$

Using (10) and (11) in the conditions $\alpha_1 x(1) + \beta_1 x'(1) = \gamma_1\int_0^\zeta y(s)\,ds + \sum_{m=1}^{\omega}\mu_m y(\eta_m)$ and $\alpha_2 y(1) + \beta_2 y'(1) = \gamma_2\int_0^\zeta x(s)\,ds + \sum_{m=1}^{\omega}\xi_m x(\eta_m)$, we obtain a system of equations in the unknown constants c_1 and d_1 given by

$$\begin{cases} U_1 c_1 - V_1 d_1 = A_1, \\ -V_2 c_1 + U_2 d_1 = A_2, \end{cases} \tag{12}$$

where

$$U_1 = (\alpha_1+\beta_1), \qquad U_2 = (\alpha_2+\beta_2),$$

$$V_1 = \left(\gamma_1\frac{\zeta^2}{2}+\sum_{m=1}^{\omega}\mu_m\eta_m\right), \qquad V_2 = \left(\gamma_2\frac{\zeta^2}{2}+\sum_{m=1}^{\omega}\xi_m\eta_m\right),$$

$$
\begin{aligned}
A_1 ={}& \alpha_1 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \widehat{f_1}(s)\,ds + \beta_1 \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \widehat{f_1}(s)\,ds \\
&- \gamma_1 \int_0^\zeta \int_0^s \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} \widehat{f_2}(u)\,du\,ds \\
&- \sum_{m=1}^{\omega} \mu_m \int_0^{\eta_m} \frac{(\eta_m-s)^{\delta-1}}{\Gamma(\delta)} \widehat{f_2}(s)\,ds + a_1\alpha_1 - b_1\gamma_1\zeta - b_1 \sum_{m=1}^{\omega} \mu_m, \\
A_2 ={}& \alpha_2 \int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)} \widehat{f_2}(s)\,ds + \beta_2 \int_0^1 \frac{(1-s)^{\delta-2}}{\Gamma(\delta-1)} \widehat{f_2}(s)\,ds \\
&- \gamma_2 \int_0^\zeta \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \widehat{f_1}(u)\,du\,ds \\
&- \sum_{m=1}^{\omega} \xi_m \int_0^{\eta_m} \frac{(\eta_m-s)^{q-1}}{\Gamma(q)} \widehat{f_1}(s)\,ds + b_1\alpha_2 - a_1\gamma_2\zeta - a_1 \sum_{m=1}^{\omega} \xi_m.
\end{aligned}
\tag{13}
$$

Solving system (12) for c_1 and d_1, we find that

$$
c_1 = \frac{U_2A_1 + V_1A_2}{U_1U_2 - V_1V_2}, \qquad d_1 = \frac{V_2A_1 + U_1A_2}{U_1U_2 - V_1V_2}.
$$

Substituting the values of c_1 and d_1 in (10) and (11) respectively together with notations (13) leads to solutions (5) and (6). The converse can be proved by direct computation. The proof is completed. □

3 Existence and uniqueness results

Let $\mathcal{S} = \{x | x \in C([a,b],\mathbb{R})\}$ be the space equipped with the norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$. Obviously, $(\mathcal{S}, \|\cdot\|)$ is a Banach space and, consequently, the product space $(\mathcal{S}\times\mathcal{S}, \|\cdot\|)$ is a Banach space with the norm $\|(x+y)\| = \|x\| + \|y\|$ for $(x,y) \in \mathcal{S}\times\mathcal{S}$.

In view of Lemma 2, we define an operator $\mathcal{Q} : \mathcal{S}\times\mathcal{S} \to \mathcal{S}\times\mathcal{S}$ by

$$
\mathcal{Q}(x,y)(t) := \big(\mathcal{Q}_1(x,y)(t), \mathcal{Q}_2(x,y)(t)\big),
$$

where

$$
\begin{aligned}
&\mathcal{Q}_1(x,y)(t) \\
&\quad= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_1\big(s,x(s),y(s)\big)\,ds - \sum_{i=1}^{k} \int_0^t \frac{(t-s)^{q+p_i-1}}{\Gamma(q+p_i)} g_i\big(s,x(s),y(s)\big)\,ds + a_1 \\
&\qquad - \frac{t}{\Lambda_1}\Bigg[\sigma_1 \int_0^1 \Bigg(\frac{(1-s)^{q-1}}{\Gamma(q)} f_1\big(s,x(s),y(s)\big) - \sum_{i=1}^{k} \frac{(1-s)^{q+p_i-1}}{\Gamma(q+p_i)} g_i\big(s,x(s),y(s)\big)\Bigg)\,ds \\
&\qquad + \sigma_2 \int_0^1 \Bigg(\frac{(1-s)^{q-2}}{\Gamma(q-1)} f_1\big(s,x(s),y(s)\big) - \sum_{i=1}^{k} \frac{(1-s)^{q+p_i-2}}{\Gamma(q+p_i-1)} g_i\big(s,x(s),y(s)\big)\Bigg)\,ds \\
&\qquad - \sigma_3 \Bigg(\int_0^\zeta \int_0^s \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} f_2\big(u,x(u),y(u)\big)\,du\,ds \\
&\qquad + \sum_{j=1}^{l} \int_0^\zeta \int_0^s \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} \int_0^u \frac{(u-w)^{\upsilon_j-1}}{\Gamma(\upsilon_j)} h_j\big(w,x(w),y(w)\big)\,dw\,du\,ds\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&- \sigma_4 \int_0^{\eta_m} \left(\frac{(\eta_m - s)^{\delta-1}}{\Gamma(\delta)} f_2\big(s, x(s), y(s)\big) + \sum_{j=1}^{l} \frac{(\eta_m - s)^{\delta+\upsilon_j-1}}{\Gamma(\delta+\upsilon_j)} h_j\big(s, x(s), y(s)\big) \right) ds \\
&+ \sigma_5 \int_0^1 \left(\frac{(1-s)^{\delta-1}}{\Gamma(\delta)} f_2\big(s, x(s), y(s)\big) - \sum_{j=1}^{l} \frac{(1-s)^{\delta+\upsilon_j-1}}{\Gamma(\delta+\upsilon_j)} h_j\big(s, x(s), y(s)\big) \right) ds \\
&+ \sigma_6 \int_0^1 \left(\frac{(1-s)^{\delta-2}}{\Gamma(\delta-1)} f_2\big(s, x(s), y(s)\big) - \sum_{j=1}^{l} \frac{(1-s)^{\delta+\upsilon_j-2}}{\Gamma(\delta+\upsilon_j-1)} h_j\big(s, x(s), y(s)\big) \right) ds \\
&- \sigma_7 \Bigg(\int_0^{\zeta} \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f_1\big(u, x(u), y(u)\big)\, du\, ds \\
&+ \sum_{i=1}^{k} \int_0^{\zeta} \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \int_0^u \frac{(u-w)^{p_i-1}}{\Gamma(p_i)} g_i\big(w, x(w), y(w)\big)\, dw\, du\, ds \Bigg) \\
&- \sigma_8 \int_0^{\eta_m} \Bigg(\frac{(\eta_m - s)^{q-1}}{\Gamma(q)} f_1\big(s, x(s), y(s)\big) \\
&+ \sum_{i=1}^{k} \frac{(\eta_m - s)^{q+p_i-1}}{\Gamma(q+p_i)} g_i\big(s, x(s), y(s)\big) \Bigg) ds + \Lambda_2 \Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{Q}_2(x,y)(t) \\
&= \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} f_2\big(s, x(s), y(s)\big)\, ds - \sum_{j=1}^{l} \int_0^t \frac{(t-s)^{\delta+\upsilon_j-1}}{\Gamma(\delta+\upsilon_j)} h_j\big(s, x(s), y(s)\big)\, ds + b_1 \\
&- \frac{t}{\Lambda_1} \Bigg[\sigma_9 \int_0^1 \Big(\frac{(1-s)^{q-1}}{\Gamma(q)} f_1\big(s, x(s), y(s)\big) - \sum_{i=1}^{k} \frac{(1-s)^{q+p_i-1}}{\Gamma(q+p_i)} g_i\big(s, x(s), y(s)\big)\, ds \\
&+ \sigma_{10} \int_0^1 \left(\frac{(1-s)^{q-2}}{\Gamma(q-1)} f_1\big(s, x(s), y(s)\big) - \sum_{i=1}^{k} \frac{(1-s)^{q+p_i-2}}{\Gamma(q+p_i-1)} g_i\big(s, x(s), y(s)\big) \right) ds \\
&- \sigma_{11} \Bigg(\int_0^{\zeta} \int_0^s \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} f_2\big(u, x(u), y(u)\big)\, du\, ds \\
&+ \sum_{j=1}^{l} \int_0^{\zeta} \int_0^s \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} \int_0^u \frac{(u-w)^{\upsilon_j-1}}{\Gamma(\upsilon_j)} h_j\big(w, x(w), y(w)\big)\, dw\, du\, ds \Bigg) \\
&- \sigma_{12} \int_0^{\eta_m} \left(\frac{(\eta_m - s)^{\delta-1}}{\Gamma(\delta)} f_2\big(s, x(s), y(s)\big) + \sum_{j=1}^{l} \frac{(\eta_m - s)^{\delta+\upsilon_j-1}}{\Gamma(\delta+\upsilon_j)} h_j\big(s, x(s), y(s)\big) \right) ds \\
&+ \sigma_{13} \int_0^1 \left(\frac{(1-s)^{\delta-1}}{\Gamma(\delta)} f_2\big(s, x(s), y(s)\big) - \sum_{j=1}^{l} \frac{(1-s)^{\delta+\upsilon_j-1}}{\Gamma(\delta+\upsilon_j)} h_j\big(s, x(s), y(s)\big) \right) ds \\
&+ \sigma_{14} \int_0^1 \left(\frac{(1-s)^{\delta-2}}{\Gamma(\delta-1)} f_2\big(s, x(s), y(s)\big) - \sum_{j=1}^{l} \frac{(1-s)^{\delta+\upsilon_j-2}}{\Gamma(\delta+\upsilon_j-1)} h_j\big(s, x(s), y(s)\big) \right) ds \\
&- \sigma_{15} \Bigg(\int_0^{\zeta} \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f_1\big(u, x(u), y(u)\big)\, du\, ds
\end{aligned}
$$

$$+\sum_{i=1}^{k}\int_0^{\zeta}\int_0^{s}\frac{(s-u)^{q-1}}{\Gamma(q)}\int_0^{u}\frac{(u-w)^{p_i-1}}{\Gamma(p_i)}g_i\big(w,x(w),y(w)\big)\,dw\,du\,ds\Bigg)$$
$$-\sigma_{16}\int_0^{\eta_m}\Bigg(\frac{(\eta_m-s)^{q-1}}{\Gamma(q)}f_1\big(s,x(s),y(s)\big)$$
$$+\sum_{i=1}^{k}\frac{(\eta_m-s)^{q+p_i-1}}{\Gamma(q+p_i)}g_i\big(s,x(s),y(s)\big)\Bigg)\,ds+\Lambda_3\Bigg].$$

In the sequel, we use the following notations:

$$\begin{aligned}
\varphi_1 &= \left[\frac{1}{\Gamma(q+1)}+\frac{1}{|\Lambda_1|}\left(\frac{|\sigma_1|+|\sigma_8|\eta_m^{q}}{\Gamma(q+1)}+\frac{|\sigma_2|}{\Gamma(q)}+\frac{|\sigma_7|\zeta^{q+1}}{\Gamma(q+2)}\right)\right],\\
\varphi_2 &= \left[\frac{1}{|\Lambda_1|}\left(\frac{|\sigma_3|\zeta^{\delta+1}}{\Gamma(\delta+2)}+\frac{|\sigma_4|\eta_m^{\delta}+|\sigma_5|}{\Gamma(\delta+1)}+\frac{|\sigma_6|}{\Gamma(\delta)}\right)\right],\\
\Omega_i &= \left[\frac{1}{\Gamma(q+p_i+1)}+\frac{1}{|\Lambda_1|}\left(\frac{|\sigma_1|+|\sigma_8|\eta_m^{q+p_i}}{\Gamma(q+p_i+1)}+\frac{|\sigma_2|}{\Gamma(q+p_i)}+\frac{|\sigma_7|\zeta^{q+p_i+1}}{\Gamma(q+p_i+2)}\right)\right],\\
\widehat{\Omega}_j &= \left[\frac{1}{|\Lambda_1|}\left(\frac{|\sigma_3|\zeta^{\delta+\upsilon_j+1}}{\Gamma(\delta+\upsilon_j+2)}+\frac{|\sigma_4|\eta_m^{\delta+\upsilon_j}+|\sigma_5|}{\Gamma(\delta+\upsilon_j+1)}+\frac{|\sigma_6|}{\Gamma(\delta+\upsilon_j)}\right)\right],
\end{aligned}\tag{14}$$

$$\begin{aligned}
\vartheta_1 &= \left[\frac{1}{|\Lambda_1|}\left(\frac{|\sigma_9|+|\sigma_{16}|\eta_m^{q}}{\Gamma(q+1)}+\frac{|\sigma_{10}|}{\Gamma(q)}+\frac{|\sigma_{15}|\zeta^{q+1}}{\Gamma(q+2)}\right)\right],\\
\vartheta_2 &= \left[\frac{1}{\Gamma(\delta+1)}+\frac{1}{|\Lambda_1|}\left(\frac{|\sigma_{11}|\zeta^{\delta+1}}{\Gamma(\delta+2)}+\frac{|\sigma_{12}|\eta_m^{\delta}+|\sigma_{13}|}{\Gamma(\delta+1)}+\frac{|\sigma_{14}|}{\Gamma(\delta)}\right)\right],\\
\Theta_i &= \left[\frac{1}{|\Lambda_1|}\left(\frac{|\sigma_9|+|\sigma_{16}|\eta_m^{q+p_i}}{\Gamma(q+p_i+1)}+\frac{|\sigma_{10}|}{\Gamma(q+p_i)}+\frac{|\sigma_{15}|\zeta^{q+p_i+1}}{\Gamma(q+p_i+2)}\right)\right],\\
\widehat{\Theta}_j &= \left[\frac{1}{\Gamma(\delta+\upsilon_j+1)}+\frac{1}{|\Lambda_1|}\left(\frac{|\sigma_{11}|\zeta^{\delta+\upsilon_j+1}}{\Gamma(\delta+\upsilon_j+2)}+\frac{|\sigma_{12}|\eta_m^{\delta+\upsilon_j}+|\sigma_{13}|}{\Gamma(\delta+\upsilon_j+1)}+\frac{|\sigma_{14}|}{\Gamma(\delta+\upsilon_j)}\right)\right].
\end{aligned}\tag{15}$$

Now we prove the existence and uniqueness of solutions for system (1) by applying the Banach contraction mapping principle.

Theorem 1 *Let $f_1, f_2, g_i, h_j : [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ $(i=1,\dots,k)$ $(j=1,\dots,l)$ be continuous functions. In addition, we assume that*:

(H_1) *There exist constants $L_1, L_2 > 0$ such that, $\forall t\in[0,1]$ and $x_\epsilon, y_\epsilon\in\mathbb{R}$, $\epsilon=1,2$,*

$$\big|f_1(t,x_1,y_1)-f_1(t,x_2,y_2)\big|\le L_1\big(|x_1-y_1|+|x_2-y_2|\big),$$
$$\big|f_2(t,x_1,y_1)-f_2(t,x_2,y_2)\big|\le L_2\big(|x_1-y_1|+|x_2-y_2|\big).$$

(H_2) *There exist constants $M_i, \widehat{M}_j > 0$ $(i=1,\dots,k)$ $(j=1,\dots,l)$ such that $\forall t\in[0,1]$ and $x_\epsilon, y_\epsilon\in\mathbb{R}$, $\epsilon=1,2$,*

$$\big|g_i(t,x_1,y_1)-g_i(t,x_2,y_2)\big|\le M_i\big(|x_1-y_1|+|x_2-y_2|\big),$$
$$\big|h_j(t,x_1,y_1)-h_j(t,x_2,y_2)\big|\le \widehat{M}_j\big(|x_1-y_1|+|x_2-y_2|\big).$$

Then system (1)–(2) *has a unique solution on* $[0,1]$, *provided that* $\Psi < 1$, *where*

$$\Psi = L_1(\varphi_1 + \vartheta_1) + L_2(\varphi_2 + \vartheta_2) + \sum_{i=1}^{k} M_i(\Omega_i + \Theta_i) + \sum_{j=1}^{l} \widehat{M}_j(\widehat{\Omega}_j + \widehat{\Theta}_j). \tag{16}$$

Proof Let us define finite numbers W_1, W_2, N_i, $\widehat{N}_j$ as follows:

$$W_1 = \sup_{t\in[0,1]} \bigl|f_1(t,0,0)\bigr|, \qquad W_2 = \sup_{t\in[0,1]} \bigl|f_2(t,0,0)\bigr|,$$
$$N_i = \sup_{t\in[0,1]} \bigl|g_i(t,0,0)\bigr|, \qquad \widehat{N}_j = \sup_{t\in[0,1]} \bigl|h_j(t,0,0)\bigr|,$$

and show that $\mathcal{Q}B_r \subset B_r$, where $B_r = \{(x,y) \in \mathcal{S} \times \mathcal{S} : \|(x,y)\| \le r\}$ with

$$r > \frac{W_1(\varphi_1 + \vartheta_1) + W_2(\varphi_2 + \vartheta_2) + \sum_{i=1}^{k} N_i(\Omega + \Theta_i) + \sum_{j=1}^{l} \widehat{N}_j(\widehat{\Omega}_j + \widehat{\Theta}_j) + |a_1| + |b_1| + (|\Lambda_2 + \Lambda_3|)/|\Lambda_1|}{1 - \Psi}.$$

For any $(x,y) \in B_r$, $t \in [0,1]$, using (H_1), we get

$$\begin{aligned}
\bigl|f_1(t,x,y)\bigr| &= \bigl|f_1(t,x,y) - f_1(t,0,0) + f_1(t,0,0)\bigr| \\
&\le \bigl|f_1(t,x,y) - f_1(t,0,0)\bigr| + \bigl|f_1(t,0,0)\bigr| \\
&\le L_1\bigl(\bigl|x(t)\bigr| + \bigl|y(t)\bigr|\bigr) + W_1 \le L_1\bigl(\|x\| + \|y\|\bigr) + W_1 \le L_1 r + W_1.
\end{aligned}$$

Similarly, we can find that

$$\bigl|f_2(t,x,y)\bigr| \le L_2 r + W_2, \qquad \bigl|g_i(t,x,y)\bigr| \le M_i r + N_i, \qquad \bigl|h_j(t,x,y)\bigr| \le \widehat{M}_j r + \widehat{N}_j.$$

Then

$$\begin{aligned}
&\bigl\|\mathcal{Q}_1(x,y)\bigr\| \\
&\quad\le \sup_{t\in[0,1]} \Biggl\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \bigl|f_1\bigl(s,x(s),y(s)\bigr)\bigr|\,ds + \sum_{i=1}^{k} \int_0^t \frac{(t-s)^{q+p_i-1}}{\Gamma(q+p_i)} \bigl|g_i\bigl(s,x(s),y(s)\bigr)\bigr|\,ds \\
&\qquad + |a_1| + \frac{t}{|\Lambda_1|} \Biggl[|\sigma_1| \int_0^1 \Biggl(\frac{(1-s)^{q-1}}{\Gamma(q)} \bigl|f_1\bigl(s,x(s),y(s)\bigr)\bigr| \\
&\qquad + \sum_{i=1}^{k} \frac{(1-s)^{q+p_i-1}}{\Gamma(q+p_i)} \bigl|g_i\bigl(s,x(s),y(s)\bigr)\bigr| \Biggr) ds \\
&\qquad + |\sigma_2| \int_0^1 \Biggl(\frac{(1-s)^{q-2}}{\Gamma(q-1)} \bigl|f_1\bigl(s,x(s),y(s)\bigr)\bigr| + \sum_{i=1}^{k} \frac{(1-s)^{q+p_i-2}}{\Gamma(q+p_i-1)} \bigl|g_i\bigl(s,x(s),y(s)\bigr)\bigr| \Biggr) ds \\
&\qquad + |\sigma_3| \Biggl(\int_0^{\zeta} \int_0^{s} \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} \bigl|f_2\bigl(u,x(u),y(u)\bigr)\bigr|\,du\,ds \\
&\qquad + \sum_{j=1}^{l} \int_0^{\zeta} \int_0^{s} \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} \int_0^{u} \frac{(u-w)^{\upsilon_j-1}}{\Gamma(\upsilon_j)} \bigl|h_j\bigl(w,x(w),y(w)\bigr)\bigr|\,dw\,du\,ds \Biggr)
\end{aligned}$$

$$
\begin{aligned}
&+|\sigma_4|\int_0^{\eta_m}\left(\frac{(\eta_m-s)^{\delta-1}}{\Gamma(\delta)}\big|f_2\big(s,x(s),y(s)\big)\big|+\sum_{j=1}^{l}\frac{(\eta_m-s)^{\delta+\upsilon_j-1}}{\Gamma(\delta+\upsilon_j)}\big|h_j\big(s,x(s),y(s)\big)\big|\right)ds \\
&+|\sigma_5|\int_0^{1}\left(\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}\big|f_2\big(s,x(s),y(s)\big)\big|+\sum_{j=1}^{l}\frac{(1-s)^{\delta+\upsilon_j-1}}{\Gamma(\delta+\upsilon_j)}\big|h_j\big(s,x(s),y(s)\big)\big|\right)ds \\
&+|\sigma_6|\int_0^{1}\left(\frac{(1-s)^{\delta-2}}{\Gamma(\delta-1)}\big|f_2\big(s,x(s),y(s)\big)\big|+\sum_{j=1}^{l}\frac{(1-s)^{\delta+\upsilon_j-2}}{\Gamma(\delta+\upsilon_j-1)}\big|h_j\big(s,x(s),y(s)\big)\big|\right)ds \\
&+|\sigma_7|\left(\int_0^{\zeta}\int_0^{s}\frac{(s-u)^{q-1}}{\Gamma(q)}\big|f_1\big(u,x(u),y(u)\big)\big|\,du\,ds\right. \\
&\left.+\sum_{i=1}^{k}\int_0^{\zeta}\int_0^{s}\frac{(s-u)^{q-1}}{\Gamma(q)}\int_0^{u}\frac{(u-w)^{p_i-1}}{\Gamma(p_i)}\big|g_i\big(w,x(w),y(w)\big)\big|\,dw\,du\,ds\right) \\
&+|\sigma_8|\int_0^{\eta_m}\left(\frac{(\eta_m-s)^{q-1}}{\Gamma(q)}\big|f_1\big(s,x(s),y(s)\big)\big|\right. \\
&\left.+\sum_{i=1}^{k}\frac{(\eta_m-s)^{q+p_i-1}}{\Gamma(q+p_i)}\big|g_i\big(s,x(s),y(s)\big)\big|\right)ds+|\Lambda_2|\Bigg]\Bigg\} \\
\le{}&(L_1r+W_1)\sup_{t\in[0,1]}\left\{\left[\frac{t^q}{\Gamma(q+1)}+\frac{t}{|\Lambda_1|}\left(\frac{|\sigma_1|}{\Gamma(q+1)}+\frac{|\sigma_2|}{\Gamma(q)}+\frac{|\sigma_7|\zeta^{q+1}}{\Gamma(q+2)}+\frac{|\sigma_8|\eta_m^{q}}{\Gamma(q+1)}\right)\right]\right. \\
&+(L_2r+W_2)\left[\frac{t}{|\Lambda_1|}\left(\frac{|\sigma_3|\zeta^{\delta+1}}{\Gamma(\delta+2)}+\frac{|\sigma_4|\eta_m^{\delta}}{\Gamma(\delta+1)}+\frac{|\sigma_5|}{\Gamma(\delta+1)}+\frac{|\sigma_6|}{\Gamma(\delta)}\right)\right] \\
&+\sum_{i=1}^{k}(M_ir+N_i)\left[\frac{t^{q+p_i}}{\Gamma(q+p_i+1)}+\frac{t}{|\Lambda_1|}\left(\frac{|\sigma_1|}{\Gamma(q+p_i+1)}+\frac{|\sigma_2|}{\Gamma(q+p_i)}+\frac{|\sigma_7|\zeta^{q+p_i+1}}{\Gamma(q+p_i+2)}\right.\right. \\
&\left.\left.+\frac{|\sigma_8|\eta_m^{q+p_i}}{\Gamma(q+p_i+1)}\right)\right]+\sum_{j=1}^{l}(\widehat{M}_jr+\widehat{N}_j)\left[\frac{t}{|\Lambda_1|}\left(\frac{|\sigma_3|\zeta^{\delta+\upsilon_j+1}}{\Gamma(\delta+\upsilon_j+2)}+\frac{|\sigma_4|\eta_m^{\delta+\upsilon_j}}{\Gamma(\delta+\upsilon_j+1)}\right.\right. \\
&\left.\left.\left.+\frac{|\sigma_5|}{\Gamma(\delta+\upsilon_j+1)}+\frac{|\sigma_6|}{\Gamma(\delta+\upsilon_j)}\right)\right]+\left(|a_1|+\frac{t|\Lambda_2|}{|\Lambda_1|}\right)\right\} \\
\le{}&\left(L_1\varphi_1+L_2\varphi_2+\sum_{i=1}^{k}M_i\Omega_i+\sum_{j=1}^{l}\widehat{M}_j\widehat{\Omega}_j\right)r+W_1\varphi_1+W_2\varphi_2+\sum_{i=1}^{k}N_i\Omega_i \\
&+\sum_{j=1}^{l}\widehat{N}_j\widehat{\Omega}_j+|a_1|+\left|\frac{\Lambda_2}{\Lambda_1}\right|.
\end{aligned}
$$

Similarly, we can find that

$$
\begin{aligned}
\big\|\mathcal{Q}_2(x,y)\big\|\le{}&\left(L_1\vartheta_1+L_2\vartheta_2+\sum_{i=1}^{k}M_i\Theta_i+\sum_{j=1}^{l}\widehat{M}_j\widehat{\Theta}_j\right)r+W_1\vartheta_1+W_2\vartheta_2+\sum_{i=1}^{k}N_i\Theta_i \\
&+\sum_{j=1}^{l}\widehat{N}_j\widehat{\Theta}_j+|b_1|+\left|\frac{\Lambda_3}{\Lambda_1}\right|.
\end{aligned}
$$

Consequently, in view of (16), we get

$$\begin{aligned}\|\mathcal{Q}(x,y)\| &\le \Psi r + W_1(\varphi_1+\vartheta_1) + W_2(\varphi_2+\vartheta_2) + \sum_{i=1}^{k} N_i(\Omega_i+\Theta_i) + \sum_{j=1}^{l} \widehat{N}_j(\widehat{\Omega}_j+\widehat{\Theta}_j)\\ &\quad + |a_1| + |b_1| + \left|\frac{\Lambda_2+\Lambda_3}{\Lambda_1}\right| \le r,\end{aligned}$$

which implies that $\mathcal{Q}B_r \subset B_r$. Next we show that the operator $\mathcal{Q}$ is a contraction. Using conditions (H_1) and (H_2), for any $(x_1,y_1),(x_2,y_2)\in\mathcal{S}\times\mathcal{S}$, $t\in[0,1]$, we get

$$\begin{aligned}&\|\mathcal{Q}_1(x_1,y_1)-\mathcal{Q}_1(x_2,y_2)\|\\
&\le \sup_{t\in[0,1]}\Biggl\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\bigl|f_1\bigl(s,x_1(s),y_1(s)\bigr)-f_1\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\,ds\\
&\quad+\sum_{i=1}^{k}\int_0^t \frac{(t-s)^{q+p_i-1}}{\Gamma(q+p_i)}\bigl|g_i\bigl(s,x_1(s),y_1(s)\bigr)-g_i\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\,ds\\
&\quad+\frac{t}{|\Lambda_1|}\Biggl[|\sigma_1|\int_0^1\Biggl(\frac{(1-s)^{q-1}}{\Gamma(q)}\bigl|f_1\bigl(s,x_1(s),y_1(s)\bigr)-f_1\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\\
&\quad+\sum_{i=1}^{k}\frac{(1-s)^{q+p_i-1}}{\Gamma(q+p_i)}\bigl|g_i\bigl(s,x_1(s),y_1(s)\bigr)-g_i\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\Biggr)\,ds\\
&\quad+|\sigma_2|\int_0^1\Biggl(\frac{(1-s)^{q-2}}{\Gamma(q-1)}\bigl|f_1\bigl(s,x_1(s),y_1(s)\bigr)-f_1\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\\
&\quad+\sum_{i=1}^{k}\frac{(1-s)^{q+p_i-2}}{\Gamma(q+p_i-1)}\bigl|g_i\bigl(s,x_1(s),y_1(s)\bigr)-g_i\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\Biggr)\,ds\\
&\quad+|\sigma_3|\Biggl(\int_0^{\zeta}\int_0^s\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}\bigl|f_2\bigl(u,x_1(u),y_1(u)\bigr)-f_2\bigl(u,x_2(u),y_2(u)\bigr)\bigr|\,du\,ds\\
&\quad+\sum_{j=1}^{l}\int_0^{\zeta}\int_0^s\frac{(s-u)^{\delta-1}}{\Gamma(\delta)}\int_0^u\frac{(u-w)^{\upsilon_j-1}}{\Gamma(\upsilon_j)}\bigl|h_j\bigl(w,x_1(w),y_1(w)\bigr)\\
&\quad-h_j\bigl(w,x_2(w),y_2(w)\bigr)\bigr|\,dw\,du\,ds\Biggr)\\
&\quad+|\sigma_4|\int_0^{\eta_m}\Biggl(\frac{(\eta_m-s)^{\delta-1}}{\Gamma(\delta)}\bigl|f_2\bigl(s,x_1(s),y_1(s)\bigr)-f_2\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\\
&\quad+\sum_{j=1}^{l}\frac{(\eta_m-s)^{\delta+\upsilon_j-1}}{\Gamma(\delta+\upsilon_j)}\bigl|h_j\bigl(s,x_1(s),y_1(s)\bigr)-h_j\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\Biggr)\,ds\\
&\quad+|\sigma_5|\int_0^1\Biggl(\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}\bigl|f_2\bigl(s,x_1(s),y_1(s)\bigr)-f_2\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\end{aligned}$$

$$\begin{aligned}
&+\sum_{j=1}^{l}\frac{(1-s)^{\delta+\upsilon_j-1}}{\Gamma(\delta+\upsilon_j)}\bigl|h_j\bigl(s,x_1(s),y_1(s)\bigr)-h_j\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\Biggr)\,ds\\
&+|\sigma_6|\int_0^1\Biggl(\frac{(1-s)^{\delta-2}}{\Gamma(\delta-1)}\bigl|f_2\bigl(s,x_1(s),y_1(s)\bigr)-f_2\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\\
&+\sum_{j=1}^{l}\frac{(1-s)^{\delta+\upsilon_j-2}}{\Gamma(\delta+\upsilon_j-1)}\bigl|h_j\bigl(s,x_1(s),y_1(s)\bigr)-h_j\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\Biggr)\,ds\\
&+|\sigma_7|\Biggl(\int_0^{\zeta}\int_0^{s}\frac{(s-u)^{q-1}}{\Gamma(q)}\bigl|f_1\bigl(u,x_1(u),y_1(u)\bigr)-f_1\bigl(u,x_2(u),y_2(u)\bigr)\bigr|\,du\,ds\\
&+\sum_{i=1}^{k}\int_0^{\zeta}\int_0^{s}\frac{(s-u)^{q-1}}{\Gamma(q)}\int_0^{u}\frac{(u-w)^{p_i-1}}{\Gamma(p_i)}\bigl|g_i\bigl(w,x_1(w),y_1(w)\bigr)\\
&-g_i\bigl(w,x_2(w),y_2(w)\bigr)\bigr|\,dw\,du\,ds\Biggr)\\
&+|\sigma_8|\int_0^{\eta_m}\Biggl(\frac{(\eta_m-s)^{q-1}}{\Gamma(q)}\bigl|f_1\bigl(s,x_1(s),y_1(s)\bigr)-f_1\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\\
&+\sum_{i=1}^{k}\frac{(\eta_m-s)^{q+p_i-1}}{\Gamma(q+p_i)}\bigl|g_i\bigl(s,x_1(s),y_1(s)\bigr)-g_i\bigl(s,x_2(s),y_2(s)\bigr)\bigr|\Biggr)\,ds\Biggr]\Biggr\}\\
\le{}& L_1\bigl(\|x_1-x_2\|+\|y_1-y_2\|\bigr)\sup_{t\in[0,1]}\Biggl\{\Biggl[\frac{t^q}{\Gamma(q+1)}+\frac{t}{|\Lambda_1|}\Biggl(\frac{|\sigma_1|}{\Gamma(q+1)}+\frac{|\sigma_2|}{\Gamma(q)}+\frac{|\sigma_7|\zeta^{q+1}}{\Gamma(q+2)}\\
&+\frac{|\sigma_8|\eta_m^{q}}{\Gamma(q+1)}\Biggr)\Biggr]+L_2\bigl(\|x_1-x_2\|+\|y_1-y_2\|\bigr)\\
&\times\Biggl[\frac{t}{|\Lambda_1|}\Biggl(\frac{|\sigma_3|\zeta^{\delta+1}}{\Gamma(\delta+2)}+\frac{|\sigma_4|\eta_m^{\delta}}{\Gamma(\delta+1)}+\frac{|\sigma_5|}{\Gamma(\delta+1)}+\frac{|\sigma_6|}{\Gamma(\delta)}\Biggr)\Biggr]\\
&+\sum_{i=1}^{k}M_i\bigl(\|x_1-x_2\|+\|y_1-y_2\|\bigr)\Biggl[\frac{t^{q+p_i}}{\Gamma(q+p_i+1)}+\frac{t}{|\Lambda_1|}\Biggl(\frac{|\sigma_1|}{\Gamma(q+p_i+1)}+\frac{|\sigma_2|}{\Gamma(q+p_i)}\\
&+\frac{|\sigma_7|\zeta^{q+p_i+1}}{\Gamma(q+p_i+2)}+\frac{|\sigma_8|\eta_m^{q+p_i}}{\Gamma(q+p_i+1)}\Biggr)\Biggr]+\sum_{j=1}^{l}\widehat{M}_j\bigl(\|x_1-x_2\|+\|y_1-y_2\|\bigr)\\
&\times\Biggl[\frac{t}{|\Lambda_1|}\Biggl(\frac{|\sigma_3|\zeta^{\delta+\upsilon_j+1}}{\Gamma(\delta+\upsilon_j+2)}+\frac{|\sigma_4|\eta_m^{\delta+\upsilon_j}}{\Gamma(\delta+\upsilon_j+1)}+\frac{|\sigma_5|}{\Gamma(\delta+\upsilon_j+1)}+\frac{|\sigma_6|}{\Gamma(\delta+\upsilon_j)}\Biggr)\Biggr]\Biggr\}\\
\le{}&\Biggl(L_1\varphi_1+L_2\varphi_2+\sum_{i=1}^{k}M_i\Omega_i+\sum_{j=1}^{l}\widehat{M}_j\widehat{\Omega}_j\Biggr)\bigl(\|x_1-x_2\|+\|y_1-y_2\|\bigr),
\end{aligned}$$

which yields

$$\begin{aligned}
&\bigl\|\mathcal{Q}_1(x_1,y_1)-\mathcal{Q}_1(x_2,y_2)\bigr\|\\
&\quad\le\Biggl(L_1\varphi_1+L_2\varphi_2+\sum_{i=1}^{k}M_i\Omega_i+\sum_{j=1}^{l}\widehat{M}_j\widehat{\Omega}_j\Biggr)\bigl(\|x_1-x_2\|+\|y_1-y_2\|\bigr).
\end{aligned}$$

Similarly, we find that

$$\begin{aligned}&\left\|\mathcal{Q}_2(x_1,y_1)-\mathcal{Q}_2(x_2,y_2)\right\|\\&\quad\le\left(L_1\vartheta_1+L_2\vartheta_2+\sum_{i=1}^{k}M_i\Theta_i+\sum_{j=1}^{l}\widehat{M}_j\widehat{\Theta}_j\right)\left(\|x_1-x_2\|+\|y_1-y_2\|\right).\end{aligned}$$

Consequently, we get

$$\left\|\mathcal{Q}(x_1,y_1)-\mathcal{Q}(x_2,y_2)\right\|\le\Psi\left(\|x_1-x_2\|+\|y_1-y_2\|\right),$$

which shows that $\mathcal{Q}$ is a contraction in view of the given condition $\Psi<1$. So, by the Banach contraction mapping principle, the operator $\mathcal{Q}$ has a unique fixed point. Therefore, system (1)–(2) has a unique solution on $[0,1]$. This completes the proof. □

In the following result, we apply the Leray–Schauder alternative [23] to prove the existence of solutions for system (1)–(2).

Lemma 3 (Leray–Schauder alternative) *Let $\mathcal{J}:\mathcal{U}\to\mathcal{U}$ be a completely continuous operator (i.e., a map that restricted to any bounded set in $\mathcal{U}$ is compact). Let*

$$\mathcal{G}(\mathcal{J})=\left\{x\in\mathcal{U}:x=\lambda\mathcal{J}(x)\text{ for some }0<\lambda<1\right\}.$$

Then either the set $\mathcal{G}(\mathcal{J})$ is unbounded, or $\mathcal{J}$ has at lest one fixed point.

Theorem 2 *Let $f_1,f_2,g_i,h_j:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ $(i=1,\dots,k)$ $(j=1,\dots,l)$ be continuous functions such that the following condition holds:*

(H_3) *There exist real constants $\kappa_\epsilon,\hat{\kappa}_\epsilon,\rho_{i,\epsilon},\hat{\rho}_{j,\epsilon}\ge0$ $(\epsilon=1,2)$ and $\kappa_0,\hat{\kappa}_0,\rho_{i,0},\hat{\rho}_{j,0}>0$ such that, for $x,y\in\mathbb{R}$,*

$$\begin{aligned}&\left|f_1(t,x,y)\right|\le\kappa_0+\kappa_1|x|+\kappa_2|y|,\qquad\left|f_2(t,x,y)\right|\le\hat{\kappa}_0+\hat{\kappa}_1|x|+\hat{\kappa}_2|y|,\\&\left|g_i(t,x,y)\right|\le\rho_{i,0}+\rho_{i,1}|x|+\rho_{i,2}|y|,\qquad\left|h_j(t,x,y)\right|\le\hat{\rho}_{j,0}+\hat{\rho}_{j,1}|x|+\hat{\rho}_{j,2}|y|.\end{aligned}$$

Then, system (1)–(2) has at least one solution on $[0,1]$ provided that

$$\begin{aligned}&(\varphi_1+\vartheta_1)\kappa_1+(\varphi_2+\vartheta_2)\hat{\kappa}_1+(\Omega_i+\Theta_i)\rho_{i,1}+(\widehat{\Omega}_j+\widehat{\Theta}_j)\hat{\rho}_{j,1}<1,\\&(\varphi_1+\vartheta_1)\kappa_2+(\varphi_2+\vartheta_2)\hat{\kappa}_2+(\Omega_i+\Theta_i)\rho_{i,2}+(\widehat{\Omega}_j+\widehat{\Theta}_j)\hat{\rho}_{j,2}<1,\end{aligned}\tag{17}$$

where Ω_i, $\widehat{\Omega}_j$ and φ_ϵ, $\epsilon=1,2$, are given by (14), and Θ_i, $\widehat{\Theta}_j$ and ϑ_ϵ, $\epsilon=1,2$, are defined by (15).

Proof In the first step, we show that the operator $\mathcal{Q}:\mathcal{S}\times\mathcal{S}\to\mathcal{S}\times\mathcal{S}$ is completely continuous. Notice that the operator $\mathcal{Q}$ is continuous in view of the continuity of the functions f_1, f_2, g_i, and h_j.

Let $\mathcal{K}\subset\mathcal{S}\times\mathcal{S}$ be bounded. Then, for all $(x,y)\in\mathcal{K}$, there exist constants τ_1, τ_2, ϖ_i, and $\widehat{\varpi}_j$ such that $|f_1(t,x(t),y(t))|\le\tau_1$, $|f_2(t,x(t),y(t))|\le\tau_2$, $|g_i(t,x(t),y(t))|\le\varpi_i$, $|h_j(t,x(t),y(t))|\le$

$\widehat{\varpi}_j$ $(i = 1,\dots,k)$ $(j = 1,\dots,l)$. Then, for any $(x,y) \in \mathcal{K}$, we have

$$
\begin{aligned}
&\|\mathcal{Q}_1(x,y)\| \\
&\le \sup_{t\in[0,1]} \Biggl\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \bigl|f_1\bigl(s,x(s),y(s)\bigr)\bigr|\,ds + \sum_{i=1}^k \int_0^t \frac{(t-s)^{q+p_i-1}}{\Gamma(q+p_i)} \bigl|g_i\bigl(s,x(s),y(s)\bigr)\bigr|\,ds \\
&\quad + |a_1| + \frac{t}{|\Lambda_1|} \Biggl[|\sigma_1| \int_0^1 \biggl(\frac{(1-s)^{q-1}}{\Gamma(q)} \bigl|f_1\bigl(s,x(s),y(s)\bigr)\bigr| \\
&\quad + \sum_{i=1}^k \frac{(1-s)^{q+p_i-1}}{\Gamma(q+p_i)} \bigl|g_i\bigl(s,x(s),y(s)\bigr)\bigr| \biggr)\,ds \\
&\quad + |\sigma_2| \int_0^1 \biggl(\frac{(1-s)^{q-2}}{\Gamma(q-1)} \bigl|f_1\bigl(s,x(s),y(s)\bigr)\bigr| + \sum_{i=1}^k \frac{(1-s)^{q+p_i-2}}{\Gamma(q+p_i-1)} \bigl|g_i\bigl(s,x(s),y(s)\bigr)\bigr| \biggr)\,ds \\
&\quad + |\sigma_3| \biggl(\int_0^\zeta \int_0^s \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} \bigl|f_2\bigl(u,x(u),y(u)\bigr)\bigr|\,du\,ds \\
&\quad + \sum_{j=1}^l \int_0^\zeta \int_0^s \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} \int_0^u \frac{(u-w)^{\upsilon_j-1}}{\Gamma(\upsilon_j)} \bigl|h_j\bigl(w,x(w),y(w)\bigr)\bigr|\,dw\,du\,ds \biggr) \\
&\quad + |\sigma_4| \int_0^{\eta_m} \biggl(\frac{(\eta_m-s)^{\delta-1}}{\Gamma(\delta)} \bigl|f_2\bigl(s,x(s),y(s)\bigr)\bigr| + \sum_{j=1}^l \frac{(\eta_m-s)^{\delta+\upsilon_j-1}}{\Gamma(\delta+\upsilon_j)} \bigl|h_j\bigl(s,x(s),y(s)\bigr)\bigr| \biggr)\,ds \\
&\quad + |\sigma_5| \int_0^1 \biggl(\frac{(1-s)^{\delta-1}}{\Gamma(\delta)} \bigl|f_2\bigl(s,x(s),y(s)\bigr)\bigr| + \sum_{j=1}^l \frac{(1-s)^{\delta+\upsilon_j-1}}{\Gamma(\delta+\upsilon_j)} \bigl|h_j\bigl(s,x(s),y(s)\bigr)\bigr| \biggr)\,ds \\
&\quad + |\sigma_6| \int_0^1 \biggl(\frac{(1-s)^{\delta-2}}{\Gamma(\delta-1)} \bigl|f_2\bigl(s,x(s),y(s)\bigr)\bigr| + \sum_{j=1}^l \frac{(1-s)^{\delta+\upsilon_j-2}}{\Gamma(\delta+\upsilon_j-1)} \bigl|h_j\bigl(s,x(s),y(s)\bigr)\bigr| \biggr)\,ds \\
&\quad + |\sigma_7| \biggl(\int_0^\zeta \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \bigl|f_1\bigl(u,x(u),y(u)\bigr)\bigr|\,du\,ds \\
&\quad + \sum_{i=1}^k \int_0^\zeta \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \int_0^u \frac{(u-w)^{p_i-1}}{\Gamma(p_i)} \bigl|g_i\bigl(w,x(w),y(w)\bigr)\bigr|\,dw\,du\,ds \biggr) \\
&\quad + |\sigma_8| \int_0^{\eta_m} \biggl(\frac{(\eta_m-s)^{q-1}}{\Gamma(q)} \bigl|f_1\bigl(s,x(s),y(s)\bigr)\bigr| \\
&\quad + \sum_{i=1}^k \frac{(\eta_m-s)^{q+p_i-1}}{\Gamma(q+p_i)} \bigl|g_i\bigl(s,x(s),y(s)\bigr)\bigr| \biggr)\,ds + |\Lambda_2| \Biggr] \Biggr\} \\
&\le \tau_1 \sup_{t\in[0,1]} \Biggl\{ \biggl[\frac{t^q}{\Gamma(q+1)} + \frac{t}{|\Lambda_1|} \biggl(\frac{|\sigma_1|}{\Gamma(q+1)} + \frac{|\sigma_2|}{\Gamma(q)} + \frac{|\sigma_7|\zeta^{q+1}}{\Gamma(q+2)} + \frac{|\sigma_8|\eta_m^q}{\Gamma(q+1)} \biggr) \biggr] \\
&\quad + \tau_2 \biggl[\frac{t}{|\Lambda_1|} \biggl(\frac{|\sigma_3|\zeta^{\delta+1}}{\Gamma(\delta+2)} + \frac{|\sigma_4|\eta_m^\delta}{\Gamma(\delta+1)} + \frac{|\sigma_5|}{\Gamma(\delta+1)} + \frac{|\sigma_6|}{\Gamma(\delta)} \biggr) \biggr] \\
&\quad + \sum_{i=1}^k \varpi_i \biggl[\frac{t^{q+p_i}}{\Gamma(q+p_i+1)} + \frac{t}{|\Lambda_1|} \biggl(\frac{|\sigma_1|}{\Gamma(q+p_i+1)} + \frac{|\sigma_2|}{\Gamma(q+p_i)} + \frac{|\sigma_7|\zeta^{q+p_i+1}}{\Gamma(q+p_i+2)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{|\sigma_8|\eta_m^{q+p_i}}{\Gamma(q+p_i+1)}\bigg)\bigg] + \sum_{j=1}^{l}\widehat{\varpi}_j\bigg[\frac{t}{|\Lambda_1|}\bigg(\frac{|\sigma_3|\zeta^{\delta+\upsilon_j+1}}{\Gamma(\delta+\upsilon_j+2)} + \frac{|\sigma_4|\eta_m^{\delta+\upsilon_j}}{\Gamma(\delta+\upsilon_j+1)} \\
&+ \frac{|\sigma_5|}{\Gamma(\delta+\upsilon_j+1)} + \frac{|\sigma_6|}{\Gamma(\delta+\upsilon_j)}\bigg)\bigg] + \bigg(|a_1| + \bigg|\frac{\Lambda_2}{\Lambda_1}\bigg|\bigg)\bigg\} \\
&= \tau_1\varphi_1 + \tau_2\varphi_2 + \sum_{i=1}^{k}\varpi_i\Omega_i + \sum_{j=1}^{l}\widehat{\varpi}_j\widehat{\Omega}_j + |a_1| + \bigg|\frac{\Lambda_2}{\Lambda_1}\bigg|,
\end{aligned}
$$

which implies that

$$
\|\mathcal{Q}_1(x,y)\| \le \tau_1\varphi_1 + \tau_2\varphi_2 + \sum_{i=1}^{k}\varpi_i\Omega_i + \sum_{j=1}^{l}\widehat{\varpi}_j\widehat{\Omega}_j + |a_1| + \bigg|\frac{\Lambda_2}{\Lambda_1}\bigg|.
$$

Similarly, we can find that

$$
\|\mathcal{Q}_2(x,y)\| \le \tau_1\vartheta_1 + \tau_2\vartheta_2 + \sum_{i=1}^{k}\varpi_i\Theta_i + \sum_{j=1}^{l}\widehat{\varpi}_j\widehat{\Theta}_j + |b_1| + \bigg|\frac{\Lambda_3}{\Lambda_1}\bigg|.
$$

Consequently, we get

$$
\begin{aligned}
\|\mathcal{Q}(x,y)\| \le{}& (\varphi_1+\vartheta_1)\tau_1 + (\varphi_2+\vartheta_2)\tau_2 + \sum_{i=1}^{k}(\Omega_i+\Theta_i)\varpi_i \\
&+ \sum_{j=1}^{l}(\widehat{\Omega}_j+\widehat{\Theta}_j)\widehat{\varpi}_j + |a_1| + |b_1| + \bigg|\frac{\Lambda_2+\Lambda_3}{\Lambda_1}\bigg|.
\end{aligned}
$$

Therefore, the operator $\mathcal{Q}$ is uniformly bounded. Next, we show that $\mathcal{Q}$ is equicontinuous. Let $t \in [0,1]$ with $t_2 < t_1$. Then we have

$$
\begin{aligned}
&\big|\mathcal{Q}_1(x,y)(t_1) - \mathcal{Q}_1(x,y)(t_2)\big| \\
&\le \bigg|\int_0^{t_2}\frac{(t_1-s)^{q-1}-(t_2-s)^{q-1}}{\Gamma(q)}f_1\big(s,x(s),y(s)\big)\,ds\bigg| + \bigg|\int_{t_2}^{t_1}\frac{(t_1-s)^{q-1}}{\Gamma(q)}f_1\big(s,x(s),y(s)\big)\,ds\bigg| \\
&+ \bigg|\sum_{i=1}^{k}\int_0^{t_2}\frac{(t_1-s)^{q+p_i-1}-(t_2-s)^{q+p_i-1}}{\Gamma(q+p_i)}g_i\big(s,x(s),y(s)\big)\,ds\bigg| \\
&+ \bigg|\int_{t_2}^{t_1}\frac{(t_1-s)^{q+p_i-1}}{\Gamma(q+p_i)}g_i\big(s,x(s)\big)\,ds\bigg| \\
&+ \frac{|t_1-t_2|}{|\Lambda_1|}\bigg[|\sigma_1|\int_0^1\bigg(\frac{(1-s)^{q-1}}{\Gamma(q)}\big|f_1\big(s,x(s),y(s)\big)\big| \\
&+ \sum_{i=1}^{k}\frac{(1-s)^{q+p_i-1}}{\Gamma(q+p_i)}\big|g_i\big(s,x(s),y(s)\big)\big|\bigg)\,ds \\
&+ |\sigma_2|\int_0^1\bigg(\frac{(1-s)^{q-2}}{\Gamma(q-1)}\big|f_1\big(s,x(s),y(s)\big)\big| + \sum_{i=1}^{k}\frac{(1-s)^{q+p_i-2}}{\Gamma(q+p_i-1)}\big|g_i\big(s,x(s),y(s)\big)\big|\bigg)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ |\sigma_3| \bigg(\int_0^\zeta \int_0^s \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} \big| f_2\big(u, x(u), y(u)\big)\big| \, du \, ds + \sum_{j=1}^{l} \int_0^\zeta \int_0^s \frac{(s-u)^{\delta-1}}{\Gamma(\delta)} \\
&\times \int_0^u \frac{(u-w)^{\upsilon_j-1}}{\Gamma(\upsilon_j)} \big| h_j\big(w, x(w), y(w)\big)\big| \, dw \, du \, ds \bigg) \\
&+ |\sigma_4| \int_0^{\eta_m} \bigg(\frac{(\eta_m - s)^{\delta-1}}{\Gamma(\delta)} \big| f_2\big(s, x(s), y(s)\big)\big| \\
&+ \sum_{j=1}^{l} \frac{(\eta_m - s)^{\delta+\upsilon_j-1}}{\Gamma(\delta+\upsilon_j)} \big| h_j\big(s, x(s), y(s)\big)\big| \bigg) \, ds + |\sigma_5| \int_0^1 \bigg(\frac{(1-s)^{\delta-1}}{\Gamma(\delta)} \big| f_2\big(s, x(s), y(s)\big)\big| \\
&+ \sum_{j=1}^{l} \frac{(1-s)^{\delta+\upsilon_j-1}}{\Gamma(\delta+\upsilon_j)} \big| h_j\big(s, x(s), y(s)\big)\big| \bigg) \, ds + |\sigma_6| \int_0^1 \bigg(\frac{(1-s)^{\delta-2}}{\Gamma(\delta-1)} \big| f_2\big(s, x(s), y(s)\big)\big| \\
&+ \sum_{j=1}^{l} \frac{(1-s)^{\delta+\upsilon_j-2}}{\Gamma(\delta+\upsilon_j-1)} \big| h_j\big(s, x(s), y(s)\big)\big| \bigg) \, ds + |\sigma_7| \bigg(\int_0^\zeta \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \\
&\times \big| f_1\big(u, x(u), y(u)\big)\big| \, du \, ds + \sum_{i=1}^{k} \int_0^\zeta \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \int_0^u \frac{(u-w)^{p_i-1}}{\Gamma(p_i)} \\
&\times \big| g_i\big(w, x(w), y(w)\big)\big| \, dw \, du \, ds \bigg) + |\sigma_8| \int_0^{\eta_m} \bigg(\frac{(\eta_m - s)^{q-1}}{\Gamma(q)} \big| f_1\big(s, x(s), y(s)\big)\big| \\
&+ \sum_{i=1}^{k} \frac{(\eta_m - s)^{q+p_i-1}}{\Gamma(q+p_i)} \big| g_i\big(s, x(s), y(s)\big)\big| \bigg) \, ds \bigg] \\
\le{}& \tau_1 \Bigg\{ \bigg[\frac{|2(t_1-t_2)^q| + |t_1^q - t_2^q|}{\Gamma(q+1)} + \frac{|t_1-t_2|}{|\Lambda_1|} \bigg(\frac{|\sigma_1|}{\Gamma(q+1)} + \frac{|\sigma_2|}{\Gamma(q)} + \frac{|\sigma_7|\zeta^{q+1}}{\Gamma(q+2)} \\
&+ \frac{|\sigma_8|\eta_m^q}{\Gamma(q+1)} \bigg) \bigg] + \tau_2 \bigg[\frac{|t_1-t_2|}{|\Lambda_1|} \bigg(\frac{|\sigma_3|\zeta^{\delta+1}}{\Gamma(\delta+2)} + \frac{|\sigma_4|\eta_m^\delta}{\Gamma(\delta+1)} + \frac{|\sigma_5|}{\Gamma(\delta+1)} + \frac{|\sigma_6|}{\Gamma(\delta)} \bigg) \bigg] \\
&+ \sum_{i=1}^{k} \varpi_i \bigg[\frac{|2(t_1-t_2)^{q+p_i}| + |-t_1^{q+p_i} + t_2^{q+p_i}|}{\Gamma(q+p_i+1)} + \frac{|t_1-t_2|}{|\Lambda_1|} \bigg(\frac{|\sigma_1|}{\Gamma(q+p_i+1)} + \frac{|\sigma_2|}{\Gamma(q+p_i)} \\
&+ \frac{|\sigma_7|\zeta^{q+p_i+1}}{\Gamma(q+p_i+2)} + \frac{|\sigma_8|\eta_m^{q+p_i}}{\Gamma(q+p_i+1)} \bigg) \bigg] \\
&+ \sum_{j=1}^{l} \widehat{\varpi}_j \bigg[\frac{|t_1-t_2|}{|\Lambda_1|} \bigg(\frac{|\sigma_3|\zeta^{\delta+\upsilon_j+1}}{\Gamma(\delta+\upsilon_j+2)} + \frac{|\sigma_4|\eta_m^{\delta+\upsilon_j}}{\Gamma(\delta+\upsilon_j+1)} \\
&+ \frac{|\sigma_5|}{\Gamma(\delta+\upsilon_j+1)} + \frac{|\sigma_6|}{\Gamma(\delta+\upsilon_j)} \bigg) \bigg] \Bigg\}.
\end{aligned}
$$

Clearly, the operator $\mathcal{Q}$ is equicontinuous. Therefore, it follows that the operator $\mathcal{Q}(x,y)$ is completely continuous.

Finally, we show that the set $\mathcal{P} = \{(x,y) \in \mathcal{S} \times \mathcal{S} | (x,y) = \lambda \mathcal{Q}(x,y), 0 \le \lambda \le 1\}$ is bounded. Let $(x,y) \in \mathcal{P}$, with $(x,y) = \lambda \mathcal{Q}(x,y)$ and for any $t \in [0,1]$, we have

$$
x(t) = \lambda \mathcal{Q}_1(x,y)(t), \qquad y(t) = \lambda \mathcal{Q}_2(x,y)(t).
$$

In view of condition (H_3), we can find that

$$\begin{aligned}\left|x(t)\right| \leq{}& \varphi_1\bigl(\kappa_0 + \kappa_1|x| + \kappa_2|y|\bigr) + \varphi_2\bigl(\hat{\kappa}_0 + \hat{\kappa}_1|x| + \hat{\kappa}_2|y|\bigr) \\ &+ \Omega_i\bigl(\rho_{i,0} + \rho_{i,1}|x| + \rho_{i,2}|y|\bigr) + \widehat{\Omega}_j\bigl(\hat{\rho}_{j,0} + \hat{\rho}_{j,1}|x| + \hat{\rho}_{j,2}|y|\bigr)\end{aligned}$$

and

$$\begin{aligned}\left|y(t)\right| \leq{}& \vartheta_1\bigl(\kappa_0 + \kappa_1|x| + \kappa_2|y|\bigr) + \vartheta_2\bigl(\hat{\kappa}_0 + \hat{\kappa}_1|x| + \hat{\kappa}_2|y|\bigr) \\ &+ \Theta_i\bigl(\rho_{i,0} + \rho_{i,1}|x| + \rho_{i,2}|y|\bigr) + \widehat{\Theta}_j\bigl(\hat{\rho}_{j,0} + \hat{\rho}_{j,1}|x| + \hat{\rho}_{j,2}|y|\bigr).\end{aligned}$$

Hence

$$\begin{aligned}\|x\| \leq{}& \varphi_1\kappa_0 + \varphi_2\hat{\kappa}_0 + \Omega_i\rho_{i,0} + \widehat{\Omega}_j\hat{\rho}_{j,0} + (\varphi_1\kappa_1 + \varphi_2\hat{\kappa}_1 + \Omega_i\rho_{i,1} + \widehat{\Omega}_j\hat{\rho}_{j,1})\|x\| \\ &+ (\varphi_1\kappa_2 + \varphi_2\hat{\kappa}_2 + \Omega_i\rho_{i,2} + \widehat{\Omega}_j\hat{\rho}_{j,2})\|y\|\end{aligned}$$

and

$$\begin{aligned}\|y\| \leq{}& \vartheta_1\kappa_0 + \vartheta_2\hat{\kappa}_0 + \Theta_i\rho_{i,0} + \widehat{\Theta}_j\hat{\rho}_{j,0} + (\vartheta_1\kappa_1 + \vartheta_2\hat{\kappa}_1 + \Theta_i\rho_{i,1} + \widehat{\Theta}_j\hat{\rho}_{j,1})\|x\| \\ &+ (\vartheta_1\kappa_2 + \vartheta_2\hat{\kappa}_2 + \Theta_i\rho_{i,2} + \widehat{\Theta}_j\hat{\rho}_{j,2})\|y\|.\end{aligned}$$

In consequence, we get

$$\begin{aligned}\|x\| + \|y\| \leq{}& (\varphi_1 + \vartheta_1)\kappa_0 + (\varphi_2 + \vartheta_2)\hat{\kappa}_0 + (\Omega_i + \Theta_i)\rho_{i,0} + (\widehat{\Omega}_j + \widehat{\Theta}_j)\hat{\rho}_{j,0} \\ &+ \bigl((\varphi_1 + \vartheta_1)\kappa_1 + (\varphi_2 + \vartheta_2)\hat{\kappa}_1 + (\Omega_i + \Theta_i)\rho_{i,1} + (\widehat{\Omega}_j + \widehat{\Theta}_j)\hat{\rho}_{j,1}\bigr)\|x\| \\ &+ \bigl((\varphi_1 + \vartheta_1)\kappa_2 + (\varphi_2 + \vartheta_2)\hat{\kappa}_2 + (\Omega_i + \Theta_i)\rho_{i,2} + (\widehat{\Omega}_j + \widehat{\Theta}_j)\hat{\rho}_{j,2}\bigr)\|y\|,\end{aligned}$$

which implies that

$$\bigl\|(x,y)\bigr\| \leq \frac{(\varphi_1 + \vartheta_1)\kappa_0 + (\varphi_2 + \vartheta_2)\hat{\kappa}_0 + (\Omega_i + \Theta_i)\rho_{i,0} + (\widehat{\Omega}_j + \widehat{\Theta}_j)\hat{\rho}_{j,0}}{E_0},$$

where

$$\begin{aligned}E_0 = \min\bigl\{&1 - \bigl((\varphi_1 + \vartheta_1)\kappa_1 + (\varphi_2 + \vartheta_2)\hat{\kappa}_1 + (\Omega_i + \Theta_i)\rho_{i,1} + (\widehat{\Omega}_j + \widehat{\Theta}_j)\hat{\rho}_{j,1}\bigr), \\ &1 - \bigl((\varphi_1 + \vartheta_1)\kappa_2 + (\varphi_2 + \vartheta_2)\hat{\kappa}_2 + (\Omega_i + \Theta_i)\rho_{i,2} + (\widehat{\Omega}_j + \widehat{\Theta}_j)\hat{\rho}_{j,2}\bigr)\bigr\}.\end{aligned}$$

Hence the set $\mathcal{P}$ is bounded. Thus, by Lemma 3, the operator $\mathcal{Q}$ has at least one fixed point, which implies that system (1)–(2) has at least one solution on $[0,1]$. □

4 Examples

Here, we present illustrative examples for the results proved in the last section.

Example 1 Consider the following system:

$$\begin{cases} {}^cD^{3/2}x(t) + \sum_{i=1}^{2} I^{p_i} g_i(t,x(t),y(t)) = f_1(t,x(t),y(t)), & 1 < q \leq 2, t \in [0,1], \\ {}^cD^{5/4}y(t) + \sum_{j=1}^{2} I^{\upsilon_j} h_j(t,x(t),y(t)) = f_2(t,x(t),y(t)), & 1 < \delta \leq 2, t \in [0,1], \end{cases} \tag{18}$$

complemented with the boundary conditions

$$\begin{cases} x(0) = 1, \qquad y(0) = 3, \\ \alpha_1 x(1) + \beta_1 x'(1) = \gamma_1 \int_0^{\zeta} y(s)\,ds + \sum_{m=1}^{3} \mu_m y(\eta_m), \\ \alpha_2 y(1) + \beta_2 y'(1) = \gamma_2 \int_0^{\zeta} x(s)\,ds + \sum_{m=1}^{3} \xi_m x(\eta_m). \end{cases} \tag{19}$$

Here, $\alpha_1 = 1/4$, $\alpha_2 = 1/2$, $\beta_1 = 3/5$, $\beta_2 = 4/5$, $\gamma_1 = \gamma_2 = 1$, $\zeta = 1/7\eta_1 = 1/5$, $\eta_2 = 2/5$, $\eta_3 = 3/5$, $\mu_1 = 1/2$, $\mu_2 = 3/4$, $\mu_3 = 1$, $\xi_1 = 1/3$, $\xi_2 = 2/3$, $\xi_3 = 1$, and

$$\begin{aligned} f_1\bigl(t,x(t),y(t)\bigr) &= \frac{1}{18\sqrt{169+t^4}}\left(\frac{|x(t)|}{1+|x(t)|} + \tan^{-1} y(t)\right) + \cos 2t, \\ f_2\bigl(t,x(t),y(t)\bigr) &= \frac{1}{8(t^2+30)} \sin x(t) + \frac{\tan^{-1} y(t)}{12\sqrt{t^2+400}} + \frac{1}{24+t^2}, \\ g_1\bigl(t,x(t),y(t)\bigr) &= \frac{1}{(118+t^2)}\bigl(\sin x(t) + \bigl|y(t)\bigr|\bigr) + 3t, \\ g_2\bigl(t,x(t),y(t)\bigr) &= \frac{1}{9\sqrt{196+t^5}}\bigl(x(t) + \cos y(t)\bigr) + 3e^{-t}, \\ h_1\bigl(t,x(t),y(t)\bigr) &= \frac{1}{6\sqrt{289+t^2}}\bigl(x(t) + \tan^{-1} x(t)\bigr) + \frac{1}{\sqrt{4}+100+t^2}\left(\frac{|y(t)|}{1+|y(t)|} + \sin 2t\right), \\ h_2\bigl(t,x(t),y(t)\bigr) &= \frac{e^{-t}}{3(4+t^2)}\bigl(\cos t + \tan^{-1} x(t)\bigr) + \frac{e^{-t}\sqrt{9}}{3(t^2+12)}\bigl(\sin y(t) + 5t\bigr). \end{aligned}$$

Clearly, we have

$$\begin{aligned} \bigl|f_1(t,x_1,y_1) - f_1(t,x_2,y_2)\bigr| &\leq \frac{1}{234}\bigl(|x_1-y_1| + |y_1-y_2|\bigr), \\ \bigl|f_2(t,x_1,y_1) - f_2(t,x_2,y_2)\bigr| &\leq \frac{1}{240}\bigl(|x_1-y_1| + |y_1-y_2|\bigr), \\ \bigl|g_1(t,x_1,y_1) - g_1(t,x_2,y_2)\bigr| &\leq \frac{1}{118}\bigl(|x_1-y_1| + |y_1-y_2|\bigr), \\ \bigl|g_2(t,x_1,y_1) - g_2(t,x_2,y_2)\bigr| &\leq \frac{1}{126}\bigl(|x_1-y_1| + |y_1-y_2|\bigr), \\ \bigl|h_1(t,x_1,y_1) - h_1(t,x_2,y_2)\bigr| &\leq \frac{1}{102}\bigl(|x_1-y_1| + |y_1-y_2|\bigr), \\ \bigl|h_2(t,x_1,y_1) - h_2(t,x_2,y_2)\bigr| &\leq \frac{1}{12e}\bigl(|x_1-y_1| + |y_1-y_2|\bigr). \end{aligned}$$

Using the given data, we have $\Lambda_1 = 0.151835$ and $\Psi = 0.788452 < 1$. Obviously the hypotheses of Theorem 1 are satisfied. Hence, by the conclusion of Theorem 1, there is a unique solution for problem (18)–(19) on $[0,1]$.

Example 2 Consider problem (18)–(19) with the following data:

$$
\begin{aligned}
f_1\big(t,x(t),y(t)\big) &= \frac{3\sin t}{(145+t^5)} + \frac{1}{158}x(t)\cos y(t) + \frac{e^{-t}}{3\sqrt{169}}\tan^{-1} y(t),\\
f_2\big(t,x(t),y(t)\big) &= \frac{2}{13\sqrt{49+t}} + \frac{\sin x(t)}{11\sqrt{144+t^2}} + \frac{y(t)}{165t^3},\\
g_1\big(t,x(t),y(t)\big) &= e^{-15t} + \frac{1}{124}x(t)\tan^{-1} y(t) + \frac{y(t)}{14\sqrt{81+t^3}},\\
g_2\big(t,x(t),y(t)\big) &= \frac{2}{17\sqrt{81+t^2}} + \frac{1}{(113+t^2)}x(t) + \frac{e^{-t}}{280}\sin y(t),\\
h_1\big(t,x(t),y(t)\big) &= \frac{3}{82\sqrt{t}} + \frac{e^{-t}}{38t^2}\cos x(t) + \frac{1}{8\sqrt{144+t^5}}y(t),\\
h_2\big(t,x(t),y(t)\big) &= e^{-13t} + \frac{3}{148}x(t)\tan^{-1} y(t) + \frac{y(t)}{11\sqrt{t^2+81}}.
\end{aligned}
\tag{20}
$$

It is easy to check that condition (H_3) is satisfied with $\kappa_0 = 3/146$, $\kappa_1 = 1/158$, $\kappa_2 = 1/39e$, $\hat{\kappa}_0 = 2/91$, $\hat{\kappa}_1 = 1/132$, $\hat{\kappa}_2 = 1/165$, $\rho_{1,0} = 1/15e$, $\rho_{1,1} = 1/124$, $\rho_{1,2} = 1/126$, $\rho_{2,0} = 2/153$, $\rho_{2,1} = 1/113$, $\rho_{2,2} = 1/280$, $\hat{\rho}_{1,0} = 3/82$, $\hat{\rho}_{1,1} = 1/38e$, $\hat{\rho}_{1,2} = 1/96$, $\hat{\rho}_{1,0} = 1/13e$, $\hat{\rho}_{2,1} = 3/148$, $\hat{\rho}_{2,2} = 1/99$. Furthermore, $(\varphi_1 + \vartheta_1)\kappa_1 + (\varphi_2 + \vartheta_2)\hat{\kappa}_1 + (\Omega_i + \Theta_i)\rho_{i,1} + (\widehat{\Omega}_j + \widehat{\Theta}_j)\hat{\rho}_{j,1} \approx 0.948955 < 1$, and $(\varphi_1 + \vartheta_1)\kappa_2 + (\varphi_2 + \vartheta_2)\hat{\kappa}_2 + (\Omega_i + \Theta_i)\rho_{i,2} + (\widehat{\Omega}_j + \widehat{\Theta}_j)\hat{\rho}_{j,2} \approx 0.846541 < 1$. Therefore the hypotheses of Theorem 2 are satisfied. Hence, by the conclusion of Theorem 2, problem (18)–(19) with data (20) has at least one solution on $[0,1]$.

5 Conclusions

Existence and uniqueness results are derived for a system of nonlinear coupled Caputo–Riemann–Liouville type fractional integro-differential equations equipped with multi-point sub-strip boundary conditions. Our results are not only new in the given configuration, but also yield some new results associated with particular choices of the parameters involved in the problem at hand. For example, our results correspond to a system of nonlinear coupled Caputo–Riemann–Liouville type fractional integro-differential equations equipped with coupled multi-point boundary conditions if we take $\gamma_1 = 0 = \gamma_2$ in the results of this paper. In case we take $\mu_m = 0 = \xi_m$ for all $m = 1,2,\dots,\omega$, we get the results for a coupled system of nonlinear Caputo–Riemann–Liouville type fractional integro-differential equations supplemented with coupled sub-strip boundary conditions.

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The authors declare that they have no competing interests.

Authors' contributions

Each of the authors, AA, AFA, SKN, and BA contributed equally to each part of this work. All authors read and approved the final manuscript.

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