Ren and Wu *Advances in Difference Equations* (2020) 2020:652
https://doi.org/10.1186/s13662-020-03098-7

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

External stability and H_∞ control of switching systems with delay and impulse

Jiaojiao Ren[1*] and Cong Wu[2]

*Correspondence: jiaojiaoren06@163.com
[1]School of Electronic Information and Electrical Engineering, Chengdu University, Chengdu, China
Full list of author information is available at the end of the article

Abstract

In this paper, we investigate the external stability and H_∞ control of switching systems with time-varying delay and impulse. First of all, a modified two-direction inequality (relation) between the switching numbers and the maximum, minimum dwell time is proposed. This new inequality is applied to proving the external stability of switching systems with delay and impulse consisting of subsystems with Hurwitz stable matrices of internal dynamics. By using this new inequality, a normal L_2 norm constraint is derived rather than weighted L_2 norm constraint. In addition, by a realizable switching law, the obtained result is extended to the switching systems comprised of subsystems with both Hurwitz stable and unstable matrices of internal dynamics. The results are finally applied to H_∞ control and illustrated by a numerical example.

Keywords: External stability; H_∞ control; Switching systems; Impulse; Maximum (minimum) dwell time

1 Introduction

External stability (ES) is defined as a property of control systems: every L_2 input generates an L_2 zero-state output [1–4], which plays an essential role in system analysis. As one type of systems, switching systems(SSs), consisting of a family of subsystems, and a switching rule that orchestrates the switching between them [5–8], have been an important framework in the area of input–output analysis. In practice, effects of time delay [9, 10] and impulse [11, 12] are usually inevitable. Therefore, there are lots of results on system and input–output analysis of delayed SSs [11–21]. For example of a discrete-time framework, the problem of robust exponential H_∞ filtering for switched fuzzy delayed systems was investigated in [15]; for example of a continuous-time framework, in [11], fault-tolerant synchronization for SSs with delay and impulse was considered.

In the field of ES, H_∞ control, H_∞ model reduction, L_2-gain analysis and disturbance attenuation for the SSs, how to obtain the L_2 norm bound constraint is a critical part of our study. Owing to the essence of SSs and average dwell time(DT) scheme, the concept of weighted L_2 norm bound constraint, instead of the normal L_2 norm bound constraint, has been proposed [16, 17, 22]. However, this weighted concept changes the original physical meaning of L_2. Recently, some researchers tried to remove this label "weighted", see

[18] in 2012, [23] in 2016 and [19] in 2017. However, there are still some doubtful problems. Specifically, Liu and Yuan adopted a normal L_2 relation between input and output in [18] but where the average DT could not be substituted into the integral in (36); In [23], Eq. (21) could not be directly derived because $\Psi(s)$ could not be guaranteed greater than zero; Syed Ali et al. employed the square of L_2 norm, see (6) in [19], but it is questionable for the cancelation of $e^{\alpha(T-s)}$ on both sides of the inequality right below (57). Very recently, a two-direction inequality ($\frac{t-\tau}{T_{\max}} \leq N_\sigma(t,\tau) \leq \frac{t-\tau}{T_{\min}}$) between the switching numbers and the maximum, minimum DT was proposed in [11] such that the label "weighted" can be removed properly. Furthermore, an improved two-direction inequality ($\frac{t-\tau}{T_{\max}} - 1 \leq N_\sigma(t,\tau) \leq \frac{t-\tau}{T_{\min}} + 1$) was provided in [24, 25]. But this improved inequality is still not precise enough. A more accurate two-direction inequality ($\max\{\frac{t-\tau}{T_{\max}} - 1, 0\} \leq N_\sigma(t,\tau) \leq \frac{t-\tau}{T_{\min}} + 1$) was given in [21]. To the best of our knowledge, this more accurate two-direction inequality has not been used to study the ES and H_∞ control of SSs with delay and impulse.

In order to study the ES of SSs comprised of subsystems with both Hurwitz stable and unstable matrices, a suitable switching law is necessary. There are a few results that have been reported in [22, 25–29]. Some state-dependent switching laws were proposed in [27, 28], while some time-dependent switching laws were presented [22, 25, 26, 29]. The authors of [29] applied fast switching and slow switching, respectively, to unstable and stable subsystems. In [26], the precondition $\inf_{t\geq t_0}[\frac{T^{-1}(t)}{T^{+}(t)}] \geq -\frac{\beta}{\alpha}$ was given to guarantee that $-\gamma t = T^{-1}(t)\alpha + T^{+}(t)\beta < 0$ holds. But this precondition cannot make sure that $-\gamma t = T^{-1}(t)\alpha + T^{+}(t)\beta$ holds or that $T^{-1}(t)\alpha + T^{+}(t)\beta$ is a linear function as desired. Only $T^{-1}(t)\alpha + T^{+}(t)\beta < 0$ can be deduced there. Another resolution was proposed in [22], a new separation of switching instants was arranged in advance. By a given parameter $c_* > 0$ without a specified sequence of time instants, the switching law in [25] is easier and clearer to implement than the one in [22]. This makes it worth to study how to adopt the switching law in [25] to investigate the ES and H_∞ control of SSs with delay and impulse.

Motivated by the above discussion, the problem of ES and H_∞ control of SSs with delay and impulse is investigated in this paper. The main contribution of the paper is as follows. First, a two-direction inequality (relation) between the switching numbers and the maximum, minimum DT is used such that the label "weighted" can be removed properly and the normal L_2 norm constraint is derived. Second, a suitable switching law is adopted to deal with the SSs with both Hurwitz stable and unstable subsystems. Third, we take the overlooked case $0 < \mu < 1$ into consideration (in almost all mentioned results above the range of μ is only larger or equal to 1), i.e. the range of switching parameters μ is extended to the set of all positive real numbers. Fourth, the non-weighted H_∞ control [30–32] of switching control systems with delay and impulse is established, in which the matrices of internal dynamics of the controlled system are not necessarily all Hurwitz as usual. Finally, the effectiveness of the results is illustrated by a numerical example.

2 Problem statement and preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional real Euclidean space. For $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm of x. We use the notation $L_2([0,\infty),\mathbb{R}^n)$ to denote the class of square integrable functions from $[0,\infty)$ to $\mathbb{R}^n$, i.e. for each $x \in L_2([0,\infty),\mathbb{R}^n)$, the L_2 norm of x is $\|x\|_{L_2} = (\int_0^\infty \|x(t)\|^2\,\mathrm{d}t)^{\frac{1}{2}} < \infty$. Define $x_t(s) = x(t+s)$, $-d \leq s \leq 0$, and $\|x_t\|_d = \sup_{-d\leq s\leq 0}\|x(t+s)\|$. The notation $P > 0$ indicates that matrix P is positive definite. $\lambda_{\max}(P)$ ($\lambda_{\min}(P)$) denotes the maximum (minimum) eigenvalue of matrix P. α_- means such a positive number that the inequality $A_i^T P_i + P_i A_i + \alpha_- P_i < 0$ holds where A_i is Hurwitz stable, while α_+ means such a positive number that $A_i^T P_i + P_i A_i - \alpha_+ P_i < 0$ holds where A_i is not Hurwitz stable.

Consider the following switching control system with delay and impulse:

$$\begin{aligned}
&\dot{x}(t) = A_{\sigma(t)}x(t) + A_{d\sigma(t)}x\bigl(t-\tau(t)\bigr) + B_{\sigma(t)}u(t), \quad t \in [t_{k-1}, t_k),\\
&\Delta x = x(t_k) - x\bigl(t_k^-\bigr) = I_{\sigma(t_k^-)}x\bigl(t_k^-\bigr), \quad t = t_k,\\
&y(t) = C_{\sigma(t)}x(t) + C_{d\sigma(t)}x\bigl(t-\tau(t)\bigr) + E_{\sigma(t)}u(t),\\
&x_{t_0} = x(t_0+\theta) = \phi(\theta), \quad \theta \in [-d, 0],
\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ represent the state vector, external input vector and output vector, respectively. $A_{\sigma(t)}$, $A_{d\sigma(t)}$, $B_{\sigma(t)}$, $C_{\sigma(t)}$, $C_{d\sigma(t)}$, $E_{\sigma(t)}$ and $I_{\sigma(t)}$ are constant matrices with appropriate dimensions, where $\sigma(t)$ is the switching signal, which takes values from $\mathscr{P} = \{1, 2, \ldots\}$ and $\sigma(t) = i \in \mathscr{P}$ means the subsystem i is active at t. The switching time instants t_k ($k = 1, 2, \ldots$) satisfy $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = \infty$. The initial function $\phi(\theta)$ is piecewise continuous on $[-d, 0]$. The delay function $\tau(t)$ is differentiable and satisfies $0 \le \tau(t) \le d$, $\dot{\tau}(t) \le \rho < 1$.

Remark 2.1 Here $\dot{x}$ is considered as the right derivative of x based on two considerations. On the one hand, the derivative at t_0 is taken to be a right derivative, since $\phi(\theta)$ may not admit a left derivative at t_0 or this left derivative even exists but may not equal the right hand function. To be consistent with the derivative at t_0, $\dot{x}(t_k)$ needs to be the notation of the right derivative at t_k. On the other hand, at t_k, the subsystem $\sigma(t_k) = \sigma(t_k^+)$ is already active, i.e. $\dot{x}(t) = A_{\sigma(t_k^+)}x(t) + A_{d\sigma(t_k^+)}x(t-\tau(t)) + B_{\sigma(t_k^+)}u(t)$. It follows that $\dot{x}(t_k)$ also denotes the right derivative at t_k.

Remark 2.2 The solution $x(t)$ is right continuous at t_k, i.e. $x(t_k^+) = x(t_k)$, since $\dot{x}(t_k)$ represents the right derivative at t_k and the right derivative is defined based on right continuity.

Definition 2.1 ([1]) The L_2 gain of an externally stable system is

$$\gamma = \sup_{u\in L_2, u\neq\theta} \frac{\|y\|_{L_2}}{\|u\|_{L_2}}.$$

(Here θ indicates the zero function.) The L_2 gain is the maximum ratio of $\|y\|_{L_2}/\|u\|_{L_2}$.

Lemma 2.1 ([21]) *For any $t \ge \tau \ge t_0$, $N_\sigma(t,\tau)$ denote the number of discontinuities of region over (τ, t), i.e., the number of region switching over (τ, t),*

$$\max\left\{\frac{t-\tau}{T_{\max}} - 1, 0\right\} \le N_\sigma(t,\tau) \le \frac{t-\tau}{T_{\min}} + 1, \tag{2}$$

where $T_{\max} = \sup_{k=1,2,\ldots}(t_k - t_{k-1})$, $T_{\min} = \inf_{k=1,2,\ldots}(t_k - t_{k-1})$ are the maximum and minimum DT of system (1).

3 Main results

At the beginning of this section, the ES of SSs with delay and impulse consisting of subsystems with Hurwitz $A_i's$ is proved. Then the obtained result is extended to the ES of SSs, in which not all subsystems are Hurwitz stable, by employing a switching law. In obtained results, the normal L_2 norm constraint, rather than a weighted L_2 norm constraint, is derived by using the new proposed relation (2). Finally, the derived results of ES is applied to H_∞ control.

3.1 External stability

In this subsection, by using Eq. (2), the ES of SSs with delay and impulse consisting of subsystems that are all Hurwitz stable, and subsystems that are not all Hurwitz stable, is proved, respectively.

3.1.1 All subsystems are Hurwitz stable

Theorem 3.1 *Given scalars $\alpha > 0$, $\mu > 0$ and the numbers of any two consecutive subsystems $i,j \in \mathscr{P}$ (σ switches from j to i), the system (1) is externally stable with a L_2 gain γ, if there exist $n \times n$ positive definite matrices P_i, Q_i, P_j, Q_j such that*

$$\begin{bmatrix} A_i^T P_i + P_i A_i + \alpha P_i + Q_i & P_i A_{di} & P_i B_i & C_i^T \\ * & -(1-\rho)e^{-\alpha d} Q_i & 0 & C_{di}^T \\ * & * & -\digamma_\mu^2 I & E_i^T \\ * & * & * & -I \end{bmatrix} < 0, \tag{3}$$

$$\begin{bmatrix} \mu P_j & (I + I_j)^T P_i \\ * & P_i \end{bmatrix} \geq 0, \qquad Q_i \leq \mu Q_j, \tag{4}$$

and

$$\alpha > \frac{\ln \mu}{T_{\min}}, \tag{5}$$

where $\digamma_\mu = \frac{\gamma}{\mu}\sqrt{\frac{\alpha - \ln\mu/T_{\min}}{\alpha - \ln\mu/T_{\max}}}$ *if* $\mu > 1$; $\digamma_\mu = \gamma$ *if* $\mu = 1$; $\digamma_\mu = \gamma\mu\sqrt{\frac{\alpha - \ln\mu/T_{\max}}{\alpha - \ln\mu/T_{\min}}}$ *if* $0 < \mu < 1$.

Proof Consider the following Lyapunov functional candidate:

$$V_{\sigma(t)}(t, x_t) = V_{1\sigma(t)}(t, x_t) + V_{2\sigma(t)}(t, x_t), \tag{6}$$

where $V_{1\sigma(t)}(t, x_t) = x^T(t) P_{\sigma(t)} x(t)$ and $V_{2\sigma(t)}(t, x_t) = \int_{t-\tau(t)}^t x^T(s)\, \mathrm{e}^{\alpha(s-t)} Q_{\sigma(t)}\, x(s)\, \mathrm{d}s$. For simplicity, we shall use $V_{\sigma(t)}(t)$ to denote $V_{\sigma(t)}(t, x_t)$.

Suppose $\sigma(t) = i$ for $t \in [t_{k-1}, t_k)$, then the derivatives of $V_{1\sigma(t)}(t)$ and $V_{2\sigma(t)}(t)$ are

$$\begin{aligned} \dot{V}_{1\sigma(t)}(t) &= \dot{V}_{1i}(t) \\ &= \dot{x}^T(t) P_i x(t) + x^T(t) P_i \dot{x}(t) \\ &= \bigl[A_i x(t) + A_{di} x\bigl(t - \tau(t)\bigr) + B_i u(t)\bigr]^T P_i x(t) \\ &\quad + x^T(t) P_i \bigl[A_i x(t) + A_{di} x\bigl(t - \tau(t)\bigr) + B_i u(t)\bigr] \\ &= x^T(t)\bigl(A_i^T P_i + P_i A_i\bigr) x(t) + x^T\bigl(t - \tau(t)\bigr) A_{di}^T P_i x(t) \\ &\quad + x^T(t) P_i A_{di} x\bigl(t - \tau(t)\bigr) + u^T(t) B_i^T P_i x(t) + x^T(t) P_i B_i u(t) \end{aligned} \tag{7}$$

and

$$\begin{aligned}\dot{V}_{2\sigma(t)}(t) &= \dot{V}_{2i}(t)\\ &= -\alpha \int_{t-\tau(t)}^{t} x^T(s)e^{\alpha(s-t)}Q_i x(s)\,\mathrm{d}s + x^T(t)Q_i x(t)\\ &\quad - \big[1-\dot{\tau}(t)\big]e^{-\alpha\tau(t)}x^T\big(t-\tau(t)\big)Q_i x\big(t-\tau(t)\big)\\ &\leq -\alpha V_{2i}(t) + x^T(t)Q_i x(t) - (1-\rho)e^{-\alpha d}x^T\big(t-\tau(t)\big)Q_i x\big(t-\tau(t)\big),\end{aligned} \tag{8}$$

respectively.

Denote $\Delta(t) = y^T(t)y(t) - \digamma_\mu^2 u^T(t)u(t)$. Then we get

$$\begin{aligned}&\dot{V}_i(t) + \alpha V_i(t) + \Delta(t)\\ &\quad\leq x^T(t)\big(A_i^T P_i + P_i A_i\big)x(t) + x^T\big(t-\tau(t)\big)A_{di}^T P_i x(t) + x^T(t)P_i A_{di} x\big(t-\tau(t)\big)\\ &\qquad + u^T(t)B_i^T P_i x(t) + x^T(t)P_i B_i u(t) - \alpha V_{2i}(t) + x^T(t)Q_i x(t)\\ &\qquad - (1-\rho)e^{-\alpha d}x^T\big(t-\tau(t)\big)Q_i x\big(t-\tau(t)\big) + \alpha x^T(t)P_i x(t) + \alpha V_{2i}(t)\\ &\qquad + x^T(t)C_i^T C_i x(t) + x^T(t)C_i^T C_{di} x\big(t-\tau(t)\big) + x^T(t)C_i^T E_i u(t)\\ &\qquad + x^T\big(t-\tau(t)\big)C_{di}^T C_i x(t) + x^T\big(t-\tau(t)\big)C_{di}^T C_{di} x\big(t-\tau(t)\big)\\ &\qquad + x^T\big(t-\tau(t)\big)C_{di}^T E_i u(t) + u^T(t)E_i^T C_i x(t) + u^T(t)E_i^T C_{di} x\big(t-\tau(t)\big)\\ &\qquad + u^T(t)E_i^T E_i u(t) - \digamma_\mu^2 u^T(t)u(t)\\ &\quad= \eta^T(t)\begin{bmatrix}(1,1) & P_iA_{di} + C_i^T C_{di} & P_iB_i + C_i^T E_i\\ * & -(1-\rho)\mathrm{e}^{-\alpha d}Q_i + C_{di}^T C_{di} & C_{di}^T E_i\\ * & * & E_i^T E_i - \digamma_\mu^2 I\end{bmatrix}\eta(t),\end{aligned}$$

where $\eta(t) = [x^T(t), x^T(t-\tau(t)), u^T(t)]^T$ and $(1,1) = A_i^T P_i + P_i A_i + \alpha P_i + Q_i + C_i^T C_i$.

From (3), by using the Schur complement [33], we obtain

$$\begin{aligned}&\begin{bmatrix}A_i^T P_i + P_i A_i + \alpha P_i + Q_i & P_i A_{di} & P_i B_i\\ * & -(1-\rho)e^{-\alpha d}Q_i & 0\\ * & * & -\digamma_\mu^2 I\end{bmatrix} - \begin{bmatrix}C_i^T\\ C_{di}^T\\ E_i^T\end{bmatrix}(-I)^{-1}\begin{bmatrix}C_i & C_{di} & E_i\end{bmatrix}\\ &\quad= \begin{bmatrix}(1,1) & P_iA_{di} + C_i^T C_{di} & P_iB_i + C_i^T E_i\\ * & -(1-\rho)e^{-\alpha d}Q_i + C_{di}^T C_{di} & C_{di}^T E_i\\ * & * & E_i^T E_i - \digamma_\mu^2 I\end{bmatrix} < 0.\end{aligned}$$

Therefore, we can deduce that

$$\dot{V}_i(t) + \alpha V_i(t) + \Delta(t) \leq 0. \tag{9}$$

Thus, integrating the inequality (9) from t_{k-1} to t, $t \in [t_{k-1}, t_k)$, produces

$$V_i(t) \leq V_i(t_{k-1})e^{-\alpha(t-t_{k-1})} - \int_{t_{k-1}}^{t} \Delta(s)e^{-\alpha(t-s)}\,\mathrm{d}s. \tag{10}$$

Since σ switches from j to i, $\sigma(t_{k-1}^-) = j$ and $\sigma(t_{k-1}^+) = \sigma(t_{k-1}) = i$. Now we consider the change of $V_{\sigma(t)}(t)$ at $t = t_{k-1}$ as follows. Firstly,

$$\begin{aligned}
&V_{1\sigma(t_{k-1}^+)}\bigl(t_{k-1}^+\bigr) - \mu V_{1\sigma(t_{k-1}^-)}\bigl(t_{k-1}^-\bigr) \\
&\quad = V_{1i}\bigl(t_{k-1}^+\bigr) - \mu V_{1j}\bigl(t_{k-1}^-\bigr) \\
&\quad = x^T\bigl(t_{k-1}^+\bigr)P_i x\bigl(t_{k-1}^+\bigr) - \mu x^T\bigl(t_{k-1}^-\bigr)P_j x\bigl(t_{k-1}^-\bigr) \\
&\quad = x^T\bigl(t_{k-1}^-\bigr)\bigl[(I+I_j)^T P_i (I+I_j) - \mu p_j\bigr]x\bigl(t_{k-1}^-\bigr).
\end{aligned}$$

Using the Schur complement again, it follows from the first inequality in (4) that $(I + I_j)^T P_i (I + I_j) - \mu p_j \le 0$ such that $V_{1i}(t_{k-1}) = V_{1i}(t_{k-1}^+) \le \mu V_{1j}(t_{k-1}^-)$. Secondly,

$$\begin{aligned}
&V_{2\sigma(t_{k-1}^+)}\bigl(t_{k-1}^+\bigr) - \mu V_{2\sigma(t_{k-1}^-)}\bigl(t_{k-1}^-\bigr) \\
&\quad = V_{2i}\bigl(t_{k-1}^+\bigr) - \mu V_{2j}\bigl(t_{k-1}^-\bigr) \\
&\quad = \int_{t_{k-1}^+ - \tau(t_{k-1}^+)}^{t_{k-1}^+} x^T(s) e^{\alpha(s - t_{k-1}^+)} Q_{\sigma(t_{k-1}^+)} x(s)\,\mathrm{d}s \\
&\qquad - \int_{t_{k-1}^- - \tau(t_{k-1}^-)}^{t_{k-1}^-} x^T(s) e^{\alpha(s - t_{k-1}^-)} \mu Q_{\sigma(t_{k-1}^-)} x(s)\,\mathrm{d}s \\
&\quad = \int_{t_{k-1} - \tau(t_{k-1})}^{t_{k-1}} x^T(s) e^{\alpha(s - t_{k-1})} Q_i x(s)\,\mathrm{d}s \\
&\qquad - \int_{t_{k-1} - \tau(t_{k-1})}^{t_{k-1}} x^T(s) e^{\alpha(s - t_{k-1})} \mu Q_j x(s)\,\mathrm{d}s \\
&\quad = \int_{t_{k-1} - \tau(t_{k-1})}^{t_{k-1}} x^T(s) e^{\alpha(s - t_{k-1})} (Q_i - \mu Q_j) x(s)\,\mathrm{d}s.
\end{aligned}$$

The second inequality in (4) and the calculation above together imply that $V_{2i}(t_{k-1}) = V_{2i}(t_{k-1}^+) \le \mu V_{2j}(t_{k-1}^-)$. Thus, one has

$$V_i(t_{k-1}) \le \mu V_j\bigl(t_{k-1}^-\bigr). \tag{11}$$

Using the technique in (2.7) of [22], it follows from (10) and (11) that

$$\begin{aligned}
V_i(t) &\le \mu V_j\bigl(t_{k-1}^-\bigr) e^{-\alpha(t - t_{k-1})} - \int_{t_{k-1}}^{t} \Delta(s) e^{-\alpha(t-s)}\,\mathrm{d}s \\
&\le \mu \biggl[V_j(t_{k-2}) e^{-\alpha(t_{k-1} - t_{k-2})} - \int_{t_{k-2}}^{t_{k-1}} \Delta(s) e^{-\alpha(t_{k-1}-s)}\,\mathrm{d}s \biggr] \\
&\quad \times e^{-\alpha(t - t_{k-1})} - \int_{t_{k-1}}^{t} \Delta(s) e^{-\alpha(t-s)}\,\mathrm{d}s \\
&\le \cdots \\
&= \mu^{N_\sigma(t,t_0)} V_{\sigma(t_0)}(t_0) e^{-\alpha(t-t_0)} - \mu^{N_\sigma(t,t_0)} \int_{t_0}^{t_1} \Delta(s) e^{-\alpha(t-s)}\,\mathrm{d}s \\
&\quad - \mu^{N_\sigma(t,t_1)} \int_{t_1}^{t_2} \Delta(s) e^{-\alpha(t-s)}\,\mathrm{d}s - \cdots - \mu^{N_\sigma(t,t_{k-1})} \int_{t_{k-1}}^{t} \Delta(s) e^{-\alpha(t-s)}\,\mathrm{d}s
\end{aligned} \tag{12}$$

$$
\begin{aligned}
&= V_{\sigma(t_0)}(t_0)e^{-\alpha(t-t_0)+N_\sigma(t,t_0)\ln\mu} - \int_{t_0}^{t_1} \Delta(s)e^{-\alpha(t-s)+N_\sigma(t,s)\ln\mu}\,\mathrm{d}s \\
&\quad - \int_{t_1}^{t_2} \Delta(s)e^{-\alpha(t-s)+N_\sigma(t,s)\ln\mu}\,\mathrm{d}s - \cdots - \int_{t_{k-1}}^{t} \Delta(s)e^{-\alpha(t-s)+N_\sigma(t,s)\ln\mu}\,\mathrm{d}s \\
&\le V_{\sigma(t_0)}(t_0)e^{-\alpha(t-t_0)+N_\sigma(t,t_0)\ln\mu} - \int_{t_0}^{t} \Delta(s)e^{-\alpha(t-s)+N_\sigma(t,s)\ln\mu}\,\mathrm{d}s.
\end{aligned}
$$

Under the zero initial condition, we acquire

$$
\int_{t_0}^{t} \Delta(s)e^{-\alpha(t-s)+N_\sigma(t,s)\ln\mu}\,\mathrm{d}s \le -V_i(t) \le 0. \tag{13}
$$

That is,

$$
\begin{aligned}
&\int_{t_0}^{t} y^T(s)y(s)e^{-\alpha(t-s)+N_\sigma(t,s)\ln\mu}\,\mathrm{d}s \\
&\quad \le F_\mu^2 \int_{t_0}^{t} u^T(s)u(s)e^{-\alpha(t-s)+N_\sigma(t,s)\ln\mu}\,\mathrm{d}s.
\end{aligned} \tag{14}
$$

To obtain the ES, we discuss (14) in three cases, that is, $\mu > 1$, $\mu = 1$ and $0 < \mu < 1$. For the case of $\mu > 1$ ($\ln\mu > 0$). It follows from (2) that $\frac{\ln\mu}{T_{\max}}(t-s) - \ln\mu \le N_\sigma(t,s)\ln\mu \le \frac{\ln\mu}{T_{\min}}(t-s) + \ln\mu$. Then inequality (14) can be rearranged as follows:

$$
\begin{aligned}
&\frac{1}{\mu}\int_{t_0}^{t} y^T(s)y(s)e^{-(\alpha-\frac{\ln\mu}{T_{\max}})(t-s)}\,\mathrm{d}s \\
&\quad \le \mu F_\mu^2 \int_{t_0}^{t} u^T(s)u(s)e^{-(\alpha-\frac{\ln\mu}{T_{\min}})(t-s)}\,\mathrm{d}s,
\end{aligned} \tag{15}
$$

where $\alpha - \frac{\ln\mu}{T_{\max}} > 0$ and $\alpha - \frac{\ln\mu}{T_{\min}} > 0$ due to (5) and $\alpha > 0$.

Integrating both sides of (15) from $t = t_0$ to ∞ and interchanging the order of integrals lead to

$$
\begin{aligned}
&\int_{t_0}^{\infty}\int_{t_0}^{t} y^T(s)y(s)e^{-(\alpha-\frac{\ln\mu}{T_{\max}})(t-s)}\,\mathrm{d}s\,\mathrm{d}t \\
&\quad \le \mu^2 F_\mu^2 \int_{t_0}^{\infty}\int_{t_0}^{t} u^T(s)u(s)e^{-(\alpha-\frac{\ln\mu}{T_{\min}})(t-s)}\,\mathrm{d}s\,\mathrm{d}t,
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&\int_{t_0}^{\infty} y^T(s)y(s)e^{-(\alpha-\frac{\ln\mu}{T_{\max}})(-s)} \int_{s}^{\infty} e^{-(\alpha-\frac{\ln\mu}{T_{\max}})t}\,\mathrm{d}t\,\mathrm{d}s \\
&\quad \le \mu^2 F_\mu^2 \int_{t_0}^{\infty} u^T(s)u(s)e^{-(\alpha-\frac{\ln\mu}{T_{\min}})(-s)} \int_{s}^{\infty} e^{-(\alpha-\frac{\ln\mu}{T_{\min}})t}\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned} \tag{17}
$$

By substituting $F_\mu = \frac{\gamma}{\mu}\sqrt{\frac{\alpha-\ln\mu/T_{\min}}{\alpha-\ln\mu/T_{\max}}}$ into (17), we obtain

$$
\int_{t_0}^{\infty} y^T(s)y(s)\,\mathrm{d}s \le \gamma^2 \int_{t_0}^{\infty} u^T(s)u(s)\,\mathrm{d}s. \tag{18}
$$

The remaining arguments for the other two cases, $\mu = 1$ and $0 < \mu < 1$, are analogous to the above analysis and will not be reproduced here. Consider $u(s) = 0$, for $s \in [0, t_0]$, then we have

$$\int_0^\infty y^T(s)y(s)\,\mathrm{d}s \le \gamma^2 \int_0^\infty u^T(s)u(s)\,\mathrm{d}s, \tag{19}$$

which implies that the system (1) is externally stable with a L_2 gain γ. □

Remark 3.1 The switching parameter $0 < \mu < 1$ was usually overlooked, as mentioned also in [11]. Only $\mu \ge 1$ was considered in [16–18, 22] and in [19] that was $\eta \ge 1$. If the sets of those matrices (such as P_i, Q_i in this work) were finite, as assumed in [16–19, 22], it would be not easy to consider this overlooked case. However, in [11, 25] and this work, the switching index set, see $\mathscr{P}$ below (1), is not restricted to be finite such that the just mentioned sets of matrices can be infinite. Moreover, in conditions like $Q_i \le \mu Q_j$, i and j are only any two consecutive (j is just after i) rather than two arbitrary indices as in [16–19, 22]. These together make the case $0 < \mu < 1$ feasible, i.e. the range of μ can be all positive real numbers.

Remark 3.2 It follows from (12) and Theorem 3.1, the zero-input state $\|x(t)\| \le K_\mu \|x_{t_0}\|_d\ e^{-\alpha_\mu(t-t_0)}$, where $K_\mu = \sqrt{\frac{\mu[\lambda_{\max}(P_{\sigma(t_0)})+d\lambda_{\max}(Q_{\sigma(t_0)})]}{\inf_{i\in\mathscr{P}}(\lambda_{\min}(P_i))}}$, $\alpha_\mu = \frac{1}{2}(\alpha - \frac{\ln\mu}{T_{\min}})$ if $\mu > 1$; $K_\mu = \sqrt{\frac{\lambda_{\max}(P_{\sigma(t_0)})+d\lambda_{\max}(Q_{\sigma(t_0)})}{\inf_{i\in\mathscr{P}}(\lambda_{\min}(P_i))}}$, $\alpha_\mu = \frac{1}{2}\alpha$ if $\mu = 1$; $K_\mu = \sqrt{\frac{\lambda_{\max}(P_{\sigma(t_0)})+d\lambda_{\max}(Q_{\sigma(t_0)})}{\mu\inf_{i\in\mathscr{P}}(\lambda_{\min}(P_i))}}$, $\alpha_\mu = \frac{1}{2}(\alpha - \frac{\ln\mu}{T_{\max}})$ if $0 < \mu < 1$.

3.1.2 Not all subsystems are Hurwitz stable

In the following, we consider to investigate the ES of SSs consisting of subsystems that are not all Hurwitz stable.

For a subsystem A_i (Hurwitz stable), similar to (6)–(10), A_{di}, B_i, C_i, D_i on $[t_{k-1}, t_k)$, replacing α with α_- under the condition (3), then, for $t \in [t_{k-1}, t_k)$, we can derive that

$$V_i(t) \le V_i(t_{k-1})e^{-\alpha_-(t-t_{k-1})} - \int_{t_{k-1}}^t \Delta(s)e^{-\alpha_-(t-s)}\,\mathrm{d}s. \tag{20}$$

For a subsystem A_i (not Hurwitz), there always exist $P_i > 0$ and $\alpha_+ > 0$ (as long as α_+ is large enough), such that $A_i^TP_i + P_iA_i - \alpha_+P_i < 0$. If the condition

$$\begin{bmatrix} A_i^TP_i + P_iA_i - \alpha_+P_i + Q_i & P_iA_{di} & P_iB_i & C_i^T \\ * & -(1-\rho)e^{-\alpha d}Q_i & 0 & C_{di}^T \\ * & * & -F_\mu^2 I & E_i^T \\ * & * & * & -I \end{bmatrix} < 0 \tag{21}$$

holds, we can deduce that

$$\dot{V}_i(t) + \alpha V_{2i}(t) - \alpha_+ V_{1i}(t) + \Delta(t) \le 0, \tag{22}$$

then

$$\dot{V}_i(t) - \alpha_+ V_i(t) + \Delta(t) \le 0. \tag{23}$$

Therefore, the following inequality holds:

$$V_i(t) \leq V_i(t_{k-1})e^{\alpha_+(t-t_{k-1})} - \int_{t_{k-1}}^{t} \Delta(s)e^{\alpha_+(t-s)}\,\mathrm{d}s. \tag{24}$$

Under the inequality in (4), using the same arguments as in (12), (13) and (14), from t_0 to t, $t \in [t_{k-1}, t_k)$, we can get

$$\begin{aligned} V_i(t) &\leq V_{\sigma(t_0)}(t_0)e^{\alpha_+T_+(t,t_0)-\alpha_-T_-(t,t_0)+N_\sigma(t,t_0)\ln\mu} \\ &\quad - \int_{t_0}^{t} \Delta(s)e^{\alpha_+T_+(t,s)-\alpha_-T_-(t,s)+N_\sigma(t,s)\ln\mu}\,\mathrm{d}s, \end{aligned} \tag{25}$$

And then

$$\begin{aligned} &\int_{t_0}^{t} y^T(s)y(s)e^{\alpha_+T_+(t,s)-\alpha_-T_-(t,s)+N_\sigma(t,s)\ln\mu}\,\mathrm{d}s \\ &\quad \leq F_\mu^2 \int_{t_0}^{t} u^T(s)u(s)e^{\alpha_+T_+(t,s)-\alpha_-T_-(t,s)+N_\sigma(t,s)\ln\mu}\,\mathrm{d}s, \end{aligned} \tag{26}$$

where $T_+(t,\tau)$, $T_-(t,\tau)$ denote the total active time of those subsystems with Hurwitz $A_i's$, not Hurwitz $A_i's$ over (τ, t), respectively.

Taking the case $\mu > 1$ as an example, the exponential index on the left side of (26): $\alpha_+T_+(t,s) - \alpha_-T_-(t,s) + N_\sigma(t,s)\ln\mu$ may be reduced to be $-(\alpha_- - \frac{\ln\mu}{T_{\max}})(t-s)$, while the same index on the right side of (26) may not be immediately increased to be in the form: $-\lambda(t-s)$, $\lambda > 0$. Now, we choose a scalar $\alpha_* \in (0, \alpha_-)$ arbitrarily and propose the following switching law.

Switching Law 3.1 ([25]) *Given a sequence of time instants* $t_1 < t_2 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = \infty$, *where* t_k, $k = 1, 2, \ldots$, *is the switching instant and* $t_1 > t_0$, *such that, for given scalars* $\alpha_+ > 0$, $\alpha_- > 0$, $0 < \alpha_* < \alpha_-$ *and* $c_* > 0$, *the inequality* $\alpha_+T_+(t,\tau) - \alpha_-T_-(t,\tau) \leq c_* - \alpha_*(t-\tau)$ *holds, for any* $t \geq \tau \geq t_0$.

Theorem 3.2 *Given scalars* $\alpha > 0$, $0 < \alpha_- < \alpha$, $\alpha_+ > 0$, $c_* > 0$, $\mu > 0$ *and the numbers of any two consecutive subsystems* $i, j \in \mathscr{P}$ (σ *switches from* j *to* i), *the system* (1), *under Switching Law* 3.1, *is externally stable with a* L_2 *gain* γ, *if there exist real* $n \times n$ *positive definite matrices* P_i, Q_i, P_j, Q_j *such that, for each subsystem* (A_i *is Hurwitz*),

$$\begin{bmatrix} A_i^TP_i + P_iA_i + \alpha_-P_i + Q_i & P_iA_{di} & P_iB_i & C_i^T \\ * & -(1-\rho)e^{-\alpha d}Q_i & 0 & C_{di}^T \\ * & * & -F_\mu^2 I & E_i^T \\ * & * & * & -I \end{bmatrix} < 0; \tag{27}$$

for each subsystem (A_i *is not Hurwitz*),

$$\begin{bmatrix} A_i^TP_i + P_iA_i - \alpha_+P_i + Q_i & P_iA_{di} & P_iB_i & C_i^T \\ * & -(1-\rho)e^{-\alpha d}Q_i & 0 & C_{di}^T \\ * & * & -F_\mu^2 I & E_i^T \\ * & * & * & -I \end{bmatrix} < 0, \tag{28}$$

$$\begin{bmatrix} \mu P_j & (I+I_j)^T P_i \\ * & P_i \end{bmatrix} \geq 0, \qquad Q_i \leq \mu Q_j, \tag{29}$$

and

$$\alpha_* > \frac{\ln \mu}{T_{\min}}, \tag{30}$$

where $F_\mu = \frac{\gamma}{\mu}\sqrt{\frac{\alpha_* - \ln\mu/T_{\min}}{e^{c_*}(\alpha_- - \ln\mu/T_{\max})}}$ *if* $\mu > 1$; $F_\mu = \gamma\sqrt{\frac{\alpha_*}{e^{c_*}\alpha_-}}$ *if* $\mu = 1$; $F_\mu = \gamma\mu\sqrt{\frac{\alpha_* - \ln\mu/T_{\max}}{e^{c_*}(\alpha_- - \ln\mu/T_{\min})}}$ *if* $0 < \mu < 1$.

Proof Under Switching Law 3.1, one has

$$e^{\alpha_+ T_+(t,s) - \alpha_- T_-(t,s)} \leq e^{c_* - \alpha_*(t-s)}. \tag{31}$$

To obtain the ES, we also investigate (26) in three cases, that is $\mu > 1$, $\mu = 1$ and $0 < \mu < 1$. For the case of $\mu > 1$. It follows from (2) that $\frac{\ln\mu}{T_{\max}}(t-s) - \ln\mu \leq N_\sigma(t,s)\ln\mu \leq \frac{\ln\mu}{T_{\min}}(t-s) + \ln\mu$. From (26), we obtain

$$\int_{t_0}^{t} y^T(s)y(s)e^{-\alpha_-(t-s)+N_\sigma(t,s)\ln\mu}\,\mathrm{d}s \leq F_\mu^2 \int_{t_0}^{t} u^T(s)u(s)e^{-\alpha_*(t-s)+N_\sigma(t,s)\ln\mu}\,\mathrm{d}s. \tag{32}$$

Then

$$\frac{1}{\mu}\int_{t_0}^{t} y^T(s)y(s)e^{-(\alpha_- - \frac{\ln\mu}{T_{\max}})(t-s)}\,\mathrm{d}s \leq \mu F_\mu^2 \int_{t_0}^{t} u^T(s)u(s)e^{-(\alpha_* - \frac{\ln\mu}{T_{\min}})(t-s)}\,\mathrm{d}s. \tag{33}$$

Using $\alpha_* > \frac{\ln\mu}{T_{\min}}$ (then $\alpha_* - \frac{\ln\mu}{T_{\min}} > 0$ and $\alpha_- - \frac{\ln\mu}{T_{\max}} > 0$ due to $\alpha_* < \alpha_-$) and selecting $F_\mu = \frac{\gamma}{\mu}\sqrt{\frac{\alpha_* - \ln\mu/T_{\min}}{e^{c_*}(\alpha_- - \ln\mu/T_{\max})}}$, then following a similar process to (16) to (19), we can conclude that the control system (1), under Switching Law 3.1 is externally stable with a L_2 gain γ. □

Remark 3.3 It follows from (25), Switching Law 3.1 and Theorem 3.2 that the state $\|x(t)\| \leq \hat{K}_\mu \|x_{t_0}\|_d\, e^{-\hat{\alpha}_\mu(t-t_0)}$, where $\hat{K}_\mu = e^{\frac{c_*}{2}}\sqrt{\frac{\mu[\lambda_{\max}(P_{\sigma(t_0)}) + d\lambda_{\max}(Q_{\sigma(t_0)})]}{\inf_{i\in\mathscr{P}}(\lambda_{\min}(P_i))}}$, $\hat{\alpha}_\mu = \frac{1}{2}(\alpha_* - \frac{\ln\mu}{T_{\min}})$ if $\mu > 1$; $\hat{K}_\mu = e^{\frac{c_*}{2}}\sqrt{\frac{\lambda_{\max}(P_{\sigma(t_0)}) + d\lambda_{\max}(Q_{\sigma(t_0)})}{\inf_{i\in\mathscr{P}}(\lambda_{\min}(P_i))}}$, $\hat{\alpha}_\mu = \frac{1}{2}\alpha_*$ if $\mu = 1$; $\hat{K}_\mu = e^{\frac{c_*}{2}}\sqrt{\frac{\lambda_{\max}(P_{\sigma(t_0)}) + d\lambda_{\max}(Q_{\sigma(t_0)})}{\mu\inf_{i\in\mathscr{P}}(\lambda_{\min}(P_i))}}$, $\hat{\alpha}_\mu = \frac{1}{2}(\alpha_* - \frac{\ln\mu}{T_{\max}})$ if $0 < \mu < 1$.

In fact, for the case $0 < \mu < 1$, without Switching Law 3.1, we also have the following conclusion.

Corollary 3.1 *Given scalars* $\alpha > 0$, $0 < \alpha_- < \alpha$, $\alpha_+ > 0$, $0 < \mu < 1$ *and the numbers of any two consecutive subsystems* $i, j \in \mathscr{P}$ (σ *switches from* j *to* i), *the system* (1) *is externally stable with a* L_2 *gain* γ, *if there exist real* $n \times n$ *positive definite matrices* P_i, Q_i, P_j, Q_j *such that, for each subsystem* (A_i *is Hurwitz*),

$$\begin{bmatrix} A_i^T P_i + P_i A_i + \alpha_- P_i + Q_i & P_i A_{di} & P_i B_i & C_i^T \\ * & -(1-\rho)e^{-\alpha d}Q_i & 0 & C_{di}^T \\ * & * & -F_\mu^2 I & E_i^T \\ * & * & * & -I \end{bmatrix} < 0; \tag{34}$$

for each subsystem (A_i is not Hurwitz),

$$\begin{bmatrix} A_i^T P_i + P_i A_i - \alpha_+ P_i + Q_i & P_i A_{di} & P_i B_i & C_i^T \\ * & -(1-\rho)e^{-\alpha d} Q_i & 0 & C_{di}^T \\ * & * & -F_\mu^2 I & E_i^T \\ * & * & * & -I \end{bmatrix} < 0, \tag{35}$$

$$\begin{bmatrix} \mu P_j & (I+I_j)^T P_i \\ * & P_i \end{bmatrix} \geq 0, \qquad Q_i \leq \mu Q_j, \tag{36}$$

and

$$\alpha_+ < \frac{|\ln \mu|}{T_{\max}}, \tag{37}$$

where $F_\mu = \gamma \mu \sqrt{\frac{-\alpha_+ + |\ln \mu|/T_{\max}}{\alpha_- + |\ln \mu|/T_{\min}}}$.

Proof Taking a similar process to (20) to (26), we have

$$\begin{aligned} &\int_{t_0}^t y^T(s) y(s) e^{\alpha_+ T_+(t,s) - \alpha_- T_-(t,s) + N_\sigma(t,s) \ln \mu} \, \mathrm{d}s \\ &\quad \leq F_\mu^2 \int_{t_0}^t u^T(s) u(s) e^{\alpha_+ T_+(t,s) - \alpha_- T_-(t,s) + N_\sigma(t,s) \ln \mu} \, \mathrm{d}s. \end{aligned} \tag{38}$$

Since $\frac{t-s}{T_{\max}} - 1 \leq N_\sigma(t,s) \leq \frac{t-s}{T_{\min}} + 1$, $\frac{-|\ln \mu|}{T_{\min}}(t-s) + \ln \mu \leq N_\sigma(t,s) \ln \mu \leq \frac{-|\ln \mu|}{T_{\max}}(t-s) - \ln \mu$. Then

$$\begin{aligned} &\mu \int_{t_0}^t y^T(s) y(s) e^{-(\alpha_- + \frac{|\ln \mu|}{T_{\min}})(t-s)} \, \mathrm{d}s \\ &\quad \leq \frac{1}{\mu} F_\mu^2 \int_{t_0}^t u^T(s) u(s) e^{-(-\alpha_+ + \frac{|\ln \mu|}{T_{\max}})(t-s)} \, \mathrm{d}s. \end{aligned} \tag{39}$$

Suppose $\alpha_+ < \frac{|\ln \mu|}{T_{\max}}$ and choose $F_\mu = \gamma \mu \sqrt{\frac{-\alpha_+ + |\ln \mu|/T_{\max}}{\alpha_- + |\ln \mu|/T_{\min}}}$. Then following a similar process to (16) to (19), we can prove that the control system (1), consisting of both subsystems with Hurwitz A_i's and subsystems with A_i's that are not Hurwitz, is externally stable with a L_2 gain γ. □

Remark 3.4 It follows from (25) and the Corollary 3.1 that, for the case $0 < \mu < 1$, we have the zero-input state

$$\|x(t)\| \leq \sqrt{\frac{\lambda_{\max}(P_{\sigma(t_0)}) + d\lambda_{\max}(Q_{\sigma(t_0)})}{\mu \inf_{i \in \mathscr{P}}(\lambda_{\min}(P_i))}} \|x_{t_0}\|_d e^{-\frac{1}{2}(-\alpha_+ + \frac{|\ln \mu|}{T_{\max}})(t-t_0)}.$$

Remark 3.5 The Corollary 3.1 does not use Switching Law 3.1, the right side of inequality (38) is enlarged as $F_\mu^2 \int_{t_0}^t u^T(s) u(s) e^{\alpha_+(t-s) + N_\sigma(t,s) \ln \mu} \, \mathrm{d}s$ instead of $F_\mu^2 \int_{t_0}^t u^T(s) u(s) \times$

$e^{-\alpha_*(t-s)+N_\sigma(t,s)\ln\mu}\,\mathrm{d}s$. As an additional condition, $\alpha_+ < \frac{|\ln\mu|}{T_{\max}}$ is imposed. Therefore, Theorem 3.2 is less conservative.

Remark 3.6 By employing the new proposed two-direction inequality (2), after the steps (15)–(19), (33) and (39), the normal L_2 norm constraint instead of the weighted L_2 norm constraint is obtained. However, only the weighted L_2 norm constraint was derived in [16, 17, 22]. This can be easily seen in the proof steps (2.10)–(2.15) in [22], (33)–(34) in [16] and (49)–(50) in [17]. The "average dwell time" method is adopted in these references, that is, the one-direction inequality $N_\sigma(0,\tau) \le \frac{\tau}{\tau_a}$ is applied.

3.2 Application to H_∞ control

In this section, we apply the derived results of ES to H_∞ control. The so-called H_∞ control is named from the H_∞ functions defined on the H_∞ (Hardy) space (see page 1 in [30]): $H_\infty := \{F : C \to C | F \text{ is analytic}, \sup_{\mathrm{Re}(s)>0} |F(s)| < \infty\}$ equipped with the norm $\|F\|_\infty := \sup_{\mathrm{Re}(s)>0} |F(s)|$, for $F \in H_\infty$. As is well known (also see page 4 in [31]), transfer functions for finite dimensional linear control systems are rational functions with real coefficients. Thus, we may consider the subset of H_∞ consisting of real-rational functions: $RH_\infty \subset H_\infty$. In fact [31], a transfer function $F(s) \in RH_\infty$ if and only if F is proper ($\lim_{s\to\infty} F(s) < \infty$) and stable ($F$ has no poles in the closed right half complex plane). In this case, $\|F\|_\infty = \sup_{\omega\in R} |F(j\omega)| = \sup_{u\in L_2, u\neq\theta} \|y\|_{L_2}/\|u\|_{L_2}$, where u, y denote the input, output of the considered control system [1]. Therefore, the transfer function of a linear control system $F(s)$ is a real-rational H_∞ function implies that the control system is externally stable, i.e. every L_2 input only excites an L_2 zero-state output. If the input is considered to be a disturbance, then ES measures the robustness of the zero-state output on the disturbance, i.e. it ensures that the zero-state output excited by the energy-bounded (because the square of the L_2 norm of a signal can be considered as the energy content of the signal) disturbance will not blow up.

For linear control systems, H_∞ control is to find a control (consisting of measured variables) such that the norm of the transfer function from the disturbance (input) u_d to the output y (something we want to minimize) $\|F_{d\to y}\|_\infty$ is minimized, i.e. the zero-state output excited by disturbance y_d is minimized [32]. For nonlinear control systems including the system considered in this paper, they do not have transfer functions as the linear ones do. However, the same name H_∞ control is employed for the following control objective: to find a control such that the controlled system is asymptotically stable when no disturbances are present, and moreover, has finite L_2 gain from u_d to y, under the zero initial condition (is externally stable from u_d to y), see page 6 in [31]. As we see above, for either linear or nonlinear control systems, H_∞ control has the same physical meaning: the H_∞ controller starts to stabilize the system after the energy-bounded disturbance has already decayed to zero, then maintains the stabilized system to be externally stable from the disturbance to the output such that effect of the disturbance on the output is attenuated during the steady period. This specializes the practical importance of H_∞ control in industry roared with noises. However, by the existing average dwell time approach, those results obtained for switched system are only on weak noise attenuation index of weighted form than cannot truly reflect the practical meaning of H_∞ problems [17].

In this section, by adopting maximum, minimum dwell time and the new proposed two-direction inequality (2), we state the details for the non-weighted H_∞ control of switching control systems with delay and impulse. Consider (1) with both control input u_c and disturbance input u_d as follows:

$$\begin{aligned}
&\dot{x}(t) = A_{\sigma(t)}x(t) + A_{d\sigma(t)}x\bigl(t-\tau(t)\bigr) + B_{\sigma(t)}\bigl(u_c(t)+u_d(t)\bigr), \quad t \in [t_{k-1}, t_k),\\
&\Delta x = x(t_k) - x\bigl(t_k^-\bigr) = I_{\sigma(t_k^-)}x\bigl(t_k^-\bigr), \quad t = t_k,\\
&y(t) = C_{\sigma(t)}x(t) + C_{d\sigma(t)}x\bigl(t-\tau(t)\bigr) + E_{\sigma(t)}\bigl(u_c(t)+u_d(t)\bigr),\\
&x_{t_0} = x(t_0+\theta) = \phi(\theta), \quad \theta \in [-d, 0].
\end{aligned} \tag{40}$$

As just discussed, the control (input) u_c is comprised of measured variables. A special (and common) form is $u_c = K_{\sigma(t)}x$ (state feedback), where $K_{\sigma(t)}$ is the control gain to be designed and x is the state variable that is pre-assumed to be measurable (available). Generally, u_c may be also designed as a function of x_t (delayed state feedback), $x(t_k^-)$ (impulsive state feedback) and y (output feedback) as needed, if they can be measured.

Controlled by $u_c = K_{\sigma(t)}x$, the system above can be rewritten as

$$\begin{aligned}
&\dot{x}(t) = \bar{A}_{\sigma(t)}x(t) + A_{d\sigma(t)}x\bigl(t-\tau(t)\bigr) + B_{\sigma(t)}u_d(t), \quad t \in [t_{k-1}, t_k),\\
&\Delta x = x(t_k) - x\bigl(t_k^-\bigr) = I_{\sigma(t_k^-)}x\bigl(t_k^-\bigr), \quad t = t_k,\\
&y(t) = \bar{C}_{\sigma(t)}x(t) + C_{d\sigma(t)}x\bigl(t-\tau(t)\bigr) + E_{\sigma(t)}u_d(t),\\
&x_{t_0} = x(t_0+\theta) = \phi(\theta), \quad \theta \in [-d, 0],
\end{aligned} \tag{41}$$

where $\bar{A}_{\sigma(t)} = A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)}$ and $\bar{C}_{\sigma(t)} = C_{\sigma(t)} + E_{\sigma(t)}K_{\sigma(t)}$. The H_∞ control problem is to find $K_{\sigma(t)}$ such that (41) is asymptotically stable when $u_d = 0$, and is externally stable from u_d to y, i.e. $\|y(t)\|_{L_2} \le \gamma^*\|u_d(t)\|_{L_2}$ for some prescribed constant γ^*, when $\phi(\theta) = 0$.

Comparing (41) with (1), it can be concluded that the objectives of H_∞ control are all achieved (since exponential stability implies asymptotic stability and the ES is already proven) in different cases if $A_{\sigma(t)}$, $C_{\sigma(t)}$ are replaced by $\bar{A}_{\sigma(t)}$, $\bar{C}_{\sigma(t)}$, respectively, in the corresponding theorems. To avoid tediousness, we only state a theorem for the H_∞ control corresponding to Theorem 3.2. The result corresponding to Theorem 3.1 is left to the reader.

Theorem 3.3 *Given scalars $\alpha > 0$, $0 < \alpha_- < \alpha$, $\alpha_+ > 0$, $c_* > 0$, $\mu > 0$ and the numbers of any two consecutive subsystems $i, j \in \mathscr{P}$ (σ switches from j to i), the H_∞ control problem of* (40), *under the switching law S3.1, is solved, if there exist real $n \times n$ positive definite matrices P_i, Q_i, P_j, Q_j and appropriate dimensions matrices K_i, K_j such that, for each subsystem ($\bar{A}_i$ is Hurwitz),*

$$\begin{bmatrix}
\bar{A}_i^T P_i + P_i\bar{A}_i + \alpha_- P_i + Q_i & P_iA_{di} & P_iB_i & \bar{C}_i^T\\
* & -(1-\rho)e^{-\alpha d}Q_i & 0 & C_{di}^T\\
* & * & -\digamma_\mu^2 I & E_i^T\\
* & * & * & -I
\end{bmatrix} < 0; \tag{42}$$

for each subsystem ($\bar{A}_i$ is not surely Hurwitz),

$$\begin{bmatrix} \bar{A}_i^T P_i + P_i\bar{A}_i - \alpha_+ P_i + Q_i & P_i A_{di} & P_i B_i & \bar{C}_i^T \\ * & -(1-\rho)e^{-\alpha d} Q_i & 0 & C_{di}^T \\ * & * & -\digamma_\mu^2 I & E_i^T \\ * & * & * & -I \end{bmatrix} < 0, \tag{43}$$

$$\begin{bmatrix} \mu P_j & (I+I_j)^T P_i \\ * & P_i \end{bmatrix} \geq 0, \qquad Q_i \leq \mu Q_j, \tag{44}$$

and

$$\alpha_* > \frac{\ln\mu}{T_{\min}}, \tag{45}$$

where $\digamma_\mu = \frac{\gamma^*}{\mu}\sqrt{\frac{\alpha_* - \ln\mu/T_{\min}}{e^{c_*}(\alpha_- - \ln\mu/T_{\max})}}$, *if* $\mu > 1$; $\digamma_\mu = \gamma^*\sqrt{\frac{\alpha_*}{e^{c_*}\alpha_-}}$ *if* $\mu = 1$; $\digamma_\mu = \gamma^*\mu\sqrt{\frac{\alpha_* - \ln\mu/T_{\max}}{e^{c_*}(\alpha_- - \ln\mu/T_{\min})}}$ *if* $0 < \mu < 1$.

Remark 3.7 It follows from Remark 3.3, under the same conditions as in Theorem 3.3, the state $\|x(t)\| \leq \tilde{K}_\mu \|x_{t_0}\|_d e^{-\tilde{\alpha}_\mu(t-t_0)}$, where $\tilde{K}_\mu = e^{\frac{c_*}{2}}\sqrt{\frac{\mu[\lambda_{\max}(P_{\sigma(t_0)}) + d\lambda_{\max}(Q_{\sigma(t_0)})]}{\inf_{i\in\mathscr{P}}(\lambda_{\min}(P_i))}}$, $\tilde{\alpha}_\mu = \frac{1}{2}(\alpha_* - \frac{\ln\mu}{T_{\min}})$ if $\mu > 1$; $\tilde{K}_\mu = e^{\frac{c_*}{2}}\sqrt{\frac{\lambda_{\max}(P_{\sigma(t_0)}) + d\lambda_{\max}(Q_{\sigma(t_0)})}{\inf_{i\in\mathscr{P}}(\lambda_{\min}(P_i))}}$, $\tilde{\alpha}_\mu = \frac{1}{2}\alpha_*$ if $\mu = 1$; $\tilde{K}_\mu = e^{\frac{c_*}{2}}\sqrt{\frac{\lambda_{\max}(P_{\sigma(t_0)}) + d\lambda_{\max}(Q_{\sigma(t_0)})}{\mu\inf_{i\in\mathscr{P}}(\lambda_{\min}(P_i))}}$, $\tilde{\alpha}_\mu = \frac{1}{2}(\alpha_* - \frac{\ln\mu}{T_{\max}})$ if $0 < \mu < 1$.

Remark 3.8 After substituting $\bar{A}_i$, $\bar{C}_i$, nonlinear terms like $P_i B_i K_i$ and $K_i^T B_i^T P_i$ appear in (42) and (43). We can first left and right multiply (42), (43) by $\operatorname{diag}(P_i^{-1}, I, I, I)$. Then we apply the Schur complement and use $-Q_i^{-1} \leq -2\epsilon_i I + \epsilon_i^2 Q_i$, $\epsilon_i > 0$ to derive the following linear matrix inequalities of P_i^{-1}, Q_i and Y_i:

$$\begin{bmatrix} (1,1)_1 & A_{di} & B_i & P_i^{-1}C_i^T + Y_i^T E_i^T & P_i^{-1} \\ * & -(1-\rho)e^{-\alpha d}Q_i & 0 & C_{di}^T & 0 \\ * & * & -\digamma_\mu^2 I & E_i^T & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -2\epsilon_i I + \epsilon_i^2 Q_i \end{bmatrix} < 0,$$

$$\begin{bmatrix} (1,1)_2 & A_{di} & B_i & P_i^{-1}C_i^T + Y_i^T E_i^T & P_i^{-1} \\ * & -(1-\rho)e^{-\alpha d}Q_i & 0 & C_{di}^T & 0 \\ * & * & -\digamma_\mu^2 I & E_i^T & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -2\epsilon_i I + \epsilon_i^2 Q_i \end{bmatrix} < 0,$$

where $(1,1)_1 = A_i P_i^{-1} + P_i^{-1}A_i^T + \alpha_- P_i^{-1} + B_i Y_i + Y_i^T B_i^T$, $(1,1)_2 = A_i P_i^{-1} + P_i^{-1}A_i^T - \alpha_+ P_i^{-1} + B_i Y_i + Y_i^T B_i^T$ and $Y_i = K_i P_i^{-1}$, obeying (42), (43), respectively. It follows that $K_i = Y_i P_i$.

As for the first inequality in (44), we can first left and right multiply it by $\operatorname{diag}(P_j^{-1}, I)$. Then left and right multiply it by $\operatorname{diag}(I, P_i^{-1})$ to derive an equivalent linear matrix inequality as

$$\begin{bmatrix} \mu P_j^{-1} & P_j^{-1}(I+I_j)^T \\ * & P_i^{-1} \end{bmatrix} \geq 0.$$

Thus, all matrix inequalities in Theorem 3.3 are already transformed to linear matrix inequalities of P_i^{-1}, Q_i and Y_i that can be solved by Matlab.

Remark 3.9 It is arbitrary to pre-assume that a certain $\bar{A}_i$ is Hurwitz or not surely Hurwitz, as long as feasible solutions for the inequalities can be found; see Example 4.1, where $\bar{A}_1$ is pre-assumed to be Hurwitz.

4 Illustrative example

In this section, we provide an example of H_∞ control with numerical simulations to illustrate previous results.

Example 4.1 Consider the switching control system with delay and impulse initialized as follows:

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad I_1 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad C_{d1} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad E_1 = 0.1,$$
$$A_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad C_{d2} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad E_2 = 0.1.$$

Let $\alpha_- = 1$, $\alpha_+ = 2$, $\alpha_* = 0.25 < \alpha_-$ and $c_* = 2.25$, then we can select the switching signal $\sigma(t)$ satisfied Switching Law 3.1. From Remark 3.6 in [25], here we choose a simple switching law realization as shown in Fig. 1. The first active subsystem is a Hurwitz stable subsystem.

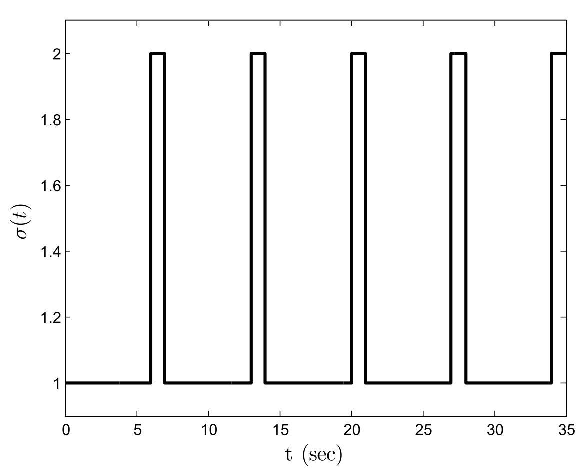

Figure 1 Switching law $\sigma(t)$

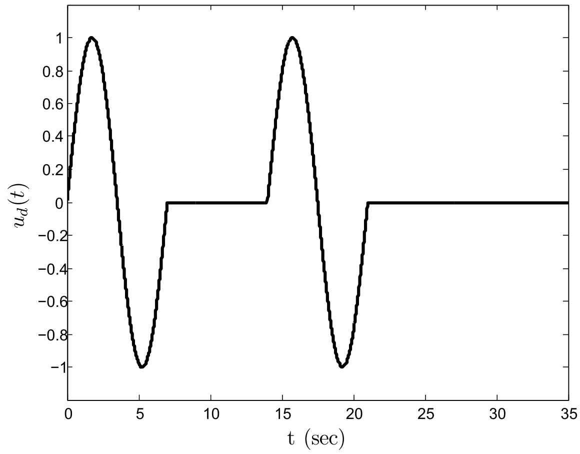

Figure 2 Disturbance input $u_d(t)$

And the switching time is periodic. Let $t_{2j} - t_{2j-1} = T_+ = 1$, $t_{2j-1} - t_{2j-2} = T_- = 6$, $j = 1, 2, \ldots$, then $T_+ \le c_*/(\alpha_+ + \alpha_*)$ and $T_- \ge 2T_+(\alpha_+ + \alpha_*)/(\alpha_- - \alpha_*)$ are satisfied. Therefore, $T_{\max} = T_- = 6$, $T_{\min} = T_+ = 1$. In fact, any switching signal can be selected as long as Switching Law 3.1 is satisfied.

Select the disturbance input $u_d(t) = \sin(\frac{2\pi}{7}t)[H(t) - H(t-7)] + \sin(\frac{2\pi}{7}(t-14))[H(t-14) - H(t-21)] \in L_2[0,\infty)$ as shown in Fig. 2 and the initial condition $\phi(\theta) = \phi(\theta_1, \theta_2) = [\theta_1 + 1, \theta_2 + 1]^T$, where $H(t)$ is the Heaviside step function. Given the delay function $\tau(t) = 0.1 + 0.1\sin t$, then we can select $d = 0.2$ and $\rho = 0.1$ such that $\tau(t) \le d$, $\dot{\tau}(t) \le \rho < 1$ as required.

All other common parameters for three cases are given as $\gamma^* = 1$, $\alpha = 1.01(\alpha > \alpha_-)$, $\epsilon_1 = 0.6$ and $\epsilon_2 = 0.6$.

Case 1: $0 < \mu < 1$. Select $\mu = 0.98$, then $F_\mu = 0.1586$ and $\alpha_* > \ln\mu/T_{\min} = -0.0202$. In this case, we consider $(A_1 + B_1K_j, A_{d1}, B_1, I_1, C_1 + E_1K_j, C_{d1}, E_1)$, $j = 1, 3, 5, \ldots, 9$ as the controlled subsystem $j = 1, 3, 5, \ldots, 9$, and $(A_2 + B_2K_i, A_{d2}, B_2, I_2, C_2 + E_2K_i, C_{d2}, E_2)$, $j = 2, 4, 6, \ldots, 10$, as the controlled subsystem $j = 2, 4, 6, \ldots, 10$. Note that in the switching signal, see Fig. 1, the lower '1's represent $1, 3, 5, \ldots, 9$ and the upper '2's represent $2, 4, 6, \ldots, 10$, from left to right. Employing the linear matrix inequality tool box of Matlab, we derive

$$P_1 = \begin{bmatrix} 2.5714 & -1.2904 \\ -1.2904 & 2.5714 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 2.0723 & -0.1300 \\ -0.1300 & 2.0723 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 1.7038 & -0.8411 \\ -0.8411 & 1.7038 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 1.0502 & -0.6317 \\ -0.6317 & 1.0502 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 1.2712 & -0.5577 \\ -0.5577 & 1.2712 \end{bmatrix}, \qquad Q_3 = \begin{bmatrix} 0.8672 & -0.4842 \\ -0.4842 & 0.8672 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 1.3189 & -0.5085 \\ -0.5085 & 1.3189 \end{bmatrix}, \qquad Q_4 = \begin{bmatrix} 0.7349 & -0.3881 \\ -0.3881 & 0.7349 \end{bmatrix},$$

$$P_5 = \begin{bmatrix} 1.0562 & -0.3770 \\ -0.3770 & 1.0562 \end{bmatrix}, \qquad Q_5 = \begin{bmatrix} 0.6212 & -0.3020 \\ -0.3020 & 0.6212 \end{bmatrix},$$

$$P_6 = \begin{bmatrix} 1.1485 & -0.3693 \\ -0.3693 & 1.1485 \end{bmatrix}, \qquad Q_6 = \begin{bmatrix} 0.5253 & -0.2344 \\ -0.2344 & 0.5253 \end{bmatrix},$$

$$P_7 = \begin{bmatrix} 0.9375 & -0.2832 \\ -0.2832 & 0.9375 \end{bmatrix}, \qquad Q_7 = \begin{bmatrix} 0.4342 & -0.1686 \\ -0.1686 & 0.4342 \end{bmatrix},$$

$$P_8 = \begin{bmatrix} 1.0295 & -0.2778 \\ -0.2778 & 1.0295 \end{bmatrix}, \qquad Q_8 = \begin{bmatrix} 0.3507 & -0.1115 \\ -0.1115 & 0.3507 \end{bmatrix},$$

$$P_9 = \begin{bmatrix} 0.8405 & -0.2137 \\ -0.2137 & 0.8405 \end{bmatrix}, \qquad Q_9 = \begin{bmatrix} 0.2616 & -0.0498 \\ -0.0498 & 0.2616 \end{bmatrix},$$

$$P_{10} = \begin{bmatrix} 0.8597 & -0.1445 \\ -0.1445 & 0.8597 \end{bmatrix}, \qquad Q_{10} = \begin{bmatrix} 0.1602 & 0.0211 \\ 0.0211 & 0.1602 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} -13.5184 & -13.5184 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} -2.4877 & -2.4877 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} -7.3631 & -7.3631 \end{bmatrix}, \qquad K_4 = \begin{bmatrix} -2.4773 & -2.4773 \end{bmatrix},$$

$$K_5 = \begin{bmatrix} -6.9145 & -6.9145 \end{bmatrix}, \qquad K_6 = \begin{bmatrix} -2.4720 & -2.4720 \end{bmatrix},$$

$$K_7 = \begin{bmatrix} -6.5315 & -6.5315 \end{bmatrix}, \qquad K_8 = \begin{bmatrix} -2.4724 & -2.4724 \end{bmatrix},$$

$$K_9 = \begin{bmatrix} -6.0558 & -6.0558 \end{bmatrix}, \qquad K_{10} = \begin{bmatrix} -2.4844 & -2.4844 \end{bmatrix}.$$

Then the conditions for this case in Theorem 3.3, where $\bar{A}_1$ (or $A_1 + B_1Kj$) is pre-assumed to be Hurwitz, are satisfied. From Remark 3.7, $K_\mu = 8.1513$ and $\alpha_\mu = 0.1267$.

Case 2: $\mu = 1$. In this case, $\digamma_\mu = 0.1623$ and $\alpha_* > \ln \mu / T_{\min} = 0$. Using Matlab, we obtain

$$P_1 = \begin{bmatrix} 0.9213 & -0.2664 \\ -0.2664 & 0.9213 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 0.4469 & -0.3123 \\ -0.3123 & 0.4469 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 1.0734 & -0.3074 \\ -0.3074 & 1.0734 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 0.4469 & -0.1087 \\ -0.1087 & 0.4469 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} -6.7553 & -6.7553 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} -2.4085 & -2.4085 \end{bmatrix}.$$

Thus, all conditions in Theorem 3.3 (where $\bar{A}_1$ is pre-assumed to be Hurwitz) hold. It follows from Remark 3.7 that $K_\mu = 4.3379$ and $\alpha_\mu = 0.1250$.

Case 3: $\mu > 1$. Select $\mu = 1.01$, then $\digamma_u = 0.1576$ and $\alpha_* > \ln \mu / T_{\min} = 0.0100$. By Matlab, we get

$$P_1 = \begin{bmatrix} 1.0091 & -0.3389 \\ -0.3389 & 1.0091 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 0.5889 & -0.3123 \\ -0.3123 & 0.5889 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 1.1750 & -0.3900 \\ -0.3900 & 1.1750 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 0.5889 & -0.3123 \\ -0.3123 & 0.5889 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} -6.7104 & -6.7104 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} -2.4849 & -2.4849 \end{bmatrix}.$$

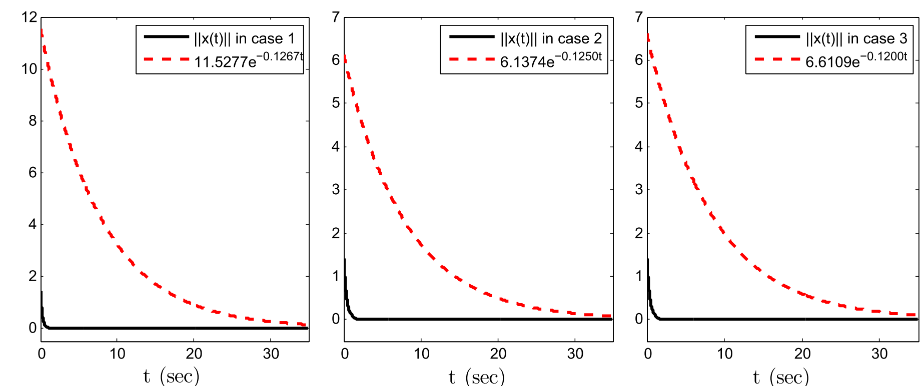

Figure 3 $\|x(t)\|$ and its exponential bound in three cases

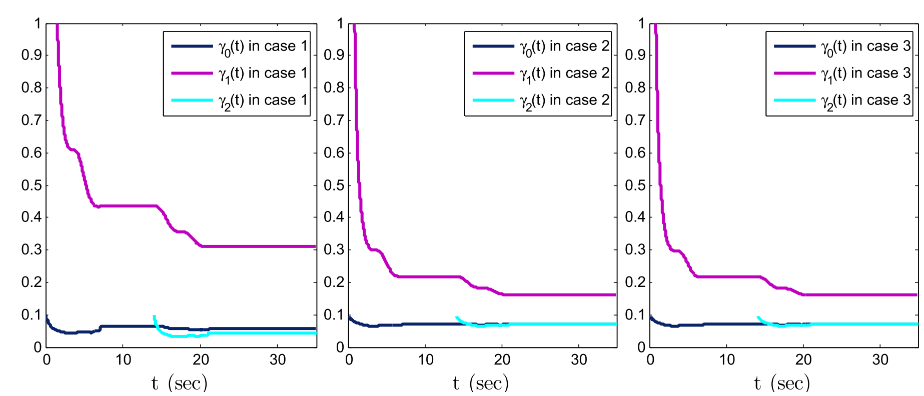

Figure 4 Ratios in three cases

Therefore, conditions in Theorem 3.3 (where $\bar{A}_1$ is pre-assumed to be Hurwitz) are all satisfied. According to Remark 3.7, $K_\mu = 4.6746$ and $\alpha_\mu = 0.1200$.

As expected, the zero-input ($u_d(t) \equiv 0$) states with the initial condition $\phi(\theta)$ in these three cases are bounded by $K_\mu \|x_{t_0}\|_d e^{-\alpha_\mu (t-t_0)}$: $11.5277e^{-0.1267t}$ (case 1), $6.6109e^{-0.1200t}$ (case 2) and $6.1347e^{-0.1250t}$ (case 3), see Fig. 3, respectively; and $\gamma_0(t)$ defined as $(\int_0^t y^2(s)\,\mathrm{d}s / \int_0^t u_d^2(s)\,\mathrm{d}s)^{\frac{1}{2}}$, under the zero initial condition, for each case, is less than $\gamma = 1$; see Fig. 4. Therefore, the H_∞ control goal has been achieved.

As introduced before, the H_∞ controller $u_c = K_{\sigma(t)} x$ starts to stabilize the system (the state x_1 with the initial condition $\phi(\theta)$ completely equaling x_2 in each case is almost stabilized between $t = 7$ s and $t = 14$ s, and after $t = 21$ s, see Fig. 5) after the energy-bounded disturbance u_d has already decayed to zero (between $t = 7$ s and $t = 14$ s, and after $t = 21$ s; see Fig. 2), then maintains the stabilized system to be externally stable from the disturbance to the output such that the effect of the disturbance on the output is attenuated during the steady period, see $\gamma_2(t)$ of each case in Fig. 4, defined as $(\int_{14}^t y^2(s)\,\mathrm{d}s / \int_{14}^t u_d^2(s)\,\mathrm{d}s)^{\frac{1}{2}}$ under the initial condition $\phi(\theta)$, which is less than $\gamma^* = 1$ and is of almost the same form as $\gamma_0(t)$. However, before the system is stabilized for the first time, i.e. during the transient period, the ES cannot be ensured, see $\gamma_1(t)$ of each case in Fig. 4, defined as $(\int_0^t y^2(s)\,\mathrm{d}s / \int_0^t u_d^2(s)\,\mathrm{d}s)^{\frac{1}{2}}$ under the initial condition $\phi(\theta)$, which is larger than γ around $t = 0$ s.

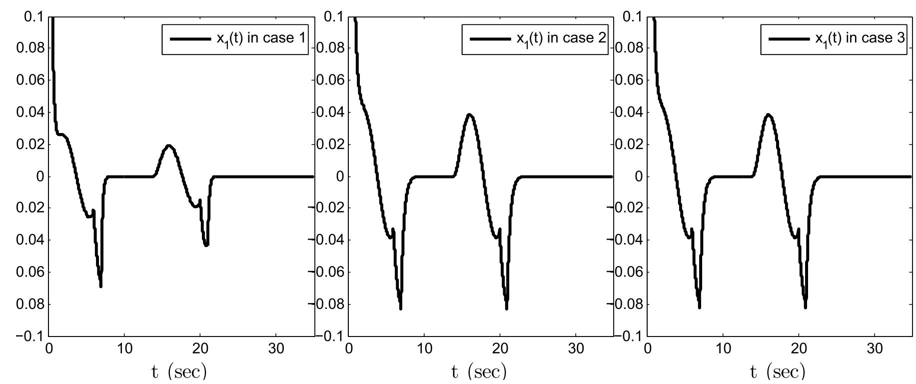

Figure 5 States in three cases

5 Conclusion

The ES and H_∞ control problem of SSs with delay and impulse has been investigated in this paper. After introducing the definitions of the maximum, minimum DT, we have applied the relation between the number of switchings and the maximum, minimum DT to prove the ES of SSs consisting of subsystems that are all Hurwitz stable. For those SSs comprised of subsystems that are not all Hurwitz stable, a realizable switching law has been employed to study their ES. And then the normal L_2 norm constraint has been derived. The label "weighted" has been removed properly in this paper. Finally, these results have been applied to H_∞ control and illustrated by a numerical example. In the future, we will first study the ES of nonlinear switching control systems without impulse or with impulse, then investigate the stability of SSs with switching signals driven by stochastic processes.

Acknowledgements
The authors would like to thank the editor and the anonymous referees for their helpful comments.

Funding
This work was supported by Opening Fund of Geomathematics Key Laboratory of Sichuan Province (scsxdz2019yb03).

Availability of data and materials
The data and material used to support the findings of this study are included within the article.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors drafted the manuscript, and they read and approved the submitted version.

Author details
[1]School of Electronic Information and Electrical Engineering, Chengdu University, Chengdu, China. [2]Institute of Nuclear Physics and Chemistry, China Academy of Engineering Physics, 621900 Mianyang, China.

Publisher's Note

Received: 30 March 2020 Accepted: 3 November 2020 Published online: 23 November 2020

References

1. Morris, K.A.: Introduction to Feedback Control. Academic Press, San Diego (2001)
2. Lin, P.Y., Liu, J.Y.: Analysis of resistance factors for LFRD of soil nail walls against external stability failures. Acta Geotech. **12**, 157–169 (2017)
3. Zheng, W., Sandip, R., Ali, S.: On the disturbance response and external stability of a saturating static-feedback-controlled double integrator. Automatica **44**, 2191–2196 (2008)

4. Khalil, H.K.: Nonlinear Systems, 3rd edn. Prentice Hall, New York (2002)
5. Liberzon, D.: Switching in Systems and Control. Birkhaüser, Boston (2003)
6. Zhang, L.X., Ning, Z.P., Shi, Y.: Analysis and synthesis for a class of stochastic switching systems against delayed mode switching: a framework of integrating mode weights. Automatica **99**, 99–111 (2019)
7. Vargas, A.N., Caruntu, C.F., Ishihara, J.Y.: Stability of switching linear systems with switching signals driven by stochastic processes. J. Franklin Inst. **356**, 31–41 (2019)
8. Sun, Y.G., Tian, Y.Z., Xie, X.J.: Stabilization of positive switched linear systems and its application in consensus of multiagent systems. IEEE Trans. Autom. Control **62**(12), 6608–6613 (2017)
9. Li, P.S., Lam, J., Lu, R.Q., Kwok, K.W.: Stability and L_2 synthesis of a class of periodic piecewise time-varying systems. IEEE Trans. Autom. Control **64**(8), 3378–3384 (2019)
10. Li, P.S., Lam, J., Kwok, K.W., Lu, R.Q.: Stability and stabilization of periodic piecewise linear systems: a matrix polynomial approach. Automatica **94**, 1–8 (2018)
11. Liu, X.Z., Wu, C.: Fault-tolerant synchronization for nonlinear switching systems with time-varying delay. Nonlinear Anal. Hybrid Syst. **23**, 91–110 (2017)
12. Zhou, Y.H., Li, C.D., Huang, T.W., Wang, X.: Impulsive stabilization and synchronization of Hopfield-type neural networks with impulse time window. Neural Comput. Appl. **28**, 775–782 (2017)
13. Tian, Y.Z., Cai, Y.L., Sun, Y.G.: Asymptotic behavior of switched delay systems with nonlinear disturbances. Appl. Math. Comput. **268**, 522–533 (2015)
14. Li, C.D., Huang, T.W., Feng, G., Chen, G.R.: Exponential stability of time-controlled switching systems with time delay. J. Franklin Inst. **349**, 216–233 (2012)
15. Wang, G., Xie, R.Q., Zhang, H.B., Yu, G.J., Dang, C.Y.: Robust exponential H_∞ filtering for discrete-time switched fuzzy systems with time-varying delay. Circuits Syst. Signal Process. **35**, 117–138 (2016)
16. Sun, X.M., Zhao, J., Hill, D.J.: Stability and L_2-gain analysis for switched delay systems: a delay-dependent method. Automatica **42**, 1769–1774 (2006)
17. Wu, L.G., Zheng, W.X.: Weighted H_∞ model reduction for linear switched systems with time-varying delay. Automatica **45**, 186–193 (2009)
18. Liu, X.Z., Yuan, S.: Reduced-order fault detection filter design for switched nonlinear systems with time delay. Nonlinear Dyn. **67**, 601–617 (2012)
19. Syed, A.M., Saravanan, S., Cao, J.: Finite-time boundedness, L_2 gain analysis and control of Markovian jump switched neural networks with additive time-varying delays. Nonlinear Anal. Hybrid Syst. **23**, 27–43 (2017)
20. Yuan, S., Liu, X.Z.: Fault estimator design for a class of switched systems with time-varying delay. Int. J. Syst. Sci. **42**, 2125–2135 (2011)
21. Liu, X.Z., Zhu, H., Zhong, S.M., Wu, C.: Exponential H_∞ synchronization of switching fuzzy systems with time-varying delay and impulses. Fuzzy Sets Syst. **365**, 116–139 (2019)
22. Zhai, G.S., Hu, B., Yasuda, K., Michel, A.N.: Disturbance attenuation properties of time-controlled switched systems. J. Franklin Inst. **338**, 765–779 (2001)
23. Shi, P., Su, X., Li, F.: Dissipativity-based filtering for fuzzy switched systems with stochastic perturbation. IEEE Trans. Autom. Control **61**, 1694–1699 (2016)
24. Hua, D.L., Wang, W.Q., Yao, J., Ren, Y.H.: Non-weighted H_∞ performance for 2-D FMLSS switched system with maximum and minimum dwell time. J. Franklin Inst. **356**, 5729–5753 (2019)
25. Wu, C., Liu, X.Z.: External stability of switching control systems. Syst. Control Lett. **106**, 24–31 (2017)
26. Zhang, H.B., Mao, Y.B., Gao, Y.S.: Exponential stability and asynchronous stabilization of switched systems with stable and unstable subsystems. Asian J. Control **15**, 1–8 (2013)
27. Long, L.J.: Input/output-to-state stability for switched nonlinear systems with unstable subsystems. Int. J. Robust Nonlinear Control **29** 3093–3110 (2019)
28. Sun, Y.W., Zhao, J., Dimirovski, G.M.: Adaptive control for a class of state-constrained high-order switched nonlinear systems with unstable subsystems. Nonlinear Anal. Hybrid Syst. **32**, 91–105 (2019)
29. Wei, J.M., Zhi, H.M., Liu, K., Mu, X.W.: Stability of mode-dependent linear switched singular systems with stable and unstable subsystems. J. Franklin Inst. **356**, 3102–3114 (2019)
30. Francis, A.B.: A Course in H_∞ Control Theory. Springer, Berlin (1987)
31. Teolis, C.A.: Robust H_∞ output feedback control. Ph.D. thesis, University of Maryland (1994)
32. Tannenbaum, A.: Feedback stabilization of linear dynamical plants with uncertainty in the gain factor. Int. J. Control **32**, 1–16 (1980)
33. Boyd, S., Ghaoui, L.E., Feron, E., Balakrishnan, V.: Linear Matrix Inequalities in System and Control Theory. SIAM, Philadelphia (1994)