Shi and Ruan *Advances in Difference Equations* (2020) 2020:588
https://doi.org/10.1186/s13662-020-02908-2

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

On the reasonability of linearized approximation and Hopf bifurcation control for a fractional-order delay Bhalekar–Gejji chaotic system

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Abstract

In this paper, we study the reasonability of linearized approximation and Hopf bifurcation control for a fractional-order delay Bhalekar–Gejji (BG) chaotic system. Since the current study on Hopf bifurcation for fractional-order delay systems is carried out on the basis of analyses for stability of equilibrium of its linearized approximation system, it is necessary to verify the reasonability of linearized approximation. Through Laplace transformation, we first illustrate the equivalence of stability of equilibrium for a fractional-order delay Bhalekar–Gejji chaotic system and its linearized approximation system under an appropriate prior assumption. This semianalytically verifies the reasonability of linearized approximation from the viewpoint of stability. Then we theoretically explore the relationship between the time delay and Hopf bifurcation of such a system. By introducing the delayed feedback controller into the proposed system, the influence of the feedback gain changes on Hopf bifurcation is also investigated. The obtained results indicate that the stability domain can be effectively controlled by the proposed delayed feedback controller. Moreover, numerical simulations are made to verify the validity of the theoretical results.

Keywords: Fractional-order BG system; Hopf bifurcation; Delayed feedback control; Linearization; Stability

1 Introduction

For a long time, researches on fractional-order calculus were mainly concentrated in the field of pure mathematics [1]. The main reason for abandoning fractional-order models in practical application was their computational complexity. With the development of computer technology, the application of fractional-order calculus has attracted the interest of many researchers in different fields, including mathematics, physics, chemistry, engineering, and even financial and social sciences. It has been found in some fields that using fractional-order is more appropriate than employing integer-order to describe the dynamical behavior and characteristics [2–6] of models, such as the classical predator–prey system [7, 8], the fractional-order model of the dengue virus infection [9], the simulated tidal wave fractional-order model [10], the fractional-order viscoelastic non-Newtonian fluid model [11], and so on. It is proved that the fractional-order dynamical system can more accurately respond to the time variation of general nature [12–14].

In order to describe those phenomena whose evolutions not only depend on the state of current time, but also on the state at a previous time, the time delay needs to be introduced into many differential equations which are defined as delayed differential equation (DDE). DDEs arise in many fields, for example, metal cutting, epidemiology, neuroscience, population dynamics [15], biological systems [16, 17], financial system [18], and traffic models [19]. The time delay is often considered as the parameter of fractional-order systems according to the actual background of the problems [20–22]. In 1992, Pyragas originally proposed the delayed feedback controller to study the control issue of nonlinear autonomous differential equations [23]. Now the delayed feedback has become an adjustment mechanism that is widely used in many nonlinear control systems. For example, in order to stabilize the unstable periodic orbit by the difference between the current state and the delay state, a suitable delayed feedback controller can be designed to achieve the desired dynamic behavior [24–26].

In 2011, Bhalekar and Daftardar-Gejji [27] constructed a three-dimensional chaotic dynamic system (shortly called as BG system subsequently):

$$\begin{cases} \dot{x}(t) = dx(t) - y^2(t), \\ \dot{y}(t) = c(z(t) - y(t)), \\ \dot{z}(t) = ay(t) - bz(t) + x(t)y(t), \end{cases} \tag{1.1}$$

where a, b, c, d are constants. Then Bhalekar [28] further explored the forming mechanism of the BG system. In recent years, some valuable results about the BG system have been obtained. Aqeel and Ahmad [29] studied the Hopf bifurcation and chaos of the integer-order BG system. Deshpande et al. [30] found that the fractional-order BG system allows chaotic solutions and the fractional-order could be regarded as the control parameter of chaos. Shahzad et al. [31] added a single time delay into the third equation of (1.1) and studied a delay Bhalekar–Gejji chaotic system of the form:

$$\begin{cases} \dot{x}(t) = dx(t) - y^2(t), \\ \dot{y}(t) = d(z(t) - y(t)), \\ \dot{z}(t) = ay(t) - bz(t - \tau) + x(t)y(t). \end{cases} \tag{1.2}$$

They derived some algebraic sufficient conditions that guarantee the globally and asymptotically stable synchronization and antisynchronization between two identical time delay Bhalekar–Gejji chaotic systems. To the best of our knowledge, there are few literature sources discussing the Hopf bifurcation of the fractional-order BG system with time delay. This motivates us to investigate the effect of time-delay and fractional-order on the occurrence of the Hopf bifurcation and to introduce an appropriate delayed state-feedback controller to control the Hopf bifurcation.

In this paper, we consider the fractional-order time delay Bhalekar–Gejji system of the form:

$$\begin{cases} D^{q_1}x(t) = dx(t) - y^2(t), \\ D^{q_2}y(t) = c(z(t) - y(t)), \\ D^{q_3}z(t) = ay(t) - bz(t-\tau) + x(t)y(t), \end{cases} \tag{1.3}$$

where $q_i \in (0,1]$ $(i = 1,2,3)$ and a, b, c, d are parameters; c is generally taken as positive, while d is a negative real number; $\tau \geq 0$ is the time delay.

The main purpose of this paper is to seek for the conditions of the occurrence of Hopf bifurcation for system (1.3) by using time delay as the bifurcation parameter based on the approach of stability analysis [32]. Especially, we give a semianalytical verification of the reasonability of linearized approximation of system (1.3) from the viewpoint of stability. In addition, we also design a delayed feedback controller to control the emergence of Hopf bifurcation and further study the effect of feedback gain on the bifurcation control of the proposed system.

This paper is organized as follows. In Sect. 2, we introduce the relevant preliminary knowledge of fractional calculus and fractional-order dynamical system. In Sect. 3, we analyze system (1.3) to get the conditions of the occurrence of Hopf bifurcation and the value range of delay in which Hopf bifurcation appears. We also verify the reasonability of the linearized approximation by the equivalence of stability of equilibrium points between the original system (1.3) and its linearized system. In Sect. 4, the delayed feedback controller is added to system (1.3) to control the Hopf bifurcation. In Sect. 5, numerical simulations are performed to verify the validity of the theoretical results by choosing appropriate values of the constants a, b, c, d, τ. Finally, necessary conclusions and a discussion are presented in Sect. 6.

2 Preliminaries

In this section, some preliminary knowledge of fractional calculus and fractional-order dynamical system are introduced. In fact, the concept of fractional derivative has many classical definitions. This paper is based on the most widely used definition of Caputo fractional derivative.

Definition 1 ([1, 33]) The fractional-order integral of order $\alpha > 0$ of a real-valued function $x(t)$ is defined as

$${}_{t_0}^{C}D_t^{-\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_t^{t_0}(t-\tau)^{\alpha-1}x(\tau)\,d\tau, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Gamma function, $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,ds$.

Definition 2 ([1, 33]) The Caputo fractional derivative can be written as

$${}_{t_0}^{C}D_t^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}\frac{x^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}\,d\tau, \quad m-1 < \alpha \leq m, \tag{2.2}$$

where $x(t) \in C^n([t_0,\infty),\mathbb{R})$. In particular, if $0 < \alpha \leq 1$, (2.2) can be written as

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\frac{x'(\tau)}{(t-\tau)^{\alpha}}\,d\tau, \quad 0<\alpha\leq 1, t>0. \tag{2.3}$$

For brevity, in what follows, we use the notation $D^{\alpha}x(t)$ to denote the Caputo fractional-order derivative operator ${}_{t_0}^{C}D_t^{\alpha}x(t)$.

Definition 3 ([34]) The Laplace transform of Caputo fractional derivative of order α ($n-1<\alpha\leq n$) for a function $x(t)\in C^n([a,\infty),\mathbb{R})$ is

$$\mathcal{L}\{\mathcal{D}^{\alpha}x(t);s\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1}s^{\alpha-k-1}x^{(k)}(a), \tag{2.4}$$

where $F(s)$ is the Laplace transform of $x(t)$, and $x^{(k)}(a)$ ($k=0,1,\dots,n-1$) are the initial conditions. Obviously, if $x^{(k)}(a)=0$ for $k=0,1,\dots,n-1$, (2.4) can be written as

$$\mathcal{L}\{\mathcal{D}_t^{q}x(t);s\} = s^{q}F(s). \tag{2.5}$$

Definition 4 ([35]) Consider the following n-dimensional fractional-order system with time delay:

$$D^{\alpha}x_i(t) = f_i\bigl(x_1(t),\dots,x_n(t);\tau\bigr), \quad i=1,2,\dots,n, \tag{2.6}$$

where $0<\alpha\leq 1$ and the time delay $\tau\geq 0$. System (2.6) undergoes a Hopf bifurcation at the equilibrium $x^*=(x_1^*,x_2^*,\dots,x_n^*)$ when $\tau=\tau_0$ if the following three conditions are satisfied:

(C1) When $\tau=0$, all the eigenvalues λ_j ($j=1,2,\dots,n$) of the coefficient matrix J of the linearized system of (2.6) satisfy $|\arg(\lambda_j)|>\frac{\alpha\pi}{2}$.

(C2) The characteristic equation of the linearized system of (2.6) has a pair of purely imaginary roots $\pm\omega_0$ when $\tau=\tau_0$.

(C3) $\mathrm{Re}[\frac{ds(\tau)}{d\tau}]|_{\tau=\tau_0,\omega=\omega_0}>0$, where $\mathrm{Re}[\cdot]$ denotes the real part of the complex number and s refers to the eigenvalue of the associated characteristic equation of the linearized system.

3 Reasonability of linearized approximation and Hopf bifurcation for fractional-order delay BG system

In this paper, we only consider the nonzero real equilibrium points of the fractional-order delay BG system (1.3). Hence, we need to assume that

(H_1) $d(b-a)>0$.

It is obvious that system (1.3) has two nonzero real equilibrium points:

$$\bigl(b-a,\sqrt{d(b-a)},\sqrt{d(b-a)}\bigr), \qquad \bigl(b-a,-\sqrt{d(b-a)},-\sqrt{d(b-a)}\bigr).$$

We use the delay τ as a bifurcation parameter to find the conditions on the occurrence of Hopf bifurcation at the equilibria of system (1.3).

For brevity, the nonzero equilibrium point is denoted as (x^*, y^*, z^*). Using the transformations $u(t) = x(t) - x^*$, $v(t) = y(t) - y^*$, $w(t) = z(t) - z^*$, system (1.3) can be reduced to

$$\begin{cases} D^{q_1}u(t) = d(u(t) + x^*) - (v(t) + y^*)^2, \\ D^{q_2}v(t) = c(w(t) + z^* - v(t) - y^*), \\ D^{q_3}w(t) = a(v(t) + y^*) - b(w(t-\tau) + z^*) + (u(t) + x^*)(v(t) + y^*), \end{cases} \tag{3.1}$$

that is,

$$\begin{cases} D^{q_1}u(t) = du(t) - 2y^*v(t) - v(t)^2, \\ D^{q_2}v(t) = -cv(t) + cw(t), \\ D^{q_3}w(t) = y^*u(t) + (a + x^*)v(t) - bw(t-\tau) + u(t)v(t). \end{cases} \tag{3.2}$$

System (3.2) has two equilibria

$$\left(u^*, v^*, w^*\right) = (0,0,0), \left(-\frac{y^{*2}}{d}, -y^*, -y^*\right).$$

The linearized system of (3.2) at the origin is

$$\begin{cases} D^{q_1}\bar{u}(t) = d\bar{u}(t) - 2y^*\bar{v}(t), \\ D^{q_2}\bar{v}(t) = -c\bar{v}(t) + c\bar{w}(t), \\ D^{q_3}\bar{w}(t) = y^*\bar{u}(t) + (a + x^*)\bar{v}(t) - b\bar{w}(t-\tau). \end{cases} \tag{3.3}$$

3.1 Analysis of reasonability of linearized approximation

Since the stability change of an equilibrium involves the appearance of Hopf bifurcation, we need to verify the reasonability of the above linearized approximation by the equivalence of stability of equilibrium for systems (3.2) and (3.3).

Following a similar idea as in [36], we prove the equivalence of stability of equilibrium for systems (3.2) and (3.3) in the sense that

$$\lim_{t\to+\infty} \bar{u}(t) = 0, \qquad \lim_{t\to+\infty} \bar{v}(t) = 0, \qquad \lim_{t\to+\infty} \bar{w}(t) = 0$$

is equivalent to

$$\lim_{t\to+\infty} u(t) = u^*, \qquad \lim_{t\to+\infty} v(t) = v^*, \qquad \lim_{t\to+\infty} w(t) = w^*,$$

where the initial values are taken as $u(t) = \bar{u}(t) = \rho(t) > 0$, $v(t) = \bar{v}(t) = \phi(t) > 0$ and $w(t) = \bar{w}(t) = \psi(t) > 0$ $(t \in [-\tau, 0])$.

Set $e_1(t) = u(t) - \bar{u}(t)$, $e_2(t) = v(t) - \bar{v}(t)$, $e_3(t) = w(t) - \bar{w}(t)$. By (3.2) and (3.3), we obtain the error system

$$\begin{cases} D^{q_1}e_1(t) = de_1(t) - 2y^*e_2(t) - (e_2(t) + \bar{v}(t))^2, \\ D^{q_2}e_2(t) = -ce_2(t) + ce_3(t), \\ D^{q_3}e_3(t) = y^*e_1(t) + (a + x^*)e_2(t) - be_3(t-\tau) + (e_1(t) + \bar{u}(t))(e_2(t) + \bar{v}(t)) \end{cases} \tag{3.4}$$

and

$$\begin{cases} D^{q_1} e_1(t) = de_1(t) - 2y^* e_2(t) - v(t)^2, \\ D^{q_2} e_2(t) = -ce_2(t) + ce_3(t), \\ D^{q_3} e_3(t) = y^* e_1(t) + (a + x^*) e_2(t) - be_3(t - \tau) + u(t)v(t). \end{cases} \tag{3.5}$$

We have two basic assertions.

Assertion (a) If the solutions $\bar{u}(t)$, $\bar{v}(t)$, $\bar{w}(t)$ of system (3.3) satisfy

$$\lim_{t\to+\infty} \bar{u}(t) = 0, \qquad \lim_{t\to+\infty} \bar{v}(t) = 0, \qquad \lim_{t\to+\infty} \bar{w}(t) = 0,$$

then the solutions $u(t)$, $v(t)$, $w(t)$ of system (3.2) satisfy

$$\lim_{t\to+\infty} u(t) = u^*, \qquad \lim_{t\to+\infty} v(t) = v^*, \qquad \lim_{t\to+\infty} w(t) = w^*.$$

Taking the Laplace transform [34, 37] of both sides of the error system (3.4) gives

$$\begin{cases} s^{q_1} F_1(s) = dF_1(s) - 2y^* F_2(s) - \mathscr{L}[(e_2(t) + \bar{v}(t))^2], \\ s^{q_2} F_2(s) = -cF_2(s) + cF_3(s), \\ s^{q_3} F_3(s) = y^* F_1(s) + (a + x^*) F_2(s) - be^{-s\tau} F_3(s) + \mathscr{L}[(e_1(t) + \bar{u}(t))(e_2(t) + \bar{v}(t))], \end{cases} \tag{3.6}$$

where $F_k(s) = \mathscr{L}[e_k(t)]$ $(k = 1, 2, 3)$, $\mathscr{L}[\cdot]$ is the Laplace transform operator.
By (3.6), one gets

$$\begin{cases} s^{q_1} sF_1(s) = dsF_1(s) - 2y^* sF_2(s) - s\mathscr{L}[(e_2(t) + \bar{v}(t))^2], \\ s^{q_2} sF_2(s) = -csF_2(s) + csF_3(s), \\ s^{q_3} sF_3(s) = y^* sF_1(s) + (a + x^*) sF_2(s) - be^{-s\tau} sF_3(s) \\ \qquad\qquad + s\mathscr{L}[(e_1(t) + \bar{u}(t))(e_2(t) + \bar{v}(t))]. \end{cases} \tag{3.7}$$

Similar to the prior assumption made in theoretical analysis of [38], we make the following prior assumption: $e_i(t)$ $(i = 1, 2, 3)$ are bounded. Then by the final-value theorem of the Laplace transformation [37] and (3.7), we have

$$\begin{cases} de_1^* - 2y^* e_2^* - e_2^{*2} = 0, \\ e_2^* = e_3^*, \\ y^* e_1^* + (a + x^*) e_2^* - be_3^* + e_1^* e_2^* = 0, \end{cases} \tag{3.8}$$

where $e_i^* := \lim_{t\to+\infty} e_i(t)$ $(i = 1, 2, 3)$.
By (3.8), we obtain

$$(e_1^*, e_2^*, e_3^*) = (0, 0, 0) \quad \text{or} \quad \left(-\frac{y^{*2}}{d}, -y^*, -y^*\right), \tag{3.9}$$

which implies

$$\lim_{t\to+\infty} u(t) = u^*, \qquad \lim_{t\to+\infty} v(t) = v^*, \qquad \lim_{t\to+\infty} w(t) = w^*.$$

On the other hand, by taking the Laplace transform [34, 37] of both sides of the error system (3.5), we have

$$\begin{cases} s^{q_1}F_1(s) = dF_1(s) - 2y^*F_2(s) - \mathscr{L}[v(t)^2], \\ s^{q_2}F_2(s) = -cF_2(s) + cF_3(s), \\ s^{q_3}F_3(s) = y^*F_1(s) + (a + x^*)F_2(s) - be^{-s\tau}F_3(s) + \mathscr{L}[u(t)v(t)]. \end{cases} \tag{3.10}$$

Similarly, by (3.10), we can also prove that $(e_1^*, e_2^*, e_3^*) = (u^*, v^*, w^*)$. Hence, we have the following result.

Assertion (b) If the solutions $u(t)$, $v(t)$, $w(t)$ of system (3.2) satisfy

$$\lim_{t\to+\infty} u(t) = u^*, \qquad \lim_{t\to+\infty} v(t) = v^*, \qquad \lim_{t\to+\infty} w(t) = w^*,$$

then the solutions $\bar{u}(t)$, $\bar{v}(t)$, $\bar{w}(t)$ of system (3.3) satisfy

$$\lim_{t\to+\infty} \bar{u}(t) = 0, \qquad \lim_{t\to+\infty} \bar{v}(t) = 0, \qquad \lim_{t\to+\infty} \bar{w}(t) = 0.$$

Thus, by Assertions (a) and (b), we verified the reasonability of the above linearized approximation from the viewpoint of stability of equilibrium.

3.2 Hopf bifurcation analysis

The linearized system of (3.2) at the origin can be expressed as

$$\begin{cases} D^{q_1}u(t) = c_{11}u(t) + c_{12}v(t), \\ D^{q_2}v(t) = c_{22}v(t) + c_{23}w(t), \\ D^{q_3}w(t) = c_{31}u(t) + c_{32}v(t) + c_{33}w(t-\tau), \end{cases} \tag{3.11}$$

where $c_{11} = d$, $c_{12} = -2y^*$, $c_{22} = -c$, $c_{23} = c$, $c_{31} = y^*$, $c_{32} = a + x^*$, $c_{33} = -b$.

It is easy to obtain the associated characteristic equation by using Laplace transform on system (3.11):

$$\begin{vmatrix} s^{q_1} - c_{11} & -c_{12} & 0 \\ 0 & s^{q_2} - c_{22} & -c_{23} \\ -c_{31} & -c_{32} & s^{q_3} - c_{33}e^{-s\tau} \end{vmatrix} = 0. \tag{3.12}$$

Equation (3.12) can be equivalently rewritten as

$$E_1(s) + E_2(s)e^{-s\tau} = 0, \tag{3.13}$$

where

$$\begin{aligned} E_1(s) &= -s^{q_1}c_{23}c_{32} + s^{q_3}c_{11}c_{22} + c_{11}c_{23}c_{32} - c_{31}c_{12}c_{23} - s^{q_1+q_3}c_{22} - s^{q_2+q_3}c_{11} \\ &\quad + s^{q_1+q_2+q_3}, \\ E_2(s) &= -c_{33}\bigl(-s^{q_1}c_{22} - s^{q_2}c_{11} + c_{11}c_{22} + s^{q_1+q_2}\bigr). \end{aligned}$$

Assume that $s = i\omega = \omega(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ is a root of Eq. (3.13), $\omega > 0$. Substituting $s = i\omega$ into Eq. (3.13) and separating the real and imaginary parts, then it results in

$$\begin{cases} \alpha_2\cos\omega\tau + \alpha_4\sin\omega\tau = -\alpha_1, \\ \alpha_4\cos\omega\tau - \alpha_2\sin\omega\tau = -\alpha_3, \end{cases} \tag{3.14}$$

where α_i $(i = 1,2,3,4)$ are defined in Appendix A.

Solving (3.14), one obtains

$$\begin{cases} \cos\omega\tau = -\frac{\alpha_1\alpha_2+\alpha_3\alpha_4}{\alpha_2^2+\alpha_4^2} = P_1(\omega), \\ \sin\omega\tau = \frac{\alpha_2\alpha_3-\alpha_1\alpha_4}{\alpha_2^2+\alpha_4^2} = P_2(\omega). \end{cases} \tag{3.15}$$

With the formula $P_1^2(\omega) + P_2^2(\omega) = 1$, we can calculate ω easily. We might as well suppose that ω_i $(i = 1,2,\dots,n)$ are positive solutions. There are four cases of τ_i as follows:

I. When $P_1(\omega_i) > 0$, $P_2(\omega_i) > 0$, and $k = 0,1,2,\dots$,

$$\tau_i^{(k)} = \frac{\arccos P_1(\omega_i) + 2k\pi}{\omega_i} = \frac{\arcsin P_2(\omega_i) + 2k\pi}{\omega_i}.$$

II. When $P_1(\omega_i) < 0$, $P_2(\omega_i) > 0$, and $k = 0,1,2,\dots$,

$$\tau_i^{(k)} = \frac{\arccos P_1(\omega_i) + 2k\pi}{\omega_i} = \frac{\pi - \arcsin P_2(\omega_i) + 2k\pi}{\omega_i}.$$

III. When $P_1(\omega_i) > 0$, $P_2(\omega_i) < 0$, and $k = 0,1,2,\dots$,

$$\tau_i^{(k)} = \frac{2\pi - \arccos P_1(\omega_i) + 2k\pi}{\omega_i} = \frac{2\pi + \arcsin P_2(\omega_i) + 2k\pi}{\omega_i}.$$

IV. When $P_1(\omega_i) < 0$, $P_2(\omega_i) < 0$, and $k = 0,1,2,\dots$,

$$\tau_i^{(k)} = \frac{2\pi - \arccos P_1(\omega_i) + 2k\pi}{\omega_i} = \frac{\pi - \arcsin P_2(\omega_i) + 2k\pi}{\omega_i}.$$

According to the actual meaning of time delay τ, we are only interested in the first positive real value of τ. Define the bifurcation point as follows:

$$\tau_0 = \min\{\tau_i^{(k)}\}, \qquad \omega_0 = \omega_i, \quad i = 1,2,\dots,n, k = 0,1,2,\dots, \tag{3.16}$$

where $\tau_i^{(k)}$ is defined in cases I–IV and ω_i corresponds to $\min\{\tau_i^{(k)}\}$.

In order to find the bifurcation point, we need to have an in-depth study of Eq. (3.13). Differentiating both sides of Eq. (3.13) with respect to τ, one gets

$$E_1'(s)\frac{ds}{d\tau} + E_2'(s)e^{-s\tau}\frac{ds}{d\tau} + E_2(s)e^{-s\tau}\left(-\tau\frac{ds}{d\tau} - s\right) = 0,$$

where $E_i'(s)$ are the derivatives of $E_i(s)$ $(i = 1,2)$. Hence,

$$\frac{ds}{d\tau} = \frac{A(s)}{B(s)}, \tag{3.17}$$

where

$$
\begin{aligned}
A(s) &= -c_{33}\bigl(-s^{q_1}c_{22} - s^{q_2}c_{11} + c_{11}c_{22} + s^{q_1+q_2}\bigr)se^{-s\tau},\\
B(s) &= -s^{q_1-1}q_1c_{23}c_{32} + s^{q_3-1}q_3c_{11}c_{22} - s^{q_1+q_3-1}(q_1+q_3)c_{22} - s^{q_2+q_3-1}(q_2+q_3)c_{11}\\
&\quad + s^{q_1+q_2+q_3-1}(q_1+q_2+q_3) - c_{33}\bigl(-s^{q_1-1}q_1c_{22} - s^{q_2-1}q_2c_{11}\\
&\quad + s^{q_1+q_2-1}(q_1+q_2)\bigr)e^{-s\tau} + c_{33}\tau\bigl(-s^{q_1}c_{22} - s^{q_2}c_{11} + c_{11}c_{22} + s^{q_1+q_2}\bigr)e^{-s\tau}.
\end{aligned}
$$

Substituting $s = i\omega = \omega(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ into $A(s)$, $B(s)$, and letting A_1, A_2 and B_1, B_2 be the real and imaginary parts of $A(s)$, $B(s)$, respectively, it can be deduced from Eq. (3.17) that

$$
\operatorname{Re}\left[\frac{ds}{d\tau}\right] = \frac{A_1B_1 + A_2B_2}{B_1^2 + B_2^2}, \tag{3.18}
$$

where A_i, B_i $(i = 1, 2)$ are defined in Appendix B.

Basing on the aforementioned analysis, we get

Lemma 1 *Let $s(\tau) = \gamma(\tau) + i\omega(\tau)$ be the root of Eq.* (3.13) *near $\tau = \tau_i^{(k)}$ satisfying $\gamma(\tau_i^{(k)}) = 0$, $\omega(\tau_i^{(k)}) = \omega_i$, then the transversality condition*

$$
\operatorname{Re}\left[\frac{ds}{d\tau}\right]\bigg|_{(\tau=\tau_0,\omega=\omega_0)} > 0 \tag{3.19}
$$

holds if the following assumption is satisfied:

(H_2) $\frac{A_1B_1+A_2B_2}{B_1^2+B_2^2} > 0$,

where A_i, B_i $(i = 1, 2)$ are defined in Appendix B.

Next, to verify Assumption $(C1)$ in Definition 4, we need the following lemma.

Lemma 2 *If the following assumptions hold*:

(H_3) $c_{11} + c_{22} + c_{33} < 0$.

(H_4) $c_{11}^2c_{22} + c_{11}^2c_{33} + c_{11}c_{22}^2 + 2c_{11}c_{22}c_{33} + c_{11}c_{33}^2 - c_{12}c_{23}c_{31} + c_{22}^2c_{33} - c_{22}c_{23}c_{32} + c_{22}c_{33}^2 - c_{23}c_{32}c_{33} < 0$.

(H_5) $c_{11}c_{22}c_{33} - c_{11}c_{23}c_{32} + c_{31}c_{12}c_{23} < 0$,

then all the eigenvalues λ_j $(j = 1, 2, 3)$ of the coefficient matrix J of the linearized system (3.11) *of system* (3.2) *with $\tau = 0$ satisfy $|\arg(\lambda_j)| > \frac{q_i\pi}{2}$ $(i, j = 1, 2, 3)$.*

Proof Neglecting the time delay, i.e., $\tau = 0$, the characteristic equation of coeffitient matrix J of the linearized system (3.11) becomes

$$
\begin{vmatrix}
\lambda - c_{11} & -c_{12} & 0\\
0 & \lambda - c_{22} & -c_{23}\\
-c_{31} & -c_{32} & \lambda - c_{33}
\end{vmatrix} = 0, \tag{3.20}
$$

which is equivalent to

$$
\begin{aligned}
&\lambda^3 - (c_{11} + c_{22} + c_{33})\lambda^2 + (c_{11}c_{22} + c_{11}c_{33} + c_{22}c_{33} - c_{32}c_{23})\lambda\\
&\quad - c_{11}c_{22}c_{33} + c_{11}c_{23}c_{32} - c_{31}c_{12}c_{23} = 0.
\end{aligned}
$$

If Assumptions (H_3)–$(H5)$ are satisfied, it is easy to check from Routh–Hurwitz criterion that three eigenvalues λ_j $(j = 1,2,3)$ of Eq. (3.20) have negative real parts. Therefore, $|\arg(\lambda_j)| > \frac{q_i\pi}{2}$ $(i,j = 1,2,3)$. □

Remark 1 It is apparent that the derived conditions in Lemma 2 are only sufficient conditions. According to Definition 4, if conditions (H_3)–(H_5) are replaced by other conditions which can guarantee that all the roots of Eq. (3.20) satisfy $|\arg(\lambda_j)| > \frac{q_i\pi}{2}$ $(i,j = 1,2,3)$, then Lemma 2 may still hold.

Basing on Definition 4, we achieve the first primary theorem of this paper.

Theorem 1 *Suppose* (H_1)–(H_5) *hold, when* $0 < q_i \leq 1$ $(i = 1,2,3)$ *and the time delay* $\tau \geq 0$. *The fractional-order delay system* (1.3) *undergoes a Hopf bifurcation at the nonzero equilibrium point* (x^*, y^*, z^*) *when* $\tau = \tau_0$, *where* τ_0 *is defined by formula* (3.16).

4 Delayed feedback control of fraction-order delay BG systems

In this section, a delayed feedback controller $k[y(t) - y(t - \tau)]$ is added to the second equation of uncontrolled system (1.3), and then the delay feedback control system can be acquired as

$$\begin{cases} D^{q_1}x(t) = dx(t) - y^2(t), \\ D^{q_2}y(t) = c(z(t) - y(t)) + k[y(t) - y(t - \tau)], \\ D^{q_3}z(t) = ay(t) - bz(t - \tau) + x(t)y(t). \end{cases} \tag{4.1}$$

For the sake of revealing the relationship between the controller and Hopf bifurcation, we still use the delay τ as a parameter in Eq. (4.1). Analogous to the previous analysis, by performing transformations $u(t) = x(t) - x^*$, $v(t) = y(t) - y^*$, $w(t) = z(t) - z^*$, with the help of the linearized scheme, the linearization of the controlled system (4.1) has the form:

$$\begin{cases} D^{q_1}u(t) = c_{11}u(t) + c_{12}v(t), \\ D^{q_2}v(t) = c_{22}v(t) + c_{23}w(t) + k[v(t) - v(t - \tau)], \\ D^{q_3}w(t) = c_{31}u(t) + c_{32}v(t) + c_{33}w(t - \tau), \end{cases} \tag{4.2}$$

where c_{11}, c_{12}, c_{22}, c_{23}, c_{31}, c_{32}, and c_{33} are defined as system (3.11).

Therefore, the associated characteristic equation of system (4.2) is

$$\begin{vmatrix} s^{q_1} - c_{11} & -c_{12} & 0 \\ 0 & s^{q_2} - c_{22} - k + ke^{-s\tau} & -c_{23} \\ -c_{31} & -c_{32} & s^{q_3} - c_{33}e^{-s\tau} \end{vmatrix} = 0. \tag{4.3}$$

Obviously, Eq. (4.3) is equivalent to

$$F_1(s) + F_2(s)e^{-s\tau} + F_3(s)e^{-2s\tau} = 0, \tag{4.4}$$

where

$$F_1(s) = ks^{q_3}c_{11} - s^{q_1}c_{23}c_{32} + s^{q_3}c_{11}c_{22} + c_{11}c_{23}c_{32} - c_{31}c_{12}c_{23} - ks^{q_1+q_3} - s^{q_2+q_3}c_{11} - s^{q_1+q_3}c_{22} + s^{q_1+q_2+q_3},$$

$$F_2(s) = ks^{q_1+q_3} + ks^{q_1}c_{33} - ks^{q_3}c_{11} - kc_{11}c_{33} - s^{q_1+q_2}c_{33} + s^{q_1}c_{22}c_{33} + s^{q_2}c_{11}c_{33} - c_{11}c_{22}c_{33},$$

$$F_3(s) = -kc_{33}\left(s^{q_1} - c_{11}\right).$$

By multiplying $e^{s\tau}$ on both sides of Eq. (4.4), it is obvious that

$$F_1(s)e^{s\tau} + F_2(s) + F_3(s)e^{-s\tau} = 0. \tag{4.5}$$

Assume that $s = i\omega = \omega(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ is a root of Eq. (4.5), $\omega > 0$. Substituting $s = i\omega$ into Eq. (4.5) and separating the real and imaginary parts, we have

$$\begin{cases} \beta_1\cos\omega\tau + \beta_3\sin\omega\tau = -\beta_5, \\ \beta_2\cos\omega\tau + \beta_4\sin\omega\tau = -\beta_6, \end{cases} \tag{4.6}$$

where β_i $(i = 1, 2, \dots, 6)$ are defined in Appendix C.

From Eq. (4.6), we have

$$\begin{cases} \cos\omega\tau = -\frac{\beta_6\beta_3-\beta_5\beta_4}{\beta_4\beta_1-\beta_3\beta_2} = Q_1(\omega), \\ \sin\omega\tau = \frac{\beta_5\beta_2-\beta_6\beta_1}{\beta_4\beta_1-\beta_3\beta_2} = Q_2(\omega). \end{cases} \tag{4.7}$$

Consistent with the previous section, with the formula $Q_1^2(\omega) + Q_2^2(\omega) = 1$, we can calculate ω easily. We might as well suppose that ω_i $(i = 1, 2, \dots, n)$ are positive solutions, and can get the same four cases of $\tau_i^{(k)}$ $(k = 0, 1, 2, \dots)$ as in Sect. 3.

Next we define the bifurcation point

$$\tau_0^* = \min\left\{\tau_i^{(k)}\right\}, \qquad \omega_0^* = \omega_i, \quad i = 1, 2, \dots, n, k = 0, 1, 2, \dots, \tag{4.8}$$

where ω_i corresponds to $\min\{\tau_i^{(k)}\}$. It needs to be noticed that the calculation of $\tau_i^{(k)}$ and ω_i is relying on the feedback grain coefficient k (see Appendix C).

Differentiating both sides of Eq. (4.5) with respect to τ, one obtains

$$F_1'(s)\frac{ds}{d\tau} + F_2'(s)e^{-s\tau}\frac{ds}{d\tau} + F_2(s)e^{-s\tau}\left(-\tau\frac{ds}{d\tau} - s\right) + F_3'(s)e^{-2s\tau}\frac{ds}{d\tau} + F_3(s)e^{-2s\tau}\left(-2\tau\frac{ds}{d\tau} - 2s\right) = 0,$$

where $F_i'(s)$ are the derivatives of $F_i(s)$ $(i = 1, 2, 3)$.

Therefore,

$$\frac{ds}{d\tau} = \frac{C(s)}{D(s)}, \tag{4.9}$$

where

$$\begin{aligned} C(s) &= s\left[F_2(s)e^{-s\tau} + 2F_3(s)e^{-2s\tau}\right], \\ D(s) &= F_1'(s) + \left[F_2'(s) - \tau F_2(s)\right]e^{-s\tau} + \left[F_3'(s) - 2\tau F_3(s)\right]e^{-2s\tau}. \end{aligned} \tag{4.10}$$

It can be deduced from Eq. (4.9) that

$$\operatorname{Re}\left[\frac{ds}{d\tau}\right] = \frac{C_1D_1 + C_2D_2}{D_1^2 + D_2^2}, \tag{4.11}$$

where C_1, C_2, D_1, D_2 are the real and imaginary parts of $C(s)$ and $D(s)$, respectively, and the exact expressions are given in Appendix D.

Thus, we obtain the following lemma:

Lemma 3 *Let $s(\tau) = \delta(\tau) + i\omega(\tau)$ be the root of Eq.* (4.5) *near $\tau = \tau_i^{(k)}$ satisfying $\delta(\tau_i^{(k)}) = 0$, $\omega(\tau_i^{(k)}) = \omega_i^*$. Then the transversality condition*

$$\operatorname{Re}\left[\frac{ds}{d\tau}\right]\bigg|_{(\tau=\tau_0^*,\omega=\omega_0^*)} > 0 \tag{4.12}$$

holds if the following assumption is satisfied:

(H_6) $\frac{C_1D_1+C_2D_2}{D_1^2+D_2^2} > 0$,

where C_i, D_i $(i = 1, 2)$ are defined as in (4.11).

Based on the previous discussion, the following theorem can be concluded.

Theorem 2 *Suppose* (H_1), (H_3)–(H_6) *hold, when* $0 < q_i \le 1$ $(i = 1, 2, 3)$ *and the delay* $\tau \ge 0$. *The delay feedback control system* (4.1) *undergoes a Hopf bifurcation at the nonzero equilibrium* (x^*, y^*, z^*) *when* $\tau = \tau_0^*$, *where* τ_0^* *is defined as in* (4.8).

Remark 2 In this section, we use the same method as in Sect. 3 to discuss the delay feedback control system (4.1). In reality, the Hopf bifurcation points (τ_0^*, ω_0^*) of system (4.1) can be controlled successfully by changing the feedback gain coefficient k. We will illustrate this fact in the next section by numerical simulations.

5 Numerical simulations

Adams–Bashforth–Moulton predictor–corrector scheme [39] has been widely used in numerical simulation for fractional-order differential equation. In this section, this method is adopted in two examples to verify the efficiency and feasibility of our theoretical results, in which step-length is taken as $h = 0.001$.

5.1 Example 1

For the convenience of comparison, all the system parameters come from the literature [31]: $a = 22$, $b = 10$, $c = 10$, $d = -2.667$. Without loss of generality, let $q_1 = 0.91$, $q_2 = 0.98$, $q_3 = 0.95$, then system (1.3) can be changed into

$$\begin{cases} D^{q_1}x(t) = -2.667x(t) - y^2(t), \\ D^{q_2}y(t) = 10(z(t) - y(t)), \\ D^{q_3}z(t) = 22y(t) - 10z(t-\tau) + x(t)y(t). \end{cases} \tag{5.1}$$

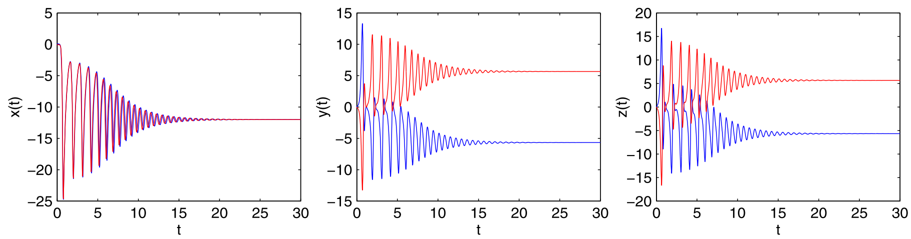


Figure 1 Waveform plots of system (5.1) with initial values (0.1, 0.1, 0.1) and (–0.1, –0.1, –0.1), $q_1 = 0.91$, $q_2 = 0.98$, $q_3 = 0.95$, and $\tau = 0.0949 < \tau_0 = 0.09517856$. The nonzero equilibrium points of system (5.1) are asymptotically stable

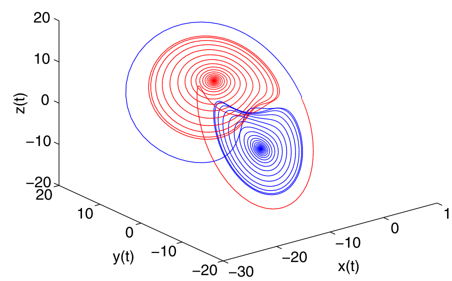


Figure 2 Portraits of system (5.1) with initial values (0.1, 0.1, 0.1) and (–0.1, –0.1, –0.1), $q_1 = 0.91$, $q_2 = 0.98$, $q_3 = 0.95$, and $\tau = 0.0949 < \tau_0 = 0.09517856$. Two nonzero equilibrium points of system (5.1) are asymptotically stable

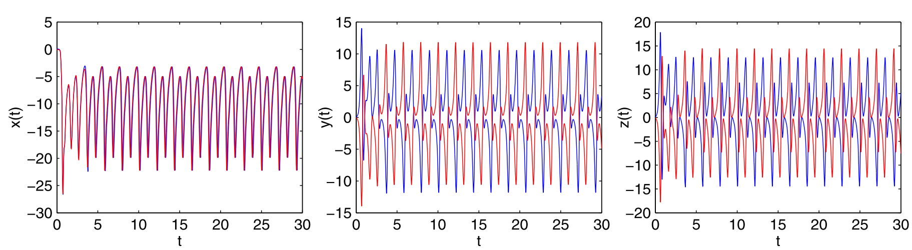


Figure 3 Waveform plots of system (5.1) with initial values (0.1, 0.1, 0.1) and (–0.1, –0.1, –0.1), $q_1 = 0.91$, $q_2 = 0.98$, $q_3 = 0.95$, and $\tau = 0.0952 > \tau_0 = 0.09517856$. The nonzero equilibrium points of system (5.1) are unstable

For system (5.1), it is easy to verify that (H_1)–(H_5) are all satisfied. The nonzero equilibrium points are $(-12, 5.657, 5.657)$ and $(-12, -5.657, -5.657)$. One can obtain the critical frequency $\omega_0 = 9.35083452$ and bifurcation point $\tau_0 = 0.09517856$. By Theorem 1, Hopf bifurcation of system (5.1) appears at τ_0. To better present our results, we give two simulations. One uses $\tau = 0.0949 < \tau_0 = 0.09517856$, which is displayed in Figs. 1 and 2. Under this condition, we can see that two nonzero equilibria are asymptotically stable. The other uses $\tau = 0.0952 > \tau_0 = 0.09517856$, which is displayed in Figs. 3 and 4. It is apparent that two nonzero equilibria are unstable and Hopf bifurcation occurs. Therefore, Theorem 1 is verified by these simulations.

5.2 Example 2

In this example, a linear delayed feedback controller is added to the uncontrolled system (5.1) so as to control the Hopf bifurcation. In order to illustrate the effects of bifurcation control via the proposed controller preferably, three fractional orders and all the system parameters are chosen the same as in Example 1, then the controlled system is shown as

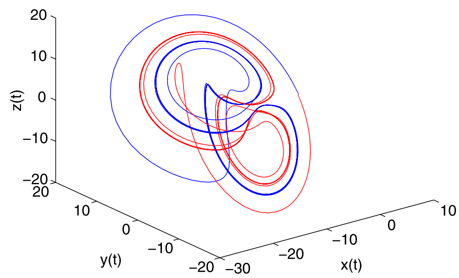


Figure 4 Portraits of system (5.1) with initial values $(0.1, 0.1, 0.1)$ and $(-0.1, -0.1, -0.1)$, $q_1 = 0.91$, $q_2 = 0.98$, $q_3 = 0.95$, and $\tau = 0.0952 > \tau_0 = 0.09517856$. Hopf bifurcation occurs

Table 1 The impact of k on the values of ω_0^* and τ_0^* for the controlled system (5.2) with $q_1 = 0.91$, $q_2 = 0.98$, $q_3 = 0.95$

Feedback gain k	Critical frequency ω_0^*	Bifurcation point τ_0^*
−3	9.653358584	0.108321963
−2.5	9.595547511	0.106433652
−2	9.539410105	0.104408781
−1.5	9.485911614	0.102254645
−1	9.436041308	0.099984585
−0.5	9.390741645	0.097617839
0	9.350834520	0.095178561
0.5	9.316958928	0.092694075
1	9.289532626	0.090192707
1.5	9.268743775	0.087701612
2	9.254570242	0.085245013
2.5	9.246818131	0.082843069
3	9.245168793	0.080511419

follows:

$$\begin{cases} D^{q_1}x(t) = -2.667x(t) - y^2(t), \\ D^{q_2}y(t) = 10(z(t) - y(t)) + k[y(t) - y(t-\tau)], \\ D^{q_3}z(t) = 22y(t) - 10z(t-\tau) + x(t)y(t). \end{cases} \tag{5.2}$$

For exhibiting the impact of feedback gain coefficient k on the Hopf bifurcation for the controlled system (5.2), we calculate a group of critical frequency ω_0^* and bifurcation points τ_0^* corresponding to ever-increasing k, see Table 1. According to Table 1, system (5.2) is controlled by the delayed feedback controller $k[y(t) - y(t-\tau)]$ effectively. When k increases from negative to positive, τ_0^* decreases gradually. This means that the stable domain wanes and the emergence of Hopf bifurcation is advanced. Moreover, by choosing three different k, Figs. 5–7 illustrate that the effect of Hopf bifurcation control is much better as k decreases.

6 Conclusions and discussion

In this paper, sufficient conditions on the emergence of Hopf bifurcation have been established for a fractional-order delay Bhalekar–Gejji chaotic system. The delay feedback control issue of Hopf bifurcations for such a system has been investigated by theoretical analysis and numerical simulation.

This paper mainly focused on three aspects: the reasonability of the linearized approximation, delay-induced Hopf bifurcation, and delay feedback control of Hopf bifurcation for a fractional-order delay Bhalekar–Gejji chaotic system. Comparing to the previous

Figure 5 Waveform plot of system (5.2) with initial values (0.1, 0.1, 0.1), $q_1 = 0.91$, $q_2 = 0.98$, $q_3 = 0.95$, and $\tau = 0.1$, the feedback gain $k = 0$, $k = -1.5$, $k = -3$. The effect of bifurcation control for the controlled system (5.2) becomes better as the feedback gain k decreases

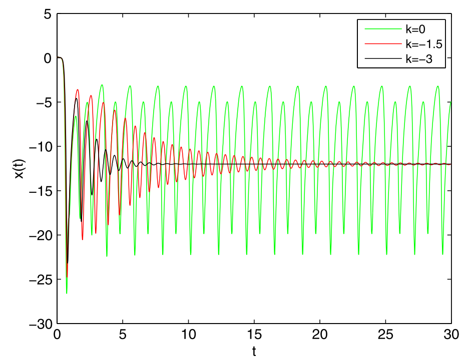


Figure 6 Waveform plot of system (5.2) with initial values (0.1, 0.1, 0.1), $q_1 = 0.91$, $q_2 = 0.98$, $q_3 = 0.95$, and $\tau = 0.1$, the feedback gain $k = 0$, $k = -1.5$, $k = -3$. The effect of bifurcation control for the controlled system (5.2) becomes better as the feedback gain k decreases

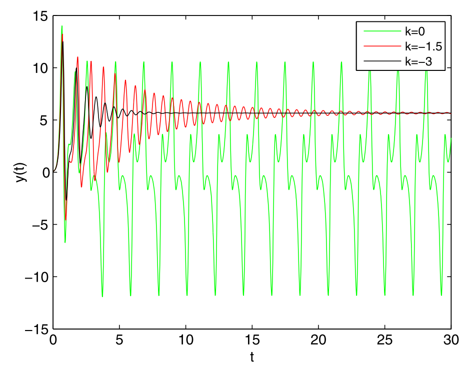
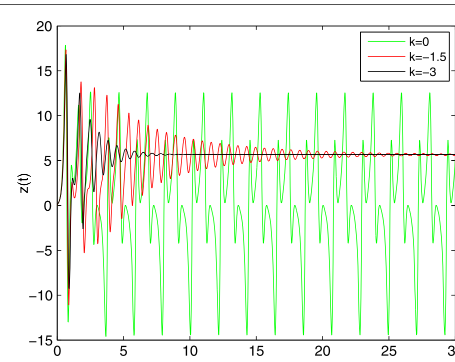


Figure 7 Waveform plot of system (5.2) with initial values (0.1, 0.1, 0.1), $q_1 = 0.91$, $q_2 = 0.98$, $q_3 = 0.95$, and $\tau = 0.1$, the feedback gain $k = 0$, $k = -1.5$, $k = -3$. The effect of bifurcation control for the controlled system (5.2) becomes better as the feedback gain k decreases



similar works, we semianalytically verified the reasonability of the linearized approximation by the equivalence of stable equilibrium for the converted systems (3.2) and (3.3) under an appropriate prior assumption. To some extent, this provides a theoretical support for the definition of Hopf bifurcation for fractional-order delay systems proposed by [35].

We find that the time delay has an important influence on the stability of the fractional-order delay Bhalekar–Gejji chaotic system. The delay can be used as the bifurcation parameter to derive the asymptotic stability interval of the system and in which conditions the system will exhibit dynamic behavior such as Hopf bifurcation. In addition, for the fractional-order delay Bhalekar–Gejji chaotic system, the feedback gain can control the bifurcation value and expand the stability range of the system.

In the simulations, we can also observe the stability of equilibrium points of the proposed system. In fact, it is difficult to analytically prove stability of equilibrium points of the original system (1.3), which is not only because of the complexity of the system, but also due to the lack of a well-developed stability theory of nonlinear fractional-order delay systems.

In the subsequent research, we would also like to explore the effect of the fractional order and time delay on the occurrence of chaos for the fractional-order delay Bhalekar–Gejji chaotic system.

Appendix A

The expressions of α_1, α_2, α_3, and α_4 in Eq. (3.14) are computed as follows:

$$\begin{aligned}
\alpha_1 ={}& -\omega^{q_1}\cos\left(\frac{q_1\pi}{2}\right)c_{23}c_{32} + \omega^{q_3}\cos\left(\frac{q_3\pi}{2}\right)c_{11}c_{22} + c_{11}c_{23}c_{32} - c_{31}c_{12}c_{23} \\
&- \omega^{q_1+q_3}\cos\left(\frac{(q_1+q_3)\pi}{2}\right)c_{22} - \omega^{q_2+q_3}\cos\left(\frac{(q_2+q_3)\pi}{2}\right)c_{11} \\
&+ \omega^{q_1+q_2+q_3}\cos\left(\frac{(q_1+q_2+q_3)\pi}{2}\right), \\
\alpha_2 ={}& \omega^{q_1}\cos\left(\frac{q_1\pi}{2}\right)c_{22}c_{33} + \omega^{q_2}\cos\left(\frac{q_2\pi}{2}\right)c_{11}c_{33} - \omega^{q_1+q_2}\cos\left(\frac{(q_3+q_2)\pi}{2}\right)c_{33} \\
&- c_{11}c_{22}c_{33}, \\
\alpha_3 ={}& \omega^{q_3}\sin\left(\frac{q_3\pi}{2}\right)c_{11}c_{22} - \omega^{q_1}\sin\left(\frac{q_1\pi}{2}\right)c_{23}c_{32} - \omega^{q_1+q_3}\sin\left(\frac{q_1\pi}{2}+\frac{q_3\pi}{2}\right)c_{22} \\
&- \omega^{q_2+q_3}\sin\left(\frac{q_2\pi}{2}+\frac{q_3\pi}{2}\right)c_{11} + \omega^{q_1+q_2+q_3}\sin\left(\frac{q_1\pi}{2}+\frac{q_2\pi}{2}+\frac{q_3\pi}{2}\right), \\
\alpha_4 ={}& \omega^{q_1}\sin\left(\frac{q_1\pi}{2}\right)c_{22}c_{33} + \omega^{q_2}\sin\left(\frac{q_2\pi}{2}\right)c_{11}c_{33} - \omega^{q_1+q_2}\sin\left(\frac{(q_1+q_2)\pi}{2}\right)c_{33}.
\end{aligned}$$

Appendix B

The expressions of A_1, A_2, B_1, and B_2 in Eq. (3.18) are given as:

$$\begin{aligned}
A_1 ={}& \omega_0 c_{33}\left(-\omega_0^{q_1}\sin\left(\frac{q_1\pi}{2}\right)c_{22} - \omega_0^{q_2}\sin\left(\frac{q_2\pi}{2}\right)c_{11} + \sin\left(\frac{(q_1+q_2)\pi}{2}\right)\omega_0^{q_1+q_2}\right) \\
&\times \cos(\tau_0\omega_0) + \omega_0 c_{33}\left(\omega_0^{q_1}\cos\left(\frac{q_1\pi}{2}\right)c_{22} + \omega_0^{q_2}\cos\left(\frac{q_2\pi}{2}\right)c_{11}\right. \\
&\left.- \cos\left(\frac{(q_1+q_2)\pi}{2}\right)\omega_0^{q_1+q_2} - c_{11}c_{22}\right)\sin(\tau_0\omega_0), \\
A_2 ={}& \omega_0 c_{33}\left(\omega_0^{q_1}\cos\left(\frac{q_1\pi}{2}\right)c_{22} + \omega_0^{q_2}\cos\left(\frac{q_2\pi}{2}\right)c_{11}\right. \\
&\left.- \cos\left(\frac{(q_1+q_2)\pi}{2}\right)\omega_0^{q_1+q_2} - c_{11}c_{22}\right) \\
&\times \cos(\tau_0\omega_0) + \omega_0 c_{33}\left(\omega_0^{q_1}\sin\left(\frac{q_1\pi}{2}\right)c_{22} + \omega_0^{q_2}\sin\left(\frac{q_2\pi}{2}\right)c_{11}\right. \\
&\left.- \sin\left(\frac{(q_1+q_2)\pi}{2}\right)\omega_0^{q_1+q_2}\right)\sin(\tau_0\omega_0),
\end{aligned}$$

$$
\begin{aligned}
B_1 ={}& \cos(\tau_0\omega_0)\frac{1}{\omega_0}\biggl(c_{11}c_{22}c_{33}\omega_0\tau_0 + \omega_0^{q_1+q_2+1}\cos\biggl(\frac{(q_1+q_2)\pi}{2}\biggr)c_{33}\tau_0 \\
& + \omega_0^{q_2}c_{11}c_{33}\biggl(\sin\biggl(\frac{q_2\pi}{2}\biggr)q_2 - \cos\biggl(\frac{q_2\pi}{2}\biggr)\tau_0\omega_0\biggr) \\
& + \omega_0^{q_1}c_{22}c_{33}\biggl(\sin\biggl(\frac{q_1\pi}{2}\biggr)q_1 - \cos\biggl(\frac{q_1\pi}{2}\biggr)\tau_0\omega_0\biggr) \\
& - \sin\biggl(\frac{(q_1+q_2)\pi}{2}\biggr)\omega_0^{q_1+q_2}c_{33}(q_1+q_2)\biggr) \\
& + \sin(\tau_0\omega_0)\frac{1}{\omega_0}\biggl(\omega_0^{q_2}c_{11}c_{33}\biggl(-\cos\biggl(\frac{q_2\pi}{2}\biggr)q_2 \\
& - \sin\biggl(\frac{q_2\pi}{2}\biggr)\omega_0\tau_0\biggr) - \omega_0^{q_1}c_{22}c_{33}\biggl(\cos\biggl(\frac{q_1\pi}{2}\biggr)q_1 + \sin\biggl(\frac{q_1\pi}{2}\biggr)\omega_0\tau_0\biggr) \\
& + \cos\biggl(\frac{(q_1+q_2)\pi}{2}\biggr)\omega_0^{q_1+q_2}c_{33}(q_1+q_2) + \omega_0^{q_1+q_2+1}\sin\biggl(\frac{(q_1+q_2)\pi}{2}\biggr)c_{33}\tau_0\biggr) \\
& + \frac{1}{\omega_0}\biggl(-\omega_0^{q_1}\sin\biggl(\frac{q_1\pi}{2}\biggr)q_1c_{23}c_{32} - \omega_0^{q_1+q_3}\sin\biggl(\frac{(q_1+q_3)\pi}{2}\biggr)c_{22}(q_1+q_3) \\
& + \omega_0^{q_1+q_2+q_3}\sin\biggl(\frac{(q_1+q_2+q_3)\pi}{2}\biggr)(q_1+q_2+q_3) \\
& - \omega_0^{q_2+q_3}\sin\biggl(\frac{(q_2+q_3)\pi}{2}\biggr)c_{11}(q_2+q_3) + \omega_0^{q_3}\sin\biggl(\frac{q_3\pi}{2}\biggr)q_3c_{11}c_{22}\biggr), \\
B_2 ={}& \cos(\tau_0\omega_0)\frac{1}{\omega_0}\biggl(-\omega_0^{q_2}c_{11}c_{33}\biggl(\cos\biggl(\frac{q_2\pi}{2}\biggr)q_2 + \sin\biggl(\frac{q_2\pi}{2}\biggr)\omega_0\tau_0\biggr) \\
& - \omega_0^{q_1}c_{22}c_{33}\biggl(\cos\biggl(\frac{q_1\pi}{2}\biggr)q_1 + \sin\biggl(\frac{q_1\pi}{2}\biggr)\omega_0\tau_0\biggr) \\
& + \omega_0^{q_1+q_2}c_{33}\cos\biggl(\frac{(q_1+q_2)\pi}{2}\biggr)(q_1+q_2) \\
& + \omega_0^{q_1+q_2+1}\sin\biggl(\frac{(q_1+q_2)\pi}{2}\biggr)c_{33}\tau_0\biggr) + \sin(\tau_0\omega_0)\frac{1}{\omega_0}\biggl(-c_{11}c_{22}c_{33}\omega_0\tau_0 \\
& - \omega_0^{q_1}c_{22}c_{33}\biggl(\sin\biggl(\frac{q_1\pi}{2}\biggr)q_1 - \cos\biggl(\frac{q_1\pi}{2}\biggr)\omega_0\tau_0\biggr) - \omega_0^{q_2}c_{11}c_{33}\biggl(\sin\biggl(\frac{q_2\pi}{2}\biggr)q_2 \\
& - \cos\biggl(\frac{q_2\pi}{2}\biggr)\omega_0\tau_0\biggr) + \sin\biggl(\frac{(q_1+q_2)\pi}{2}\biggr)\omega_0^{q_1+q_2}c_{33}(q_1+q_2) \\
& - \omega_0^{q_1+q_2+1}\cos\biggl(\frac{(q_1+q_2)\pi}{2}\biggr)c_{33}\tau_0\biggr) + \frac{1}{\omega_0}\biggl(\omega_0^{q_1}\cos\biggl(\frac{q_1\pi}{2}\biggr)q_1c_{23}c_{32} \\
& + \omega_0^{q_1+q_3}\cos\biggl(\frac{(q_1+q_3)\pi}{2}\biggr)c_{22}(q_1+q_3) + \omega_0^{q_2+q_3}\cos\biggl(\frac{(q_2+q_3)\pi}{2}\biggr)c_{11}(q_2+q_3) \\
& - \omega_0^{q_1+q_2+q_3}\cos\biggl(\frac{(q_1+q_2+q_3)\pi}{2}\biggr)(q_1+q_2+q_3) - \omega_0^{q_3}\cos\biggl(\frac{q_3\pi}{2}\biggr)q_3c_{11}c_{22}\biggr).
\end{aligned}
$$

Appendix C

The expressions of β_1, β_2, β_3, β_4, β_5, and β_6 in Eq. (4.6) are:

$$
\beta_1 = \omega^{q_1+q_2+q_3}\sin\biggl(\frac{q_1\pi}{2}+\frac{q_2\pi}{2}+\frac{q_3\pi}{2}\biggr) - k\omega^{q_1}\sin\biggl(\frac{q_1\pi}{2}\biggr)c_{33} - \omega^{q_1}\sin\biggl(\frac{q_1\pi}{2}\biggr)c_{23}c_{32}
$$

$$
\begin{aligned}
&+ \omega^{q_3} \sin\left(\frac{q_3\pi}{2}\right)c_{11} + \omega^{q_3} \sin\left(\frac{q_3\pi}{2}\right)c_{11}c_{22} - k\omega^{q_1+q_3} \sin\left(\frac{q_1\pi}{2} + \frac{q_3\pi}{2}\right) \\
&- \omega^{q_1+q_3} \sin\left(\frac{q_1\pi}{2} + \frac{q_3\pi}{2}\right)c_{22} - \omega^{q_2+q_3} \sin\left(\frac{q_2\pi}{2} + \frac{q_3\pi}{2}\right)c_{11},
\end{aligned}
$$

$$
\begin{aligned}
\beta_2 = {}& c_{11}c_{23}c_{32} - c_{31}c_{12}c_{23} + kc_{11}c_{33} + \omega^{q_1+q_2+q_3} \cos\left(\frac{(q_1+q_2+q_3)\pi}{2}\right) \\
&+ k\omega^{q_3} \cos\left(\frac{q_3\pi}{2}\right)c_{11} + \omega^{q_3} \cos\left(\frac{q_3\pi}{2}\right)c_{11}c_{22} - k\omega^{q_1} \cos\left(\frac{q_1\pi}{2}\right)c_{33} \\
&- \omega^{q_1} \cos\left(\frac{q_1\pi}{2}\right)c_{23}c_{32} - k\omega^{q_1+q_3} \cos\left(\frac{(q_1+q_3)\pi}{2}\right) - \omega^{q_1+q_3} \cos\left(\frac{(q_1+q_3)\pi}{2}\right)c_{22} \\
&- \omega^{q_2+q_3} \cos\left(\frac{(q_2+q_3)\pi}{2}\right)c_{11},
\end{aligned}
$$

$$
\begin{aligned}
\beta_3 = {}& \omega^{q_1+q_2+q_3} \cos\left(\frac{q_1\pi}{2} + \frac{q_2\pi}{2} + \frac{q_3\pi}{2}\right) - \omega^{q_1} \cos\left(\frac{q_1\pi}{2}\right)c_{23}c_{32} + k\omega^{q_3} \cos\left(\frac{q_3\pi}{2}\right)c_{11} \\
&+ \omega^{q_3} \cos\left(\frac{q_3\pi}{2}\right)c_{11}c_{22} + k\omega^{q_1} \cos\left(\frac{q_1\pi}{2}\right)c_{33} - k\omega^{q_1+q_3} \cos\left(\frac{q_1\pi}{2} + \frac{q_3\pi}{2}\right) \\
&- \omega^{q_1+q_3} \cos\left(\frac{q_1\pi}{2} + \frac{q_3\pi}{2}\right)c_{22} - \omega^{q_2+q_3} \cos\left(\frac{q_2\pi}{2} + \frac{q_3\pi}{2}\right)c_{11} \\
&+ c_{11}c_{23}c_{32} - c_{31}c_{12}c_{23} - kc_{11}c_{33},
\end{aligned}
$$

$$
\begin{aligned}
\beta_4 = {}& -\omega^{q_1+q_2+q_3} \sin\left(\frac{(q_1+q_2+q_3)\pi}{2}\right) + \omega^{q_1} \sin\left(\frac{q_1\pi}{2}\right)c_{23}c_{32} - k\omega^{q_3} \sin\left(\frac{q_3\pi}{2}\right)c_{11} \\
&- \omega^{q_3} \sin\left(\frac{q_3\pi}{2}\right)c_{11}c_{22} - k\omega^{q_1} \sin\left(\frac{q_1\pi}{2}\right)c_{33} + k\omega^{q_1+q_3} \sin\left(\frac{(q_1+q_3)\pi}{2}\right) \\
&+ \omega^{q_1+q_3} \sin\left(\frac{(q_1+q_3)\pi}{2}\right)c_{22} + \omega^{q_2+q_3} \sin\left(\frac{(q_2+q_3)\pi}{2}\right)c_{11},
\end{aligned}
$$

$$
\begin{aligned}
\beta_5 = {}& -k\omega^{q_3} \sin\left(\frac{q_3\pi}{2}\right)c_{11} + k\omega^{q_1} \sin\left(\frac{q_1\pi}{2}\right)c_{33} + \omega^{q_2} \sin\left(\frac{q_2\pi}{2}\right)c_{11}c_{33} \\
&+ \omega^{q_1} \sin\left(\frac{q_1\pi}{2}\right)c_{22}c_{33} + k\omega^{q_1+q_3} \sin\left(\frac{q_1\pi}{2} + \frac{q_3\pi}{2}\right) \\
&- \omega^{q_1+q_2} \sin\left(\frac{q_1\pi}{2} + \frac{q_2\pi}{2}\right)c_{33},
\end{aligned}
$$

$$
\begin{aligned}
\beta_6 = {}& k\omega^{q_1} \cos\left(\frac{q_1\pi}{2}\right)c_{33} - k\omega^{q_3} \cos\left(\frac{q_3\pi}{2}\right)c_{11} \\
&+ \omega^{q_1} \cos\left(\frac{q_1\pi}{2}\right)c_{22}c_{33} + \omega^{q_2} \cos\left(\frac{q_2\pi}{2}\right)c_{11}c_{33} \\
&+ k\omega^{q_1+q_3} \cos\left(\frac{(q_1+q_3)\pi}{2}\right) - kc_{11}c_{33} - \omega^{q_1+q_2} \cos\left(\frac{(q_1+q_2)\pi}{2}\right)c_{33} - c_{11}c_{22}c_{33}.
\end{aligned}
$$

Appendix D

The expressions of C_1, C_2, D_1, and D_2 in Eq. (4.11) are:

$$
C_1 = -\omega_0 \cos(\tau_0\omega_0)\left(-k\omega_0^{q_1} \sin\left(\frac{q_1\pi}{2}\right)c_{33} - k\omega_0^{q_3} \sin\left(\frac{q_3\pi}{2}\right)c_{11} + \omega_0^{q_1} \sin\left(\frac{q_1\pi}{2}\right)c_{22}c_{33}\right.
$$

$$
\begin{aligned}
&+\omega_0^{q_2}\sin\left(\frac{q_2\pi}{2}\right)c_{11}c_{33}+k\omega_0^{q_1+q_3}\sin\left(\frac{(q_1+q_3)\pi}{2}\right)-\omega_0^{q_1+q_2}\sin\left(\frac{(q_1+q_2)\pi}{2}\right)c_{33}\right)\\
&-\omega_0\sin(\tau_0\omega_0)\left(k\omega_0^{q_1}\cos\left(\frac{q_1\pi}{2}\right)c_{33}+k\omega_0^{q_3}\cos\left(\frac{q_3\pi}{2}\right)c_{11}-\omega_0^{q_1}\cos\left(\frac{q_1\pi}{2}\right)c_{22}c_{33}\right.\\
&-\omega_0^{q_2}\cos\left(\frac{q_2\pi}{2}\right)c_{11}c_{33}-k\omega_0^{q_1+q_3}\cos\left(\frac{(q_1+q_3)\pi}{2}\right)+\omega_0^{q_1+q_2}\cos\left(\frac{(q_1+q_2)\pi}{2}\right)c_{33}\\
&\left.-kc_{11}c_{33}+c_{11}c_{22}c_{33}\right),\\
D_1={}&\frac{1}{\omega_0}\left(\left(k\omega_0^{q_3}c_{11}\left(\cos\left(\frac{q_3\pi}{2}\right)\omega_0\tau_0-\sin\left(\frac{q_3\pi}{2}\right)q_3\right)+\omega_0^{q_1}\sin\left(\frac{q_1\pi}{2}\right)c_{33}q_1(k+c_{22})\right.\right.\\
&+\omega_0^{q_2}c_{11}c_{33}\left(\sin\left(\frac{q_2\pi}{2}\right)q_2-\cos\left(\frac{q_2\pi}{2}\right)\omega_0\tau_0\right)-\omega_0^{q_1+1}\cos\left(\frac{q_1\pi}{2}\right)c_{33}\tau_0(k+c_{22})\\
&-\omega_0^{q_1+q_2}\sin\left(\frac{(q_1+q_2)\pi}{2}\right)c_{33}(q_1+q_2)-k\omega_0^{q_1+q_3+1}\cos\left(\frac{(q_1+q_3)\pi}{2}\right)\tau_0\\
&+k\omega_0^{q_1+q_3}\sin\left(\frac{(q_1+q_3)\pi}{2}\right)(q_1+q_3)+\omega_0^{q_1+q_2+1}\cos\left(\frac{(q_1+q_2)\pi}{2}\right)c_{33}\tau_0\\
&\left.\left.+c_{11}c_{22}c_{33}\omega_0\tau_0+kc_{11}c_{33}\omega_0\tau_0\right)\cos(\tau_0\omega_0)\right)+\frac{1}{\omega_0}\left(\left(k\omega_0^{q_3}c_{11}\left(\sin\left(\frac{q_3\pi}{2}\right)\omega_0\tau_0\right.\right.\right.\\
&\left.+\cos\left(\frac{q_3\pi}{2}\right)q_3\right)-\omega_0^{q_1+1}\sin\left(\frac{q_1\pi}{2}\right)c_{33}\tau_0(k+c_{22})-\omega_0^{q_2}c_{11}c_{33}\left(\sin\left(\frac{q_2\pi}{2}\right)\omega_0\tau_0\right.\\
&\left.-\cos\left(\frac{q_2\pi}{2}\right)q_2\right)-\omega_0^{q_1}\cos\left(\frac{q_1\pi}{2}\right)c_{33}q_1(c_{22}+k)-k\omega_0^{q_1+q_3+1}\sin\left(\frac{(q_1+q_3)\pi}{2}\right)\tau_0\\
&-k\omega_0^{q_1+q_3}\cos\left(\frac{(q_1+q_3)\pi}{2}\right)(q_1+q_3)+\omega_0^{q_1+q_2+1}\sin\left(\frac{(q_1+q_2)\pi}{2}\right)c_{33}\tau_0\\
&\left.\left.+\omega_0^{q_1+q_2}\cos\left(\frac{(q_1+q_2)\pi}{2}\right)c_{33}(q_1+q_2)\right)\sin(\tau_0\omega_0)\right)\\
&+\frac{1}{\omega_0}\left(k\omega_0^{q_1}c_{33}q_1\sin\left(2\tau_0\omega_0-\frac{q_1\pi}{2}\right)+2k\omega_0^{q_1+1}c_{33}\tau_0\cos\left(2\tau_0\omega_0-\frac{q_1\pi}{2}\right)\right.\\
&+\omega_0^{q_1+q_2+q_3}\sin\left(\frac{(q_1+q_2+q_3)\pi}{2}\right)(q_1+q_2+q_3)+\omega_0^{q_3}\sin\left(\frac{q_3\pi}{2}\right)q_3c_{11}(c_{22}+k)\\
&-\omega_0^{q_1}\sin\left(\frac{q_1\pi}{2}\right)q_1c_{23}c_{32}-\omega_0^{q_2+q_3}\sin\left(\frac{(q_2+q_3)\pi}{2}\right)c_{11}(q_2+q_3)\\
&-\omega_0^{q_1+q_3}\sin\left(\frac{(q_1+q_3)\pi}{2}\right)c_{22}(q_1+q_3)-k\omega_0^{q_1+q_3}\sin\left(\frac{(q_1+q_3)\pi}{2}\right)(q_1+q_3)\\
&\left.-2k\omega_0\tau_0\cos(2\omega_0\tau_0)c_{11}c_{33}\right),\\
C_2={}&\omega_0\left(-k\omega_0^{q_1}\cos\left(\frac{q_1\pi}{2}\right)c_{33}-k\omega_0^{q_3}\cos\left(\frac{q_3\pi}{2}\right)c_{11}+kc_{11}c_{33}+\omega_0^{q_1}\cos\left(\frac{q_1\pi}{2}\right)c_{22}c_{33}\right.\\
&+\omega_0^{q_2}\cos\left(\frac{q_2\pi}{2}\right)c_{11}c_{33}-c_{11}c_{22}c_{33}+k\omega_0^{q_1+q_3}\cos\left(\frac{(q_1+q_3)\pi}{2}\right)\\
&\left.-\omega_0^{q_1+q_2}\cos\left(\frac{(q_1+q_2)\pi}{2}\right)c_{33}\right)\cos(\tau_0\omega_0)+\omega0\left(-k\omega_0^{q_1}\sin\left(\frac{q_1\pi}{2}\right)c_{33}\right.
\end{aligned}
$$

$$
\begin{aligned}
&- k\omega_0^{q_3}\sin\left(\frac{q_3\pi}{2}\right)c_{11} + \omega_0^{q_1}\sin\left(\frac{q_1\pi}{2}\right)c_{22}c_{33} + \omega 0^{q_2}\sin\left(\frac{q_2\pi}{2}\right)c_{11}c_{33} \\
&+ k\omega_0^{q_1+q_3}\sin\left(\frac{(q_1+q_3)\pi}{2}\right) - \omega_0^{q_1+q_2}\sin\left(\frac{(q_1+q_2)\pi}{2}\right)c_{33}\bigg)\sin(\tau_0\omega_0), \\
D_2 = {}& \frac{1}{\omega_0}\left(\left(k\omega_0^{q_3}c_{11}\left(\sin\left(\frac{q_3\pi}{2}\right)\omega_0\tau_0 + \cos\left(\frac{q_3\pi}{2}\right)q_3\right)\right.\right. \\
&- \omega_0^{q_1+1}\sin\left(\frac{q_1\pi}{2}\right)c_{33}\tau_0(k+c_{22}) \\
&- \omega_0^{q_2}c_{11}c_{33}\left(\sin\left(\frac{q_2\pi}{2}\right)\omega_0\tau_0 + \cos\left(\frac{q_2\pi}{2}\right)q_2\right) - \omega_0^{q_1}\cos\left(\frac{q_1\pi}{2}\right)c_{33}q_1(c_{22}+k) \\
&- k\omega_0^{q_1+q_3+1}\sin\left(\frac{(q_1+q_3)\pi}{2}\right)\tau_0 - k\omega_0^{q_1+q_3}\cos\left(\frac{(q_1+q_3)\pi}{2}\right)(q_1+q3) \\
&+ \omega_0^{q_1+q_2+1}\sin\left(\frac{(q_1+q_2)\pi}{2}\right)c_{33}\tau_0 \\
&+ \omega_0^{q_1+q_2}\cos\left(\frac{(q_1+q_2)\pi}{2}\right)c_{33}(q_1+q_2)\bigg)\cos(\tau_0\omega_0)\bigg) \\
&+ \frac{1}{\omega_0}\left(\left(k\omega_0^{q_1+q_3+1}\cos\left(\frac{(q_1+q_3)\pi}{2}\right)\tau_0 + \omega_0^{q_1+q_2}\sin\left(\frac{(q_1+q_2)\pi}{2}\right)c_{33}(q_1+q_2)\right.\right. \\
&- \omega_0^{q_1+q_2+1}\cos\left(\frac{(q_1+q_2)\pi}{2}\right)c_{33}\tau_0 - k\omega_0^{q_1+q_3}\sin\left(\frac{(q_1+q_3)\pi}{2}\right)(q_1+q_3) \\
&+ k\omega_0^{q_1}\cos\left(\frac{q_1\pi}{2}\right)c_{33}\tau_0(\omega_0+c_{22}) + \omega_0^{q_2}c_{11}c_{33}\left(\cos\left(\frac{q_2\pi}{2}\right)\tau_0\omega_0 - \sin\left(\frac{q_2\pi}{2}\right)q_2\right) \\
&- k\omega_0^{q_3}c_{11}\left(\cos\left(\frac{q_3\pi}{2}\right)\omega_0\tau_0 - \sin\left(\frac{q_3\pi}{2}\right)q_3\right) - \omega_0^{q_1}c_{33}\sin\left(\frac{q_1\pi}{2}\right)(c_{22}+k) \\
&- kc_{11}c_{33}\omega_0\tau_0 - c_{11}c_{22}c_{33}\tau_0\bigg)\sin(\tau_0\omega_0)\bigg) + \frac{1}{\omega_0}\bigg(-2kc_{11}c_{33}\omega_0\tau_0\sin(2\tau_0\omega_0) \\
&+ \omega_0^{q_1}\cos\left(\frac{q_1\pi}{2}\right)q_1c_{23}c_{32} - \omega_0^{q_3}\cos\left(\frac{q_3\pi}{2}\right)q_3c_{11}(c_{22}+k) \\
&+ k\omega_0^{q_1}c_{33}q_1\left(\cos\left(2\tau_0\omega_0 - \frac{q_1\pi}{2}\right)\right) - 2k\omega_0^{q_1+1}c_{33}\tau_0\left(\sin\left(2\tau_0\omega_0 - \frac{q_1\pi}{2}\right)\right) \\
&- \omega_0^{q_1+q_2+q_3}\cos\left(\frac{(q_1+q_2+q_3)\pi}{2}\right)(q_1+q_2+q_3) \\
&+ \omega_0^{q_2+q_3}\cos\left(\frac{(q_2+q_3)\pi}{2}\right)c_{11}(q_2+q_3) \\
&+ \omega_0^{q_1+q_3}\cos\left(\frac{(q_1+q_3)\pi}{2}\right)(q_1+q_3)(c_{22}+k)\bigg).
\end{aligned}
$$

Acknowledgements
The authors thank the referees for their valuable suggestions.

Funding
This research is supported by the National Natural Science Foundation of China (Grant Nos. 11561034, 11761040).

Availability of data and materials
Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally in the writing of this paper. All authors read and approved final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 6 December 2019 Accepted: 17 August 2020 Published online: 20 October 2020

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