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A new variation for the relativistic Euler equations

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Abstract

The Glimm scheme is one of the so famous techniques for getting solutions of the general initial value problem by building a convergent sequence of approximate solutions. The approximation scheme is based on the solution of the Riemann problem. In this paper, we use a new strength function in order to present a new kind of total variation of a solution. Based on this new variation, we use the Glimm scheme to prove the global existence of weak solutions for the nonlinear ultra-relativistic Euler equations for a class of large initial data that involve the interaction of nonlinear waves.

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Keywords: Total variation; Nonlinear waves; Riemann solutions; Wave interaction; Glimm scheme

1 Introduction

The Cauchy problem for a one space dimension hyperbolic $m \times m$ system of conservation laws is a first order quasi-linear system of PDEs of the form

$$\vartheta_t + F(\vartheta)_x = 0, \quad (1.1)$$

$$\vartheta(0, x) = \vartheta_0(x). \quad (1.2)$$

The early work on system (1.1) was introduced by Lax [1]. His results gave the infrastructure necessary for Glimm to prove the existence of solutions with small total variations for system (1.1) [2]. Systems of conservation laws (1.1) have several important applications, such as the Euler equations of fluid dynamics, vehicular and pedestrian traffic, biological models, etc., see [3–5]. For a comprehensive overview for system (1.1) and related issues, we refer to Friedrichs [6], Evans [7], Dafermos [3], LeVeque [8, 9], Majda [10], Serre [11], Smoller [12], and Abdelrahman [13–16].

Glimm [2] gave the proof of the existence of solutions to general strictly hyperbolic systems of conservation laws with genuinely nonlinear or linearly degenerate eigenvalues. The Glimm method consists of approximating the solution at one time step by piecewise constant states and solving the resulting Riemann problems to evaluate the solution at

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later time. The aim is to estimate the strength of the waves at the wave interaction during the time evolution of the solution. The Glimm scheme has the advantage of keeping sharp resolution, because in each computational cell, the local Riemann solution is achieved by randomly keeping the Riemann wave. One further important feature of the Glimm scheme is that a family of wave interactions can be handled automatically. Actually, his estimation is based on the Riemann invariants as a measure for total variation. Moreover, in the most related papers, the measure of wave strength is determined as the change of the Riemann invariants, see for example [2, 17–22]. In contrast, in this paper we introduce a new measure of strength of the waves in a natural way [23]. One interesting application of this new strength function is presented in this paper; specifically, we use it to prove the existence of solutions to the ultra-relativistic Euler system.

This paper is concerned with the 1-D ultra-relativistic Euler equations [13–15, 17, 19, 21, 23–25],

$$W_t + F(W)_x = 0, \quad (1.3)$$

where

$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} p(3 + 4u^2) \\ 4pu\sqrt{1 + u^2} \end{pmatrix}, \quad F(W) = \begin{pmatrix} 4pu\sqrt{1 + u^2} \\ p(1 + 4u^2) \end{pmatrix}. \quad (1.4)$$

The domains Υ and Υ' of (p, u) and the (W_1, W_2) state spaces, respectively, are

$$\begin{aligned} \Upsilon &= \{(p, u) \in \mathbb{R}^+ \times \mathbb{R}\}, \\ \Upsilon' &= \{(W_1, W_2) \in \mathbb{R} \times \mathbb{R} \mid |W_2| < W_1\}. \end{aligned} \quad (1.5)$$

Proposition 1.1 ([13]) *The mapping $\Gamma : \Upsilon \rightarrow \Upsilon'$ defined as*

$$\Gamma(p, u) = \begin{pmatrix} p(3 + 4u^2) \\ 4pu\sqrt{1 + u^2} \end{pmatrix} \quad (1.6)$$

is 1-1 with the Jacobian determinant continuous as well positive in the domain Υ .

System (1.3) is strictly hyperbolic in the region Υ , because it has two real and distinct characteristic velocities

$$\lambda_1 = \frac{2u\sqrt{1 + u^2} - \sqrt{3}}{3 + 2u^2} < \lambda_3 = \frac{2u\sqrt{1 + u^2} + \sqrt{3}}{3 + 2u^2}. \quad (1.7)$$

The corresponding eigenvectors of system (1.3) are

$$r_1 = \left(\frac{-4p}{\sqrt{3}\sqrt{1 + u^2}}, 1 \right)^T, \quad r_3 = \left(\frac{4p}{\sqrt{3}\sqrt{1 + u^2}}, 1 \right)^T. \quad (1.8)$$

It is straightforward to check that

$$\begin{aligned}\nabla\lambda_1 \cdot r_1 &= \frac{2((\sqrt{1+u^2} + \sqrt{3}u)^2 + 2)}{\sqrt{1+u^2}(3+2u^2)^2} > 0, \\ \nabla\lambda_3 \cdot r_3 &= \frac{2((\sqrt{1+u^2} - \sqrt{3}u)^2 + 2)}{\sqrt{1+u^2}(3+2u^2)^2} > 0.\end{aligned}\quad (1.9)$$

Hence, system (1.3) is genuinely nonlinear at every point (p, u) in Υ .

The 1 and 3-Riemann invariants corresponding to system (1.3), respectively, are

$$w = \ln(\sqrt{1+u^2} + u) + \frac{\sqrt{3}}{4} \ln p \quad (1.10)$$

and

$$z = \ln(\sqrt{1+u^2} + u) - \frac{\sqrt{3}}{4} \ln p. \quad (1.11)$$

In [23] we studied the interaction estimates of nonlinear waves for system (1.3), which produce transmitted waves. Actually, our new approach allows us to pose a confidential strength function to measure the total wave strength, which is used to exhibit that the strength after interactions is nonincreasing. This will be the basic ingredient of this paper. The motivation of the strength for the Riemann solution of system (1.3) was presented in [23]. This yields us an opportunity to introduce a new variation of a solution at a fixed time $t > 0$.

In this paper, we consider the Glimm scheme to solve the Cauchy problem (1.3) with the initial data

$$p(0, x) = p_0(x), \quad u(0, x) = u_0(x), \quad x \in \mathbb{R}, t \geq 0, \quad (1.12)$$

where $p > 0$ and $u \in \mathbb{R}$. Given an initial condition $p(0, x) = p_0(x)$, $u(0, x) = u_0(x)$, we establish an admissible weak solution (p, u) , defined for all $t \geq 0$. Namely, we introduce a significant contribution of the Glimm scheme to construct approximate solutions to the Cauchy problem (1.3) and (1.12) using a new total variation.

The novelty of this paper is mainly presented in the following aspect: we pose a new total variation at a fixed time $t > 0$, which has never been used before. This new variation differs completely from all the other variations given in [2, 17–19, 21, 22] for instance. Our variation is given in a natural way [23]. Thus our result here is more accurate and realistic than the other results presented in [2, 17–19, 21, 22] for instance. This fact will be significantly introduced in the proof of the main result in this paper (Theorem 1.1) given in Sect. 4.

Theorem 1.1 *Let $p_0(x)$ and $u_0(x)$ be arbitrary initial data such that the following conditions hold:*

$$\text{Tot. Var}\{\ln(p_0(\cdot))\} < \infty, \quad (1.13)$$

$$\text{Tot. Var}\left\{\ln\left(\frac{\sqrt{1+u_0^2} + u_0}{\sqrt{1+u_0^2} - u_0}\right)\right\} < \infty, \quad (1.14)$$

where $\text{Tot.Var}f(\cdot)$ represents the total variation of the function $f(\theta)$, $\theta \in \mathbb{R}$. Hence there exists a bounded weak solution $p(x, t)$, $u(x, t)$ to (1.3), obeying

$$\text{Tot.Var}\{\ln(p(0, \cdot))\} < L_0, \quad (1.15)$$

$$\text{Tot.Var}\left\{\ln\left(\frac{\sqrt{1+u(0, \cdot)^2} + u(0, \cdot)}{\sqrt{1+u(0, \cdot)^2} - u(0, \cdot)}\right)\right\} < L_1, \quad (1.16)$$

where L_0 and L_1 are positive constants depending only on the initial data. Then there exists an entropy solution $(p(x, t), u(x, t))$ of the Cauchy problem (1.3) and (1.12) in the upper half-plane $t \geq 0$, $-\infty < x < \infty$.

To prove this theorem, we develop a new analysis that is similar to that first given by Nishida in [26]. We show here that the ideas of Nishida can be generalized to system (1.3). This idea is to analyze solutions by the Glimm scheme [2] during a study of interaction of waves in a Riemann invariants plane. We use the same technique for system (1.3) by using a new approach based on a new function, which measures the strength of the waves in a natural way. Recently, there has been huge progress on several fundamental problems in nonlinear partial differential equations, see for example [27–41].

The layout of the article is given as follows. Section 2 presents the fundamental concepts of parametrization of waves given in [25, Sect. 4.4] and [13]. Section 3 introduces the new strength function of the waves for system (1.3) and gives sharp estimates for these strengths. Actually, these estimates play a crucial role in order to prove the main result, namely Theorem 1.1 in a completely unified way. In Sect. 4 we employ the Glimm scheme using our new strength function to prove Theorem 1.1. Conclusions are offered in Sect. 5.

2 Preliminary

Here we review the notions in [25, Sect. 4.4] and [13], concerning system (1.3) of conservation laws.

Consider a shock wave $x = x(t)$, where (p_-, u_-) and (p_+, u_+) are the constant lower (upper) state with $p_{\pm} > 0$, respectively. The relations defining the shock waves for system (1.3) are the Rankine–Hugoniot jump conditions:

$$\begin{aligned} s[p_+(3 + 4u_+^2) - p_-(3 + 4u_-^2)] &= 4p_+u_+\sqrt{1 + u_+^2} - 4p_-u_-\sqrt{1 + u_-^2}, \\ s[4p_+u_+\sqrt{1 + u_+^2} - 4p_-u_-\sqrt{1 + u_-^2}] &= p_+(1 + 4u_+^2) - p_-(1 + 4u_-^2), \end{aligned} \quad (2.1)$$

for shock speed $s = \dot{x}$. The entropy condition is

$$s(\psi_+ - \psi_-) + (\chi_+ - \chi_-) > 0, \quad (2.2)$$

where

$$\psi(p, u) = p^{\frac{3}{4}}\sqrt{1 + u^2}, \quad \chi(p, u) = p^{\frac{3}{4}}u.$$

In [13], we proved that condition (2.2) is equivalent to $u_- > u_+$.

To prescribe waves of system (1.3), we present the two positive parameters:

$$\alpha := \frac{p_+}{p_-}, \quad \beta := \frac{\sqrt{1+u_+^2} - u_+}{\sqrt{1+u_-^2} - u_-}. \quad (2.3)$$

We define the function $K_S: \mathbb{R}^+ \mapsto \mathbb{R}^+$ by

$$K_S(\alpha) := \frac{\sqrt{1+3\alpha}\sqrt{3+\alpha} + \sqrt{3}(\alpha-1)}{4\sqrt{\alpha}} \quad (2.4)$$

for $\alpha > 0$, which is a key to giving the shock parametrization in a completely unified way.

Lemma 2.1 ([25, Sect. 4.4]) *Given $(p_{\pm}, u_{\pm}) \in \mathbb{R}^+ \times \mathbb{R}$. Define α and β according to (2.3).*

1. *Suppose that $\alpha > 1$, and*

$$\beta = K_S(\alpha). \quad (2.5)$$

Hence the lower state (p_-, u_-) and the upper state (p_+, u_+) are connected to each other by a single 1-shock.

2. *Suppose that $\alpha < 1$, and*

$$\beta K_S(\alpha) = 1. \quad (2.6)$$

Hence the lower state (p_-, u_-) and the upper state (p_+, u_+) are connected to each other by a single 3-shock.

For $\alpha > 0$, we define the function $K_R: \mathbb{R}^+ \mapsto \mathbb{R}^+$ by

$$K_R(\alpha) := \alpha^{\frac{\sqrt{3}}{4}} \quad (2.7)$$

to give the rarefaction waves (fans) parametrization.

Lemma 2.2 ([25, Sect. 4.4]) *Given $(p_{\pm}, u_{\pm}) \in \mathbb{R}^+ \times \mathbb{R}$. Define α and β according to (2.3).*

1. *Assume that $\alpha < 1$; moreover, assume that*

$$\beta = K_R(\alpha). \quad (2.8)$$

Hence the lower state (p_-, u_-) and the upper state (p_+, u_+) are connected to each other by a single 1-fan.

2. *Assume that $\alpha > 1$; moreover, assume that*

$$\beta K_R(\alpha) = 1. \quad (2.9)$$

Hence the lower state (p_-, u_-) and the upper state (p_+, u_+) are connected to each other by a single 3-fan.

Lemma 2.3 ([13]) *The functions K_S and K_R are strictly monotonically increasing.*

In [13], we solved the weak form of system (1.3) for the Riemannian initial data

$$p_0(x) = \begin{cases} p_-, & x \leq 0, \\ p_+, & x > 0, \end{cases} \quad u_0(x) = \begin{cases} u_-, & x \leq 0, \\ u_+, & x > 0. \end{cases} \quad (2.10)$$

We also proved that the solution is unique in the class of fans and shock waves.

3 Interaction estimates

In [23, Sect. 4] a new function was introduced that measures the strength of nonlinear waves in a natural way. Here we should review the construction of this strength function which considers the basic ingredient of this paper, and this will conserve the homogeneity of the analysis. We now describe briefly the construction of this function. Namely, we presented a strength function $S(\alpha, \beta)$ depending on two positive arguments α and β , given in (2.3). We call $\text{Str}(S_1)$ the strength of the 1-shock wave, for instance. We also determine the strength of the waves of the Riemann solutions.

In [23] we studied the interaction of two shocks for system (1.3). We have given new explicit shock interaction formulas, see Fig. 1, which will be extended to other incoming waves of different families as follows.

Proposition 3.1 *Given are the three states $(p_j, u_j) \in \mathbb{R}^+ \times \mathbb{R}$, $j = 1, 2, 3$. Assume that the states (p_1, u_1) and (p_2, u_2) as well as (p_2, u_2) and (p_3, u_3) , respectively, are connected by a lower 3-wave and an upper 1-wave. The intermediate state (p_*, u_*) is connected with two single waves to the specified lower and upper Riemannian initial data, specifically*

- (i) $S_3 S_1 \rightarrow S'_1 S'_3$,
- (ii) $R_3 S_1 \rightarrow S'_1 R'_3$,
- (iii) $S_3 R_1 \rightarrow R'_1 S'_3$,
- (iv) $R_3 R_1 \rightarrow R'_1 R'_3$.

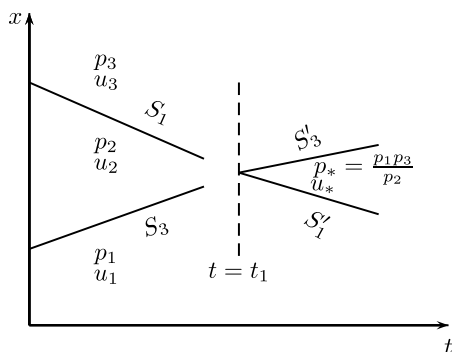
The intermediate pressure is given by

$$p_* = \frac{p_1 p_3}{p_2}. \quad (3.1)$$

Proof The proof is similar to that of Proposition 5.1 in [23]. \square

In the following proposition we present the strength estimates for the waves corresponding to system (1.3), which plays a key role in the sequel.

Figure 1 Asymptotic interaction of two shock waves



Proposition 3.2 ([23, Proposition 4.1]) *Recall α and β in (2.3), then*

$$\max(|\ln \alpha|, 2|\ln \beta|) \leq S(\alpha, \beta) \leq \max\left(|\ln \alpha|, \frac{4}{\sqrt{3}}|\ln \beta|\right)$$

for all $(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+$.

In [23], we gave asymptotic interaction estimates of the waves for system (1.3). We proved that the strength is nonincreasing. We now review briefly the interaction estimates of the waves of system (1.3). In Propositions 3.3 and 3.4 we clarify that the strength is conserved and strictly decreasing, respectively.

Proposition 3.3 ([23, Sect. 5.1]) *Given are the three states $(p_j, u_j) \in \mathbb{R}^+ \times \mathbb{R}$, $j = 1, 2, 3$. Suppose that the states (p_1, u_1) and (p_2, u_2) as well as (p_2, u_2) and (p_3, u_3) , respectively, are connected by a lower wave and an upper wave.*

- (i) $S_3 S_1 \rightarrow S'_1 S'_3$ and $\text{Str}(S_3) + \text{Str}(S_1) = \text{Str}(S'_1) + \text{Str}(S'_3)$,
- (ii) $S_3 R_1 \rightarrow R'_1 S'_3$ and $\text{Str}(S_3) + \text{Str}(R_1) = \text{Str}(R'_1) + \text{Str}(S'_3)$,
- (iii) $R_3 S_1 \rightarrow S'_1 R'_3$ and $\text{Str}(R_3) + \text{Str}(S_1) = \text{Str}(S'_1) + \text{Str}(R'_3)$,
- (iv) $R_3 R_1 \rightarrow R'_1 R'_3$ and $\text{Str}(R_3) + \text{Str}(R_1) = \text{Str}(R'_1) + \text{Str}(R'_3)$,
- (v) $S_1 \tilde{S}_1 \rightarrow S'_1 R'_3$ and $\text{Str}(S_1) + \text{Str}(\tilde{S}_1) = \text{Str}(S'_1) + \text{Str}(R'_3)$,
- (vi) $S_3 \tilde{S}_3 \rightarrow R'_1 S'_3$ and $\text{Str}(S_3) + \text{Str}(\tilde{S}_3) = \text{Str}(R'_1) + \text{Str}(S'_3)$.

Proposition 3.4 ([23, Sect. 5.2]) *Given are the three states $(p_j, u_j) \in \mathbb{R}^+ \times \mathbb{R}$, $j = 1, 2, 3$. Suppose that the states (p_1, u_1) and (p_2, u_2) as well as (p_2, u_2) and (p_3, u_3) , respectively, are connected by a lower wave and an upper wave. Thus we get:*

- (a)₁ $R_3 S_3 \rightarrow S'_1 S'_3$ and $R_3 S_3 \rightarrow S'_1 R'_3$ with
- (a)₂

$$\begin{aligned} \text{Str}(S'_1) + \text{Str}(S'_3) &< \text{Str}(R_3) + \text{Str}(S_3) \quad \text{and} \\ \text{Str}(S'_1) + \text{Str}(R'_3) &< \text{Str}(R_3) + \text{Str}(S_3), \quad \text{respectively;} \end{aligned}$$

- (b)₁ $S_1 R_1 \rightarrow S'_1 S'_3$ and $S_1 R_1 \rightarrow R'_1 S'_3$ with
- (b)₂

$$\begin{aligned} \text{Str}(S'_1) + \text{Str}(S'_3) &< \text{Str}(S_1) + \text{Str}(R_1) \quad \text{and} \\ \text{Str}(R'_1) + \text{Str}(S'_3) &< \text{Str}(S_1) + \text{Str}(R_1), \quad \text{respectively;} \end{aligned}$$

- (c)₁ $S_3 R_3 \rightarrow S'_1 S'_3$ and $S_3 R_3 \rightarrow S'_1 R'_3$ with
- (c)₂

$$\begin{aligned} \text{Str}(S'_1) + \text{Str}(S'_3) &< \text{Str}(S_3) + \text{Str}(R_3) \quad \text{and} \\ \text{Str}(S'_1) + \text{Str}(R'_3) &< \text{Str}(S_3) + \text{Str}(R_3), \quad \text{respectively;} \end{aligned}$$

- (d)₁ $R_1 S_1 \rightarrow S'_1 S'_3$ and $R_1 S_1 \rightarrow R'_1 S'_3$ with

(d)₂

$$\begin{aligned} \text{Str}(S'_1) + \text{Str}(S'_3) &< \text{Str}(R_1) + \text{Str}(S_1) \quad \text{and} \\ \text{Str}(R'_1) + \text{Str}(S'_3) &< \text{Str}(R_1) + \text{Str}(S_1), \quad \text{respectively.} \end{aligned}$$

Similarly, we can treat the interaction of three waves, and accordingly we can gain the same estimates as in Propositions 3.3 and 3.4.

4 The Glimm difference scheme

In this section we prove Theorem 1.1 using the Glimm scheme [2]. We construct a solution for the initial value problem of system (1.3) with the initial data (1.12). This scheme consists of approximating the solution at a fixed time level by piecewise constant states and solving the resulting Riemann problems herewith getting a sequence of single waves at the next time level. The aim is to estimate the behavior of the strength of these waves as the interaction of waves during the time procedure of the solution, which was given in an interesting way in Sect. 3. Specifically, we prove that the strength after interactions is nonincreasing using a new kind of total variation (measurement of strength) of a solution at a fixed time. Actually this fact is the crucial role in the proof of Theorem 1.1.

4.1 Proof of Theorem 1.1

Here we employ our contribution for the Glimm method in order to prove the existence of solution for system (1.3), namely Theorem 1.1, which is the main result in this paper.

Let Δx denote a mesh size, Δt denote a mesh size, and let $x_j = j\Delta x$, $t_n = n\Delta t$ represent the mesh points for the approximate solutions. We will use some notations in order to discretize time and space in the following way.

We assume $\Delta t > 0$ and $\Delta x > 0$ in terms of the Courant–Friedrichs–Lewy (CFL) condition due to

$$2\Delta t \leq \Delta x. \quad (4.1)$$

In fact, condition (4.1) guarantees that neighboring light cones have never interacted within the time step. Consider the initial data $U_0(x) = (p_0(x), u_0(x))$ for system (1.3) obeying $p_0(x) > 0$, $|v_0(x) = \frac{u_0(x)}{\sqrt{1+u_0^2(x)}}| < 1$, where $v = \frac{u}{\sqrt{1+u^2}}$ is the particle speed.

We start the definition of the Glimm solution $U_{\Delta x}(x, t)$ by discretizing the initial data into piecewise constant states $U_j^0 = U_0(x_j+)$. To begin the scheme, we define

$$U_{\Delta x}(x, 0) = U_j^0 \quad \text{for } x_j \leq x < x_{j+1}. \quad (4.2)$$

We suppose that the solution $U_{\Delta x}$ has been defined for $t < t_{n-1}$ and that the solution at time $t = t_{n-1}$ is

$$U_{\Delta x}(x, t_{n-1}) = U_j^{n-1} \quad \text{for } x_j \leq x < x_{j+1}. \quad (4.3)$$

To entire the definition of $U_{\Delta x}$ by induction, it suffices to define $U_{\Delta x}(x, t)$ for $t_{n-1} < t < t_n$. For $t_{n-1} < t < t_n$, let $U_{\Delta x}(x, t)$ be acquired by solving the Riemann problems at time $t = t_{n-1}$. Condition (4.1) ensures that waves never interact within one time step. Now to

repose the constant states and the corresponding Riemann problems at time level t in the approximate solution, let $a \equiv a_k \in A$ denote a (fixed) random sequence, $0 < a_k < 1$, where A represents the infinite product of intervals $(0, 1)$ endowed with Lebesgue measure $1 < k < \infty$. Hence define

$$\begin{aligned} U_{\Delta x}(x, t_n+) &= U_j^n \quad \text{for } x_j \leq x < x_{j+1}, \\ U_j^n &= U_{\Delta x}(x_j + a_n \Delta x, t_{n-1}). \end{aligned} \quad (4.4)$$

Thus the definition of $U_{\Delta x}$ is completed.

The next lemma plays an important role in proving our main result, namely Theorem 1.1.

Lemma 4.1 (Glimm [2]) *Suppose that the approximate solution $U_{\Delta x}$ obeys*

$$\text{Tot.Var}\{U_{\Delta x(\cdot, t)}\} \leq L < \infty \quad (4.5)$$

for all $t \geq 0$. Hence there exists a subsequence of mesh lengths $\Delta x \rightarrow 0$, where $U_{\Delta x} \rightarrow U$, and $U(x, t)$ also satisfies (4.5). The approximate solution converges pointwise a.e., and in L^1_{loc} , at each time, uniformly on bounded x and t sets. Furthermore, there exists a set $N \subset A$ of Lebesgue measure zero such that $a \in A - N$, hence $U(x, t)$ is a weak solution of the initial value problem (1.3) and (1.12).

Let $d_{j,n}$ denote the point $(x_j + a_n \Delta x, t_n)$. With a view to get the required estimates, it is suitable to consider curves consisting of line segments joining $d_{j,n}$ to both $d_{j+1,n \pm 1}$. Thus we cover the upper half-plane $t \geq 0$ by diamonds, the corners of which are the random points $d_{j,n}$, see [2]. We then define a mesh curve to be a piecewise linear curve lying on diamond boundaries going from west to north or south. It follows that if I is any mesh curve, then I divides the upper half-plane into an I^+ and I^- part, the I^- part being the one containing $t = 0$. We partially order the mesh curves by saying that $I_1 > I_2$ if every point of I_1 is either on I_2 or contained in I_2^+ , and we call I an immediate successor to J if $I > J$ and every mesh point of I except one is on J . In fact, all these technical procedures can be found in more detail in [17, 19, 21]. We shall get our estimates on the variation by considering our own function F defined on mesh curves.

In [2, 17–19, 21, 22] one defines the function F as the total change in Riemann invariants. In contrast, we introduce a new definition of the function F in a natural way, which has never been seen before [23], as follows:

$$F(J_i) = \sum_{J_i} S(\alpha_i, \beta_i). \quad (4.6)$$

The following lemma is a basic ingredient in order to prove Theorem 1.1 in a completely unified way.

Lemma 4.2 ([26]) *$F(J_2) \leq F(J_1)$ where J_i are two mesh curves and J_2 is an immediate successor to J_1 .*

In Propositions 3.3 and 3.4 we proved that the strength is conserved and strictly decreasing, respectively. The main outcome is that after the interactions, there is no increasing in the strength of the waves. This fact is so useful, and consider a powerful tool for use in the Glimm scheme.

Lemma 4.2 clearly follows from Propositions 3.3 and 3.4. We are now ready to complete the proof of Theorem 1.1. Using Lemma 4.2 we see that

$$F(J) \leq F(O)$$

for any mesh curve J , where O is the unique mesh curve which passes through points on $t = 0$ and $t = \Delta t$. At the moment notice that (1.15) and (1.16) imply that there exist states $p_{\pm\infty} = \lim_{x \rightarrow \pm\infty} p_0(x)$ and $u_{\pm\infty} = \lim_{x \rightarrow \pm\infty} u_0(x)$. By construction of $U_{\Delta x}$ it pursues that, for every Δx and $a \in A$, $p_{\pm\infty} = \lim_{x \rightarrow \pm\infty} p_{\Delta x}(x)$ and $u_{\pm\infty} = \lim_{x \rightarrow \pm\infty} u_{\Delta x}(x)$. This means that $(p_{\pm\infty}, u_{\pm\infty})$ are constant states manifesting in the solution at each fixed time level. Using Propositions 3.3 and 3.4 gives

$$\underbrace{\text{Tot. Var}}_{J_i} \{\alpha, \beta\} \leq \underbrace{\text{Tot. Var}}_{\text{Shocks and fans on } J_i} \{\alpha, \beta\} + C \leq F(J) + C \leq F(O) + C \leq L,$$

where L is a constant depending only on the initial data. This implies that

$$\text{Tot. Var} \{ \ln(p_0(\cdot)) \} < L_0, \text{Tot. Var} \left\{ \left(\frac{\sqrt{1+u_0^2} + u_0}{\sqrt{1+u_0^2} - u_0} \right) \right\} < L_1$$

at each time $t > 0$ in the solution $U_{\Delta x}$, where L_0 and L_1 depend only on the initial data. Hence there exists a constant $M > 0$ such that

$$\frac{1}{M} < p_{\Delta x}(t, x) < M, \\ -1 + \frac{1}{M} < v_{\Delta x}(t, x) = \frac{u_{\Delta x}(t, x)}{\sqrt{1+u_{\Delta x}^2(t, x)}} < 1 - \frac{1}{M}$$

for all x and $t > 0$ uniformly in Δx . These two bounds give the desired result (4.5) because, due to Proposition 1.1, the Jacobian determinant $|\frac{\partial(W_1, W_2)}{\partial(p, u)}|$ is bounded away from zero (uniformly in Δx) on the image of the approximate solutions $U_{\Delta x}$. Thus by Lemma 4.1 there exists a subsequence of mesh lengths $\Delta x \rightarrow 0$ such that $U_{\Delta x} \rightarrow U(p(x, t), u(x, t))$, where U satisfies (1.15), (1.16), and (4.5) and the convergence is pointwise a.e., and in L^1_{loc} at each time, uniformly on bounded x and t sets. This completes the proof of Theorem 1.1.

Remark 4.1 Roughly speaking, the new proposed strength function has so many further applications. For example, this function will be so salutary in studying numerical techniques to emphasize some interesting physical phenomena. This function will also play an important role in proving the uniqueness of weak solution of the Cauchy problems for hyperbolic systems. Moreover, finding such strength function for another 2×2 system of conservation laws is a big challenge and an interesting task.

5 Conclusions and outlook

In this work, we considered the global entropy solutions to the ultra-relativistic Euler equations for a class of large initial data which involve the interaction of shock waves and fans. Namely, we introduced a new strength for nonlinear elementary waves. Then we deduced that the total strength after interactions is nonincreasing. This interesting result enables us to introduce a new variation at a fixed time. Based on this new approach of analysis, we proved the global existence of weak solutions to the relativistic Euler system.

In the future work we plan to extend our new approach, specifically the new measurement of strength for the other hyperbolic systems of conservation laws such as shallow water system, pressure gradient system, blood flow model, etc. Finally, we believe that this new strength function has so many further interesting applications in the field of hyperbolic system of conservation laws.

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Authors' contributions

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