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Amended oscillation criteria for second-order neutral differential equations with damping term

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Abstract

The aim of this work is to improve the oscillation results for second-order neutral differential equations with damping term. We consider the noncanonical case which always leads to two independent conditions for oscillation. We are working to improve related results by simplifying the conditions, based on taking a different approach that leads to one condition. Moreover, we obtain different forms of conditions to expand the application area. An example is also given to demonstrate the applicability and strength of the obtained conditions over known ones.

Keywords: Amended criteria for oscillation; Second-order differential equations; Neutral delay; Damping term

1 Introduction

This work is concerned with studying the oscillation properties of the second-order neutral delay differential equations with damping term of the form

$$(r(t)(z'(t))^\alpha)' + p(t)(z'(t))^\alpha + q(t)f(x(\sigma(t))) = 0, \quad t \geq t_0, \quad (1)$$

where

$$z(t) = x(t) + c(t)x(\tau(t)). \quad (2)$$

Throughout this work, we assume $\alpha \in Q_{\text{odd}}^+ := \{a/b : a, b \in \mathbf{Z}^+ \text{ are odd}\}$, $r, c, \tau, \sigma, p, q \in C([t_0, \infty))$, r is positive, c, p and q are nonnegative, $\tau(t) \leq t$, $\sigma(t) \leq t$, $\sigma'(t) > 0$, $\lim_{t \rightarrow \infty} \tau(t) = \lim_{t \rightarrow \infty} \sigma(t) = \infty$, $f \in C(\mathbf{R}, \mathbf{R})$, and it satisfies the following property:

$$f(x) > kx^\beta \quad \text{for all } x \neq 0, \quad (3)$$

where $k > 0$ is a constant and $\beta \in Q_{\text{odd}}^+$. Furthermore, this study requires that

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(h)} \exp\left(\frac{-1}{\alpha} \int_{t_0}^h \frac{p(s)}{r(s)} ds\right) dh < \infty \quad (4)$$

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and

$$c(t) < \min \left\{ 1, \frac{\eta(t)}{\eta(\tau(t))} \right\},$$

where

$$\eta(t) := \int_t^\infty \left(\frac{1}{r(v)} \exp \left(- \int_{t_0}^v \frac{p(s)}{r(s)} ds \right) \right)^{1/\alpha} dv.$$

A real-valued function $x \in C([t_x, \infty), \mathbf{R})$, $t_x \geq t_0$ is a *solution* of (1) if x satisfies (1) on $[t_x, \infty)$, and it has the property that $z(t)$ and $r(t)(z'(t))^\alpha$ are continuously differentiable for $t \in [t_x, \infty)$. A nontrivial solution x is said to be *oscillatory*, if it has arbitrary large zeros. Otherwise, it is said to be *non-oscillatory*. The set of all eventually positive solutions of (1) is denoted by \mathfrak{N} , that is, if $x \in \mathfrak{N}$, then there exists a $t_1 \geq t_0$ large enough such that $x(t) > 0$ for all $t \geq t_1$. We only focus on solutions of (1), which exist on $[t_0, \infty)$ and satisfy $\sup\{|x(t)| : t_x \leq t\} > 0$ for every $t \geq t_x$.

Half-linear differential equations arise in real problems; for instance, in the study of non-Newtonian fluid theory and the turbulent flow of a polytropic gas in a porous medium; see [1, 2].

In the past few years, there has been research studying the asymptotic properties and the oscillatory behavior of solutions of differential equations with different order. This research focused on developing and improving the oscillation criteria for differential equations. References [3–9] improved the oscillation criteria for noncanonical second-order equations with delay and advanced argument. For canonical second-order delay equations, Refs. [10, 11] developed the oscillation criteria. The results in [12–21] dealt with the issue of oscillation of equations of higher order. For differential equations with damping, we present the following results that are closely related to this paper.

Tunc and Kaymaz [22] studied the oscillatory behavior of equations with damping term

$$z''(t) + r(t)z'(t) + q(t)x(\sigma(t)) = 0, \quad t \geq t_0 > 0,$$

under the condition

$$\int_{t_0}^\infty \exp \left(- \int_{t_0}^t r(s) ds \right) dt = \infty.$$

They are extended to more general second-order linear and/or nonlinear neutral differential equations with damping in [23–25]. Saker et al. [26] established Kamenev-type and Philos-type theorems for oscillation of equation with damping term

$$(a(t)x'(t))' + p(t)x'(\sigma(t)) + q(t)f(x(g(t))) = 0.$$

Results in [26] are extended and improved results in [27–29].

In this paper, we try to improve the oscillation criteria for solutions of (1) by creating new and more effective criteria. In a noncanonical case, we always have two cases of derivatives signs for corresponding function z , which often leads to two independent conditions

to ensure oscillatory. As an extension of the results of [4], we create a new criterion for oscillation of (1), which in turn is a simplification of the previous results in [26].

Although the theoretical advantage lies in reducing the number of conditions that are sufficient to verify the oscillation of solutions of differential equations, but sometimes a single condition is less effective in practical applications. So, we also follow the usual approach to creating two independent criteria of oscillation.

Remark 1 The functional inequalities in this paper are supposed to hold eventually, that is, they are satisfied for all t large enough.

2 Preliminaries

In the following, we provide some notations which help us to easily display the results. Moreover, we present the auxiliary lemmas which help in validating the main results.

Notation 1 For the sake of brevity, we define the operators $\gamma := (\alpha/(\alpha + 1))^{\alpha+1}$,

$$\mu(t) := \exp\left(\int_{t_1}^t \frac{p(s)}{r(s)} ds\right),$$

and

$$\varrho(t) := \begin{cases} 1 & \text{if } \alpha = \beta, \\ l_1 & \text{if } \alpha > \beta, \\ l_2 \eta^{\beta-\alpha}(t) & \text{if } \alpha < \beta, \end{cases}$$

where l_1, l_2 are positive constants.

Notation 2 For the sake of brevity, we define the functions

$$G(t) := k\mu(t)q(t)(1 - c(\sigma(t)))^\beta$$

and

$$Q(t) := k\mu(t)q(t)\left(1 - c(\sigma(t))\frac{\eta(\tau(\sigma(t)))}{\eta(\sigma(t))}\right)^\beta.$$

Lemma 1 Let $\Theta(\theta) = A_1\theta - A_2(\theta - A_3)^{(\alpha+1)/\alpha}$, where $A_2 > 0, A_1$ and A_3 are constants. Then the maximum value of Θ on \mathbf{R} at $\theta^* = A_3 + (\alpha A_1/((\alpha + 1)A_2))^\alpha$ is

$$\max_{\theta \in \mathbf{R}} \Theta(\theta) = \Theta(\theta^*) = A_1 A_3 + \frac{\gamma}{\alpha} \frac{A_1^{\alpha+1}}{A_2^\alpha}. \quad (5)$$

Lemma 2 Assume that $x \in \mathfrak{R}$. Then:

(i) The function z satisfies $z(t) > 0$, $(\mu(t)r(t)(z'(t))^\alpha)' \leq 0$ and one of the next cases:

$$(D_1) \quad z'(t) > 0;$$

$$(D_2) \quad z'(t) < 0.$$

(ii) If z satisfies (D_2) , then

$$z^{\beta-\alpha}(t) \geq \varrho(t) \quad (6)$$

and

$$(\mu(t)r(t)(z'(t))^\alpha)' \leq -Q(t)z^\beta(\sigma(t)). \quad (7)$$

Proof Since $x \in \mathfrak{R}$, it is obvious that there exists a $t_1 \geq t_0$ such that $x(t)$, $x(\tau(t))$ and $x(\sigma(t))$ are positive functions. As a direct conclusion from the definition (2), we find that $z(t) > 0$. Next, from (1) and (3), we obtain

$$\begin{aligned} (\mu(t)r(t)(z'(t))^\alpha)' &= -\mu(t)q(t)f(x(\sigma(t))) \\ &\leq -k\mu(t)q(t)x^\beta(\sigma(t)), \end{aligned} \quad (8)$$

which means that $\mu(t)r(t)(z'(t))^\alpha$ is a nonincreasing function and has a fixed sign. Since $\mu(t)r(t) > 0$, we get either $z'(t) > 0$ or $z'(t) < 0$.

Now, let $z'(t) < 0$ for all $t \geq t_2 \geq t_1$, where t_2 is large enough. Then there exists a constant $K_1 > 0$ such that

$$z^{\beta-\alpha}(t) \geq K_1^{\beta-\alpha} = l_1 \quad \text{for } \alpha \geq \beta.$$

On the other hand, using the fact that $(\mu(t)r(t)(z'(t))^\alpha)' \leq 0$, we get

$$\mu(t)r(t)(z'(t))^\alpha \leq \mu(t_2)r(t_2)(z'(t_2))^\alpha = -K_2 < 0 \quad (9)$$

and therefore

$$z'(t) \leq \left(\frac{-K_2}{\mu(t)r(t)} \right)^{1/\alpha}. \quad (10)$$

Integrating (10) from t to ∞ we obtain $-z(t) \leq -K_2^{1/\alpha}\eta(t)$. Thus,

$$z^{\beta-\alpha}(t) \geq K_2^{\frac{\beta-\alpha}{\alpha}} \eta^{\beta-\alpha}(t) := l_2 \eta^{\beta-\alpha}(t) \quad \text{for } \alpha < \beta,$$

i.e., (6) holds. Moreover, we have

$$z(t) \geq - \int_t^\infty z'(\theta) d\theta \geq -\eta(t)(\mu(t)r(t))^{1/\alpha} z'(t) \quad (11)$$

and so

$$\frac{d}{dt} \left(\frac{z(t)}{\eta(t)} \right) = \frac{\eta(t)z'(t) + (\mu(t)r(t))^{-1/\alpha} z(t)}{\eta^2(t)} \geq 0.$$

Hence,

$$x(t) \geq \left(1 - c(t) \frac{\eta(\tau(t))}{\eta(t)} \right) z(t),$$

which, in view of (8), gives (7). The proof of the lemma is complete. \square

Lemma 3 Let $x \in \mathfrak{N}$, the function z satisfy (D_2) and

$$\int_{t_0}^t \left(\frac{1}{\mu(v)r(v)} \int_{t_1}^v Q(s) ds \right)^{1/\alpha} dv = \infty. \quad (12)$$

Then

$$\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} z(t) = 0. \quad (13)$$

Proof Suppose that $x \in \mathfrak{N}$ and the function z satisfies (D_2) . Since z is a positive decreasing function, we have $\lim_{t \rightarrow \infty} z(t) = \epsilon \geq 0$.

Let $\epsilon > 0$. From Lemma 2, there exists a $t_1 \geq t_0$ such that

$$(\mu(t)r(t)(z'(t))^\alpha)' \leq -Q(t)z^\beta(\sigma(t)) \leq -\epsilon^\beta Q(t).$$

Integrating this inequality from t_1 to t , we have

$$-z'(t) \geq \epsilon^{\beta/\alpha} \left(\frac{1}{\mu(t)r(t)} \int_{t_1}^t Q(s) ds \right)^{1/\alpha}. \quad (14)$$

Integrating (14) from t_1 to t , we get

$$z(t) \leq z(t_1) - \epsilon^{\beta/\alpha} \int_{t_1}^t \left(\frac{1}{\mu(v)r(v)} \int_{t_1}^v Q(s) ds \right)^{1/\alpha} dv.$$

Taking $\lim_{t \rightarrow \infty}$ of this inequality and using (12), we are led to a contradiction with positivity of z . Therefore, $\epsilon = 0$ and (13) hold. This completes the proof. \square

Lemma 4 Let $x \in \mathfrak{N}$, the function z satisfy (D_2) and (12) hold. If, moreover there exists a constant $\delta \in [0, 1)$ such that

$$\eta(t) \left(\varrho(t) \int_{t_0}^t Q(s) ds \right)^{1/\alpha} \geq \delta, \quad (15)$$

then

$$\frac{d}{dt} \left(\frac{z(t)}{\eta^\delta(t)} \right) \leq 0. \quad (16)$$

Proof Suppose that $x \in \mathfrak{N}$ and the function z satisfies (D_2) . From Lemma 2, we see that (7) holds. Integrating (7) from t_1 to t , we obtain

$$\begin{aligned} \mu(t)r(t)(z'(t))^\alpha &\leq \mu(t_1)r(t_1)(z'(t_1))^\alpha - \int_{t_1}^t Q(s)z^\beta(\sigma(s)) ds \\ &\leq \mu(t_1)r(t_1)(z'(t_1))^\alpha - z^\beta(\sigma(t)) \int_{t_1}^t Q(s) ds. \end{aligned} \quad (17)$$

From Lemma 3, clearly (13) holds. Thus,

$$\mu(t_1)r(t_1)(z'(t_1))^\alpha + z^\beta(\sigma(t)) \int_{t_0}^{t_1} Q(s) ds > 0, \quad (18)$$

for $t \geq t_2$, where t_2 is large enough. Combining (17) and (18) and using the fact that $z^{\beta-\alpha}(t) \geq \varrho(t)$, we find

$$\mu(t)r(t)(z'(t))^\alpha \leq -z^\beta(\sigma(t)) \int_{t_0}^t Q(s) ds \leq -\varrho(t)z^\alpha(t) \int_{t_0}^t Q(s) ds, \quad (19)$$

which, in view of (15), gives

$$\begin{aligned} z'(t) &\leq -(\mu(t)r(t))^{-1/\alpha} z(t) \left(\varrho(t) \int_{t_0}^t Q(s) ds \right)^{1/\alpha} \\ &\leq -\frac{\delta}{\eta(t)} (\mu(t)r(t))^{-1/\alpha} z(t). \end{aligned}$$

Thus,

$$\frac{d}{dt} \left(\frac{z(t)}{\eta^\delta(t)} \right) = \frac{\eta^{\delta-1}(t)(\eta(t)z'(t) + \delta(\mu(t)r(t))^{-1/\alpha} z(t))}{\eta^{2\delta}(t)} \leq 0.$$

This completes the proof. \square

3 The one criterion theorems

In the following theorems, we obtain a criterion which ensure oscillation of (1) without verifying the extra condition.

Theorem 1 *If*

$$\int_{t_1}^{\infty} \left(\frac{1}{\mu(h)r(h)} \int_{t_1}^h \eta^\beta(\sigma(v))Q(v) dv \right)^{1/\alpha} dh = \infty, \quad (20)$$

for any $t_1 \geq t_0$, then all solutions of (1) are oscillatory.

Proof Assuming that the required result is not fulfilled. We assume that (1) has a solution $x \in \mathfrak{N}$. From Lemma 2, we see that z satisfies either (D_1) or (D_2) for all $t \geq t_1$.

Let (D_2) hold. As in the proof of Lemma 2, we arrive at (19). Integrating (19) from t to ∞ , we find

$$z(t) \geq K_2^{1/\alpha} \eta(t) \quad (21)$$

for all $t \geq t_2 \geq t_1$, where t_2 is large enough. From (7) in Lemma 2, we have

$$(\mu(t)r(t)(z'(t))^\alpha)' \leq -K_2^{\beta/\alpha} Q(t)\eta^\beta(\sigma(t)).$$

Integrating this inequality from t_2 to t , we get

$$z'(t) \leq -K_2^{\beta/\alpha^2} \left(\frac{1}{\mu(t)r(t)} \int_{t_2}^t \eta^\beta(\sigma(v))Q(v) dv \right)^{1/\alpha}.$$

Integrating again from t_2 to t , we obtain

$$z(t) \leq z(t_2) - K_2^{\beta/\alpha^2} \int_{t_2}^t \left(\frac{1}{\mu(h)r(h)} \int_{t_2}^h \eta^\beta(\sigma(v))Q(v) dv \right)^{1/\alpha} dh.$$

Taking $\lim_{t \rightarrow \infty}$ of this inequality and using (20), we are led to a contradiction with positivity of z .

Next, we let (D_1) hold. Since $z(t) \geq x(t)$, $\tau(t) \leq t$ and $z'(t) > 0$, we get $x(t) \geq (1 - c(t))z(t)$. Hence, (1) becomes

$$(\mu(t)r(t)(z'(t))^\alpha)' \leq -G(t)z^\beta(\sigma(t)). \quad (22)$$

Since $\eta(\tau(\sigma(t))) \geq \eta(\sigma(t))$, we obtain

$$1 - c(\sigma(t)) \frac{\eta(\tau(\sigma(t)))}{\eta(\sigma(t))} \leq 1 - c(\sigma(t)). \quad (23)$$

We note that the function $\int_t^\infty \eta^\beta(\sigma(v))Q(v) dv$ is unbounded due to (20) and (4). Taking into account $\eta'(t) \leq 0$ and (23), it is easy to see that

$$\int_t^\infty G(v) dv \geq \int_t^\infty Q(v) dv = \infty. \quad (24)$$

Combining (22) and (23), we have

$$\begin{aligned} \mu(t)r(t)(z'(t))^\alpha &\leq \mu(t_1)r(t_1)(z'(t_1))^\alpha - \int_{t_1}^t G(v)z^\beta(\sigma(v)) dv \\ &\leq \mu(t_1)r(t_1)(z'(t_1))^\alpha - z^\beta(\sigma(t_2)) \int_{t_1}^t G(v) dv, \end{aligned}$$

which, in view of (24), contradicts the positivity $z'(t)$. The proof of the theorem is complete. \square

Theorem 2 *If*

$$\limsup_{t \rightarrow \infty} \eta(t) \left(\varrho(t) \int_{t_1}^t Q(s) ds \right)^{1/\alpha} > 1 \quad (25)$$

for any $t_1 \geq t_0$, then all solutions of (1) are oscillatory.

Proof Assuming that the required result is not fulfilled. We assume that (1) has a solution $x \in \mathfrak{R}$. From Lemma 2, we see that z satisfies either (D_1) or (D_2) for all $t \geq t_1$.

Let (D_2) hold. From the proofs of Lemmas 2 and 4, we arrive at (11) and (17), respectively. Combining (11) and (17), we get

$$\mu(t)r(t)(z'(t))^\alpha \leq \varrho(t)\eta^\alpha(t)\mu(t)r(t)(z'(t))^\alpha \int_{t_1}^t Q(s) ds$$

and so

$$\eta(t) \left(\varrho(t) \int_{t_1}^t Q(s) ds \right)^{1/\alpha} \leq 1,$$

which contradicts (25).

Next, we let $z'(t) > 0$ for $t \geq t_1$. We note that (25) along with (4) imply (24). The rest of the proof is similar to the proof of Theorem 1. The proof of the theorem is complete. \square

4 Theorems of two independent criteria

Theorem 3 Assume that (12) holds. If

$$\limsup_{t \rightarrow \infty} \eta(t) \left(\varrho(t) \int_{t_0}^t Q(s) \, ds \right)^{1/\alpha} > 1, \quad (26)$$

then all solutions of (1) are oscillatory.

Proof For the proof of this lemma, it suffices to use (19) [from the proof of Lemma 4] instead of (17) in the proof of Theorem 2. \square

Using Lemma 4, we obtain the new results which improve the previous theorems when

$$\limsup_{t \rightarrow \infty} \eta(t) \left(\varrho(t) \int_{t_0}^t Q(s) \, ds \right)^{1/\alpha} \leq 1.$$

Theorem 4 Assume that (12) and there exists a $\delta \in [0, 1)$ such that (15) holds. If

$$\limsup_{t \rightarrow \infty} \eta(t) \left(\varrho(t) \left(\frac{\eta(\sigma(t))}{\eta(t)} \right)^{\delta\beta} \int_{t_0}^t Q(s) \, ds \right)^{1/\alpha} > 1, \quad (27)$$

then all solutions of (1) are oscillatory.

Proof Assuming that the required result is not fulfilled. We assume that (1) has a solution $x \in \mathfrak{R}$. From Lemma 2, we see that z satisfies either (D_1) or (D_2) for all $t \geq t_1$.

Let (D_2) hold. Using Lemma 4, we arrive at (16) and (19). From (16), we conclude that

$$z(\sigma(t)) \geq \frac{\eta^\delta(\sigma(t))}{\eta^\delta(t)} z(t), \quad (28)$$

which with (19) gives

$$\mu(t)r(t)(z'(t))^\alpha \leq -z^\beta(t) \left(\frac{\eta(\sigma(t))}{\eta(t)} \right)^{\delta\beta} \int_{t_0}^t Q(s) \, ds. \quad (29)$$

As in the proof of Lemma 2, we arrive at (11). Combining (11) and (29), we obtain

$$\begin{aligned} \mu(t)r(t)(z'(t))^\alpha &\leq -\varrho(t)z^\alpha(t) \left(\frac{\eta(\sigma(t))}{\eta(t)} \right)^{\delta\beta} \int_{t_0}^t Q(s) \, ds \\ &\leq \varrho(t)\eta^\alpha(t)\mu(t)r(t)(z'(t))^\alpha \left(\frac{\eta(\sigma(t))}{\eta(t)} \right)^{\delta\beta} \int_{t_0}^t Q(s) \, ds, \end{aligned}$$

then

$$\eta(t) \left(\varrho(t) \left(\frac{\eta(\sigma(t))}{\eta(t)} \right)^{\delta\beta} \int_{t_0}^t Q(s) \, ds \right)^{1/\alpha} \leq 1.$$

This contradicts (27).

Next, we let (D_1) hold. We note that (12) along with (4) imply (24). The rest of the proof is similar to the proof of Theorem 1. The proof of the theorem is complete. \square

Theorem 5 Assume that (12) and there exists a $\delta \in [0, 1)$ such that (15) holds. If there exist a function $\rho \in C^1([t_0, \infty), (0, \infty))$ and a $t_1 \in [t_0, \infty)$ such that

$$\limsup_{t \rightarrow \infty} \left\{ \frac{\eta^\alpha(t)}{\rho(t)} \int_{t_0}^t \left(\rho(v)Q(v)\varrho(v) \left(\frac{\eta(\sigma(v))}{\eta(v)} \right)^{\delta\alpha} - \gamma \frac{\mu(v)r(v)(\rho'(v))^{\alpha+1}}{\alpha^{\alpha+1}\rho^\alpha(v)} \right) dv \right\} > 1, \quad (30)$$

then all solutions of (1) are oscillatory.

Proof Assuming that the required result is not fulfilled. We assume that (1) has a solution $x \in \mathfrak{N}$. From Lemma 2, we see that z satisfies either (D_1) or (D_2) for all $t \geq t_1$.

Let (D_2) hold. From Lemma 2, we see that (6), (7) and (11) hold. We define the function

$$w(t) := \rho(t) \left(\frac{\mu(t)r(t)(z'(t))^\alpha}{z^\alpha(t)} + \frac{1}{\eta^\alpha(t)} \right). \quad (31)$$

Using (11), we see that $w(t) \geq 0$ for all $t \geq t_2 \geq t_1$. Differentiating (31), we get

$$\begin{aligned} w'(t) &= \frac{\rho'(t)}{\rho(t)} w(t) + \rho(t) \frac{(\mu(t)r(t)(z'(t))^\alpha)'}{z^\alpha(t)} - \alpha \rho(t) \frac{\mu(t)r(t)(z'(t))^{\alpha+1}}{z^{\alpha+1}(t)} \\ &\quad + \alpha \rho(t) \frac{1}{(\mu(t)r(t))^{1/\alpha} \eta^{\alpha+1}(t)}. \end{aligned}$$

Combining (6), (7) and (31), we obtain

$$\begin{aligned} w'(t) &\leq -\frac{\alpha}{(\rho(t)\mu(t)r(t))^{1/\alpha}} \left(w(t) - \frac{\rho(t)}{\eta^\alpha(t)} \right)^{1+1/\alpha} - \rho(t)Q(t)\varrho(t) \frac{z^\alpha(\sigma(t))}{z^\alpha(t)} \\ &\quad + \alpha \rho(t) \frac{1}{(\mu(t)r(t))^{1/\alpha} \eta^{\alpha+1}(t)} + \frac{\rho'(t)}{\rho(t)} w(t). \end{aligned} \quad (32)$$

Using inequality (5) with $A_1 := \rho'/\rho$, $A_2 := \alpha/(\rho\mu r)^{-1/\alpha}$, $A_3 = \rho/\eta^\alpha$ and $\theta := w$, we get

$$\frac{\rho'}{\rho} w - \frac{\alpha}{(\rho\mu r)^{1/\alpha}} \left(w - \frac{\rho}{\eta^\alpha} \right)^{1+1/\alpha} \leq \frac{\rho'}{\eta^\alpha} + \frac{\gamma}{\alpha^{\alpha+1}} \mu r \frac{(\rho')^{\alpha+1}}{\rho^\alpha},$$

which, in view of (32), gives

$$\begin{aligned} w'(t) &\leq \frac{\rho'(t)}{\eta^\alpha(t)} + \frac{\gamma}{\alpha^{\alpha+1}} \mu(t)r(t) \frac{(\rho'(t))^{\alpha+1}}{\rho^\alpha(t)} - \rho(t)Q(t)\varrho(t) \frac{z^\alpha(\sigma(t))}{z^\alpha(t)} \\ &\quad + \alpha \rho(t) \frac{1}{(\mu(t)r(t))^{1/\alpha} \eta^{\alpha+1}(t)}. \end{aligned} \quad (33)$$

As in the proof of Theorem 4, we arrive at (28). Thus, (33) becomes

$$w'(t) \leq -\rho(t)Q(t)\varrho(t) \left(\frac{\eta(\sigma(t))}{\eta(t)} \right)^{\delta\alpha} + \frac{\gamma}{\alpha^{\alpha+1}} \mu(t)r(t) \frac{(\rho'(t))^{\alpha+1}}{\rho^\alpha(t)} + \left(\frac{\rho(t)}{\eta^\alpha(t)} \right)'.$$

Integrating this inequality from t_2 to t , we have

$$\begin{aligned} & \int_{t_2}^t \left(\rho(v) Q(v) \varrho(v) \left(\frac{\eta(\sigma(v))}{\eta(v)} \right)^{\delta\alpha} - \gamma \frac{\mu(v) r(v) (\rho'(v))^{\alpha+1}}{\alpha^{\alpha+1} \rho^\alpha(v)} \right) dv \\ & \leq \left(\frac{\rho(t)}{\eta^\alpha(t)} - w(t) \right) \Big|_{t_2}^t \\ & \leq - \left(\rho(t) \frac{\mu(t) r(t) (z'(t))^\alpha}{z^\alpha(t)} \right) \Big|_{t_2}^t. \end{aligned} \quad (34)$$

From (11), we have

$$- \frac{(\mu(t) r(t))^{1/\alpha} z'(t)}{z(t)} \leq \frac{1}{\eta(t)},$$

which, in view of (34), implies

$$\frac{\eta^\alpha(t)}{\rho(t)} \int_{t_2}^t \left(\rho(v) Q(v) \varrho(v) \left(\frac{\eta(\sigma(v))}{\eta(v)} \right)^{\delta\alpha} - \gamma \frac{\mu(v) r(v) (\rho'(v))^{\alpha+1}}{\alpha^{\alpha+1} \rho^\alpha(v)} \right) dv \leq 1.$$

Taking the lim sup on both sides of this inequality, we are led to a contradiction with (30).

The rest of proof is the same and hence we omit it. The proof of the theorem is complete. \square

Example 1 Consider the differential equation

$$\left(t^{\alpha+1} \left[\left(x(t) + c_0 x \left(\frac{t}{2} \right) \right) \right]' \right)' + t \left(\left(x(t) + c_0 x \left(\frac{t}{2} \right) \right) \right)' + q_0 t^3 x(\lambda t) = 0, \quad (35)$$

where $t \geq 1$, $q_0 \in (0, \infty)$, $c_0 \in (0, 1/16)$ and $\lambda \in (0, 1]$. From Theorem 2, we deduce that Eq. (35) is oscillatory if

$$\frac{1}{16} q_0 (1 - 16c_0) > 1. \quad (36)$$

To apply Theorem 4, we first note that (12) and (15) are satisfied with

$$\delta = \frac{1}{16} q_0 (1 - 16c_0).$$

Therefore, (35) is oscillatory if

$$\frac{1}{16\lambda^{4\delta}} q_0 (1 - 16c_0) > 1.$$

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Authors' contributions

The authors contributed equally to the manuscript and read and approved the final manuscript.

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