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New general Grüss-type inequalities over σ -finite measure space with applications

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Abstract

In this paper, we establish some new integral inequalities involving general kernels. We obtain the related broad range of fractional integral inequalities. Also, we apply the Young inequality to find new forms of inequalities for generalized kernels. These new and motivated results generalize the results for fractional integrals such that fractional integral of a function with respect to an increasing function, Riemann–Liouville fractional integrals, Erdélyi–Kober fractional integrals, Hadamard fractional integrals, generalized fractional integral integrals in addition to the corresponding k -fractional integrals.

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1 Introduction

Fractional calculus deals with the study of derivative and integral operators of fractional order. This field is as important as calculus itself. In the last few decades, it attracted many researchers, who produced remarkable work (see, e.g., [1–3, 6, 7, 9–13, 18]). In particular, the uniqueness of solutions for fractional partial differential equations can be established by using fractional integral inequalities.

The Grüss inequality connects the integral of the product of two functions with the product of their integrals. Our main purpose in this paper is showing some new modifications of the Grüss inequality by using a general kernel. The Grüss inequality is one of the most fascinating inequalities and stated in the following theorem.

Theorem 1.1 ([5]) *Let \mathfrak{R} be a set of real numbers, let $m, M, n, N \in \mathfrak{R}$, and let $\Omega, \Upsilon : [\tau_1, \tau_2] \rightarrow \mathfrak{R}$ be two positive functions such that $m \leq \Omega(\mu) \leq M$ and $n \leq \Upsilon(\mu) \leq N$ for $\mu \in [\tau_1, \tau_2]$. Then*

$$\left| \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \Omega(\mu) \Upsilon(\mu) d\mu - \frac{1}{(\tau_2 - \tau_1)^2} \int_{\tau_1}^{\tau_2} \Omega(\mu) d\mu \int_{\tau_1}^{\tau_2} \Upsilon(\mu) d\mu \right|$$

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$$\leq \frac{1}{4}(M - m)(N - n), \tag{1.1}$$

where the constant $\frac{1}{4}$ cannot be improved.

Let (Δ, Σ, μ) be a measure space with positive σ -finite measure, let $k : \Delta \times \Delta \rightarrow \mathbb{R}$ be a nonnegative function, and let

$$\Theta(x) = \int_{\Delta} k(x, y) d\mu(y), \quad x \in \Delta. \tag{1.2}$$

Throughout this paper, we suppose $\Theta(x) > 0$ a.e. on Δ .

Let $U(k)$ denote the class of functions $\Lambda : \Delta \rightarrow \mathbb{R}$ with the representation

$$\bar{\Lambda}(x) = \int_{\Delta} k(x, y)\Lambda(y) d\mu(y),$$

where $\bar{\Lambda} : \Delta \rightarrow \mathbb{R}$ is a measurable function.

Definition 1.2 Let $f \in L_1([a, b])$ (the Lebesgue measure). The left-sided and right-sided Riemann–Liouville fractional integrals $I_{a^+}^\alpha f$ and $I_{b^-}^\alpha f$ of order $\alpha > 0$ are defined by

$$I_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(y)(x - y)^{\alpha-1} dy \quad (x > a)$$

and

$$I_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(y)(y - x)^{\alpha-1} dy \quad (x < b),$$

where Γ is the gamma function.

Diaz et al. [4] defined the gamma k -function as follows.

Definition 1.3 The function Γ_k , a generalization of the classical gamma function, is defined as follows:

$$\Gamma_k(t) = \lim_{n \rightarrow \infty} \frac{n!k^n(nk)^{\frac{t}{k}-1}}{(t)_{n,k}}, \quad k > 0, \Re(t) > 0,$$

where $(t)_{n,k} = t(t + k)(t + 2k) \dots (t + (n - 1)k), n \geq 1$, is the Pochhammer k symbol. Its integral representation is given by

$$\Gamma_k(t) = \int_0^\infty x^{t-1} e^{-\frac{x^k}{k}} dx, \quad \Re(t) > 0. \tag{1.3}$$

In particular, for $k = 1, \Gamma_1(t) = \Gamma(t)$.

Definition 1.4 ([12]) Let $f \in L_1([a, b])$ (the Lebesgue measure). The left-sided and right-sided Riemann–Liouville fractional integrals $I_{a^+}^{\alpha,k} f$ and $I_{b^-}^{\alpha,k} f$ of order $\alpha, k > 0$ are defined

by

$$I_{a^+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x f(y)(x-y)^{\frac{\alpha}{k}-1} dy \quad (x > a)$$

and

$$I_{b^-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b f(y)(y-x)^{\frac{\alpha}{k}-1} dy \quad (x < b),$$

where $\Gamma_k(\alpha)$ is the k -gamma function.

Definition 1.5 ([10]) Let (a, b) , $-\infty \leq a < b \leq \infty$, be a finite or infinite interval of the real line \mathfrak{R} , and let $\alpha > 0$. Let g be an increasing and positive monotone function on (a, b) . The left- and right-sided fractional integrals of a function f with respect to g in $[a, b]$ are given by

$$I_{a^+;g}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t)f(t) dt}{[g(x) - g(t)]^{1-\alpha}}, \quad x > a,$$

and

$$I_{b^-;g}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t)f(t) dt}{[g(t) - g(x)]^{1-\alpha}}, \quad x < b.$$

A more general form of Definition 1.5 is as follows.

Definition 1.6 Let $k > 0$, let (a, b) , $-\infty \leq a < b \leq \infty$, be a finite or infinite interval of the real line \mathfrak{R} , and let $\alpha > 0$. Let g be an increasing and positive monotone function on (a, b) . The left- and right-sided fractional integrals of a function f with respect to g of order $\alpha, k > 0$ in $[a, b]$ are given by

$$I_{a^+;g}^{\alpha,k} f(x) = \frac{1}{\Gamma_k(\alpha)} \int_a^x \frac{g'(t)f(t) dt}{[g(x) - g(t)]^{1-\frac{\alpha}{k}}}, \quad x > a,$$

and

$$I_{b^-;g}^{\alpha,k} f(x) = \frac{1}{\Gamma_k(\alpha)} \int_x^b \frac{g'(t)f(t) dt}{[g(t) - g(x)]^{1-\frac{\alpha}{k}}}, \quad x < b.$$

Now we continue with the definition of Hadamard-type fractional integrals.

Definition 1.7 Let (a, b) be a finite or infinite interval of the half-axis \mathfrak{R}_+ , and $\alpha > 0$. The left-sided and right-sided Hadamard-type fractional integrals of order $\alpha > 0$ are given by

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{y}\right)^{\alpha-1} \frac{f(y) dy}{y}, \quad x > a,$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\log \frac{y}{x}\right)^{\alpha-1} \frac{f(y) dy}{y}, \quad x < b,$$

respectively.

The generalized Hadamard-type fractional integrals are defined as follows.

Definition 1.8 Let (a, b) be a finite or infinite interval of \mathbb{R}^+ , and let $\alpha > 0$. The left- and right-sided Hadamard-type fractional integrals of order $\alpha > 0$ are given by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma_k(\alpha)} \int_a^x \left(\log \frac{x}{y}\right)^{\frac{\alpha}{k}-1} \frac{f(y) dy}{y}, \quad x > a,$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma_k(\alpha)} \int_x^b \left(\log \frac{y}{x}\right)^{\frac{\alpha}{k}-1} \frac{f(y) dy}{y}, \quad x < b,$$

respectively.

Note that Hadamard fractional integrals of order α are a particular case of the left- and right-sided fractional integrals of a function f with respect to the function $g(x) = \log(x)$ in $[a, b]$, where $0 \leq a < b \leq \infty$.

Now we present the definition of the *Erdélyi–Kober-type fractional integrals*. Some of these definitions and results were presented in Samko et al. [16].

Definition 1.9 Let (a, b) ($0 \leq a < b \leq \infty$) be a finite or infinite interval of the half-axis \mathbb{R}^+ . Let $\alpha > 0, \sigma > 0$, and $\eta \in \mathbb{R}$. We consider the left- and right-sided integrals of order $\alpha \in \mathbb{R}$ defined by

$$I_{a+; \sigma; \eta}^{\alpha} f(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{t^{\sigma\eta+\sigma-1} f(t) dt}{(x^{\sigma} - t^{\sigma})^{1-\alpha}} \tag{1.4}$$

and

$$I_{b-; \sigma; \eta}^{\alpha} f(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^b \frac{t^{\sigma(1-\eta-\alpha)-1} f(t) dt}{(t^{\sigma} - x^{\sigma})^{1-\alpha}}, \tag{1.5}$$

respectively. Integrals (1.4) and (1.5) are called the Erdélyi–Kober-type fractional integrals.

2 Main results

The first main result is given in the following:

Theorem 2.1 Let (Δ, Σ, μ) be a measure space with positive σ -finite measure, let $k : \Delta \times \Delta \rightarrow \mathbb{R}$ be a nonnegative function, and let $\Omega \in U(k)$ be a positive function on $[0, \infty)$. Suppose that the exist integrable functions Ψ_1, Ψ_2 on $[0, \infty)$, such that

$$\Psi_1(\xi) \leq \Omega(\xi) \leq \Psi_2(\xi) \tag{2.1}$$

for all $\xi \in [0, \infty)$. Then

$$\overline{\Psi}_1(\xi)\overline{\Omega}(\xi) + \overline{\Psi}_2(\xi)\overline{\Omega}(\xi) \geq \overline{\Psi}_1(\xi)\overline{\Psi}_2(\xi) + \overline{\Omega}(\xi)\overline{\Omega}(\xi). \tag{2.2}$$

Proof Using (2.1), for all $\gamma \geq 0$ and $\delta \geq 0$, we have

$$[\Psi_2(\gamma) - \Omega(\gamma)][\Omega(\delta) - (\Psi_1)(\delta)] \geq 0.$$

Then

$$\Psi_2(\gamma)\Omega(\delta) + \Psi_1(\delta)\Omega(\gamma) \geq \Psi_1(\delta)\Psi_2(\gamma) + \Omega(\gamma)\Omega(\delta). \tag{2.3}$$

Multiplying both sides of (2.3) by $k(\xi, \gamma)$ and integrating the resulting identity with respect to γ over Δ , we get

$$\begin{aligned} \Omega(\delta) \int_{\Delta} k(\xi, \gamma)\Psi_2(\gamma) d\mu(\gamma) + \Psi_1(\delta) \int_{\Delta} k(\xi, \gamma)\Omega(\gamma) d\mu(\gamma) \\ \geq \Psi_1(\delta) \int_{\Delta} k(\xi, \gamma)\Psi_2(\gamma) d\mu(\gamma) + \Omega(\delta) \int_{\Delta} k(\xi, \gamma)\Omega(\gamma) d\mu(\gamma), \end{aligned}$$

which can be written as

$$\Omega(\delta)\overline{\Psi}_2(\xi) + \Psi_1(\delta)\overline{\Omega}(\xi) \geq \Psi_1(\delta)\overline{\Psi}_2(\xi) + \Omega(\delta)\overline{\Omega}(\xi). \tag{2.4}$$

Now multiplying both sides of (2.4) by $k(\xi, \delta)$ and integrating the resulting identity with respect to δ over Δ , we get

$$\overline{\Psi}_1(\xi)\overline{\Omega}(\xi) + \overline{\Psi}_2(\xi)\overline{\Omega}(\xi) \geq \overline{\Psi}_1(\xi)\overline{\Psi}_2(\xi) + \overline{\Omega}(\xi)\overline{\Omega}(\xi).$$

This completes the proof. □

Corollary 2.2 *Let $m, M \in \mathfrak{N}$, with $m < M$, and let $\xi > 0$. Let Ω be a positive function such that $m \leq \Omega(\xi) \leq M$. Then*

$$m\Theta(\xi)\overline{\Omega}(\xi) + M\Theta(\xi)\overline{\Omega}(\xi) \geq mM\Theta(\xi)\Theta(\xi) + \overline{\Omega}(\xi)\overline{\Omega}(\xi),$$

where

$$\Theta(\xi) = \int_{\Delta} k(\xi, \gamma) d\mu_2(\gamma), \quad \xi \in \Delta. \tag{2.5}$$

Remark 2.3 Applying Theorem 2.1 and Corollary 2.2 with $\Delta = (a, b)$, $d\mu(\gamma) = d\gamma$, and $d\mu(\delta) = d\delta$, we have

$$k(x, y) = \begin{cases} \frac{g'(y)}{k\Gamma_k(\alpha)(g(x)-g(y))^{1-\frac{\alpha}{k}}}, & a \leq y \leq x, \\ 0, & x < y \leq b. \end{cases} \tag{2.6}$$

We get that $\Theta(x) = \frac{1}{\Gamma_k(\alpha+k)}(g(x) - g(a))^{\frac{\alpha}{k}}$. Substituting $\overline{\Psi}_1 = I_{a+;g}^{\alpha,k} \Psi_1$, $\overline{\Psi}_2 = I_{a+;g}^{\alpha,k} \Psi_2$, and $\overline{\Omega} = I_{a+;g}^{\alpha,k} \Omega$, we get Theorem 2.1 and Corollary 2.3 of [15], respectively. In particular, taking $k = 1$, we get Theorem 2.11 and Corollary 2.14 of [8], respectively.

Remark 2.4 Applying Theorem 2.1 and Corollary 2.2 with $\Delta = (a, b)$, $d\mu(\gamma) = d\gamma$, and $d\mu(\delta) = d\delta$, we have

$$k(x, y) = \begin{cases} \frac{1}{k\Gamma_k(\alpha)(x-y)^{1-\frac{\alpha}{k}}}, & a \leq y \leq x, \\ 0, & x < y \leq b. \end{cases} \tag{2.7}$$

We get that $\Theta(x) = \frac{1}{\Gamma_k(\alpha+k)}(x - a)^{\frac{\alpha}{k}}$. Substituting $\overline{\Psi}_1 = I_{a+}^{\alpha,k} \Psi_1$, $\overline{\Psi}_2 = I_{a+}^{\alpha,k} \Psi_2$, and $\overline{\Omega} = I_{a+}^{\alpha,k} \Omega$ in Corollary 2.2, we get Corollary 2.2 of [15], respectively. Moreover, taking $k = 1$, we get Theorem 2 and Corollary 3 of [17], respectively.

Remark 2.5 Applying Theorem 2.1 and Corollary 2.2 with $\Delta = (a, b)$, $d\mu(\gamma) = d\gamma$, and $d\mu(\delta) = d\delta$, we have

$$k(x, y) = \begin{cases} \frac{1}{y^k \Gamma_k(\alpha)(\log x - \log y)^{1-\frac{\alpha}{k}}}, & a \leq y \leq x, \\ 0, & x < y \leq b. \end{cases} \tag{2.8}$$

We get that $\Theta(\xi) = \frac{1}{\Gamma_k(\alpha+k)}(\log \xi - \log a)^{\frac{\alpha}{k}}$. Substituting $\overline{\Psi}_1 = J_{a+}^{\alpha} \Psi_1$, $\overline{\Psi}_2 = J_{a+}^{\alpha} \Psi_2$, and $\overline{\Omega} = J_{a+}^{\alpha} \Omega$, we get the results for the Hadamard-type fractional integrals.

Remark 2.6 Applying Theorem 2.1 and Corollary 2.2 with $\Delta = (a, b)$, $d\mu(\gamma) = d\gamma$, and $d\mu(\delta) = d\delta$, we have

$$k(x, y) = \begin{cases} \frac{1}{\Gamma(\alpha)} \frac{\sigma x^{-\sigma(\alpha+\eta)}}{(x^\sigma - y^\sigma)^{1-\alpha}} y^{\sigma\eta+\sigma-1}, & a \leq y \leq x, \\ 0, & x < y \leq b. \end{cases} \tag{2.9}$$

We get that $\Theta(x) = \frac{1}{\Gamma(\alpha+1)}(1 - (\frac{a}{x})^\sigma)_2^{\alpha} F_1(\alpha, -\eta; \alpha + 1; 1 - (\frac{a}{x})^\sigma)$. Substituting $\overline{\Psi}_1 = I_{a+;\sigma;\eta}^{\alpha} \Psi_1$, $\overline{\Psi}_2 = I_{a+;\sigma;\eta}^{\alpha} \Psi_2$, and $\overline{\Omega} = I_{a+;\sigma;\eta}^{\alpha} \Omega$, we get the results for Erdélyi–Kober fractional integral.

Theorem 2.7 Let (Δ, Σ, μ) be a measure space with positive σ -finite measure, let $k : \Delta \times \Delta \rightarrow \mathbb{R}$ be a non-negative function, let $\Omega, \Psi_1, \Psi_2, \varphi_1, \varphi_2, \Upsilon \in U(k)$, and let Ω and Υ be positive functions on $[0, \xi]$. Suppose that (2.1) holds and there exist integrable functions φ_1 and φ_2 on $[0, \xi]$ such that

$$\varphi_1(\xi) \leq \Upsilon(\xi) \leq \varphi_1(\xi). \tag{2.10}$$

Then the following inequalities hold:

- (a) $\overline{\varphi}_1(\xi)\overline{\Omega}(\xi) + \overline{\Psi}_2(\xi)\overline{\Upsilon}(\xi) \geq \overline{\varphi}_2(\xi)\overline{\Psi}_2(\xi) + \overline{\Omega}(\xi)\overline{\Upsilon}(\xi),$
- (b) $\overline{\Psi}_1(\xi)\overline{\Upsilon}(\xi) + \overline{\varphi}_2(\xi)\overline{\Omega}(\xi) \geq \overline{\Psi}_1(\xi)\overline{\varphi}_2(\xi) + \overline{\Omega}(\xi)\overline{\Upsilon}(\xi),$
- (c) $\overline{\Psi}_2(\xi)\varphi_2(\xi) + \overline{\Omega}(\xi)\overline{\Upsilon}(\xi) \geq \overline{\Psi}_2(\xi)\overline{\Upsilon}(\xi)_2(\xi) + \overline{\Omega}(\xi)\overline{\varphi}(\xi),$
- (d) $\overline{\Psi}_1(\xi)\overline{\varphi}_1(\xi) + \overline{\Omega}(\xi)\overline{\Upsilon}(\xi) \geq \overline{\Psi}_1(\xi)\overline{\Upsilon}(\xi) + \overline{\varphi}_1(\xi)\overline{\Omega}(\xi).$

Proof For all $\xi \in [0, \infty)$, from (2.1) and (2.10) it follows that

$$[\Psi_2(\gamma) - \Omega(\gamma)][(\Upsilon(\delta) - \varphi_1(\delta))] \geq 0.$$

Then

$$\Psi_2(\gamma)(\Upsilon(\delta) + \varphi_1(\delta)\Omega(\gamma) \geq \varphi_1(\delta)\Psi_2(\gamma) + \Omega(\gamma)\Upsilon(\delta).$$

Multiplying both sides by $k(\xi, \gamma)$ and integrating the resulting identity with respect to γ over the interval Δ , we have that

$$\begin{aligned} &\Upsilon(\delta) \int_{\Delta} k(\xi, \gamma)\Psi_2(\gamma) d\mu(\gamma) + \varphi_1(\delta) \int_{\Delta} k(\xi, \gamma)\Omega(\gamma) d\mu(\gamma) \\ &\geq \varphi_1(\delta) \int_{\Delta} k(\xi, \gamma)\Psi_2(\gamma) d\mu(\gamma) + \Upsilon(\delta) \int_{\Delta} k(\xi, \gamma)\Omega(\gamma) d\mu(\gamma), \end{aligned}$$

which can be written as

$$\Upsilon(\delta)\overline{\Psi}_2(\xi) + \varphi_1(\delta)\overline{\Omega}(\xi) \geq \varphi_1(\delta)\overline{\Psi}_2(\xi) + \Upsilon(\delta)\overline{\Omega}(\xi).$$

Again multiplying both sides by $k(\xi, \delta)$ and integrating the resulting identity with respect to δ over Δ , we have that

$$\overline{\varphi}_1(\xi)\overline{\Omega}(\xi) + \overline{\Psi}_2(\xi)\overline{\Upsilon}(\xi) \geq \overline{\varphi}_1(\xi)\overline{\Psi}_2(\xi) + \overline{\Omega}(\xi)\overline{\Upsilon}(\xi)$$

This completes the proof of part (a).

To prove parts (b)–(d), we will use the following inequalities:

$$\begin{aligned} &(\varphi_2(\gamma) - \Upsilon(\gamma))(\Omega(\delta) - \Psi_1(\delta)) \geq 0, \\ &(\Psi_2(\gamma) - \Omega(\gamma))(\Upsilon(\delta) - \varphi_2(\delta)) \geq 0, \\ &(\Psi_1(\gamma) - \Omega(\gamma))(\Upsilon(\delta) - \varphi_1(\delta)) \geq 0. \end{aligned}$$

□

Corollary 2.8 *Let the assumptions of Theorem 2.7 be satisfied. Suppose that there exist real constants m, M, n, N such that $m \leq \Omega(\xi) \leq M$ and $n \leq \Upsilon(\xi) \leq N$ for all $\xi \in [0, \infty)$. Then*

- (a) $n\Theta(\xi)\overline{\Omega}(\xi) + M\Theta(\xi)\overline{\Upsilon}(\xi) \geq nM\Theta(\xi)\Theta(\xi) + \overline{\Omega}(\xi)\overline{\Upsilon}(\xi),$
- (b) $m\Theta(\xi)\Upsilon(\xi) + N\Theta(\xi)\Omega(\xi) \geq mN\Theta(\xi)m\Theta(\xi) + \overline{\Omega}(\xi)\overline{\Upsilon}(\xi),$
- (c) $NM\Theta(\xi)\Theta(\xi) + \overline{\Omega}(\xi)\overline{\Upsilon}(\xi) \geq M\Theta(\xi)\overline{\Upsilon}(\xi) + N\Theta(\xi)\overline{\Omega}(\xi),$
- (d) $nm\Theta(\xi)\Theta(\xi)\overline{\varphi}_1(\xi) + \overline{\Omega}(\xi)\overline{\Upsilon}(\xi) \geq m\Theta(\xi)\overline{\Upsilon}(\xi) + nm\Theta(\xi)\overline{\Omega}(\xi),$

where $\Theta(\xi)$ is defined by (2.5).

Remark 2.9 Choosing the kernel $k(x, y)$ defined by (2.6) and substituting $\overline{\Psi}_1 = I_{a+;g}^{\alpha,k} \Psi_1$, $\overline{\Psi}_2 = I_{a+;g}^{\alpha,k} \Psi_2$, and $\overline{\Omega} = I_{a+;g}^{\alpha,k} \Omega$ in Theorem 2.7 and Corollary 2.8, we get Theorem 2.5 and

Corollary 2.6 of [15], respectively. In particular, taking $k = 1$, we get Theorem 2.15 and Corollary 2.16 of [8], respectively.

Remark 2.10 Choosing the kernel $k(x, y)$ defined by (2.7) and substituting $\overline{\Psi}_1 = I_{a^+}^{\alpha, k} \Psi_1$, $\overline{\Psi}_2 = I_{a^+}^{\alpha, k} \Psi_2$, and $\overline{\Omega} = I_{a^+}^{\alpha, k} \Omega$ in Theorem 2.7 and Corollary 2.8, we get Theorem 2.5 and Corollary 2.6 of [15], respectively. In particular, by taking $k = 1$ Theorem 2.7 leads to Theorem 2.15 of [17].

Remark 2.11 Choosing the kernel $k(x, y)$ defined by (2.8) and $\Theta(\xi) = \frac{1}{\Gamma_k(\alpha+k)} (\log \xi - \log a)^{\frac{\alpha}{k}}$ and substituting $\overline{\Psi}_1 = J_{a^+}^{\alpha} \Psi_1$, $\overline{\Psi}_2 = J_{a^+}^{\alpha} \Psi_2$, and $\overline{\Omega} = J_{a^+}^{\alpha} \Omega$ in Theorem 2.7 and Corollary 2.8, we get the inequalities for Haadnard-type fractional integrals.

Remark 2.12 Choosing the kernel $k(x, y)$ defined by (2.9) and

$$\Theta(\xi) = \frac{1}{\Gamma(\alpha + 1)} \left(1 - \left(\frac{a}{x} \right)^{\sigma} \right)^{\alpha} F_1 \left(\alpha, -\eta; \alpha + 1; 1 - \left(\frac{a}{x} \right)^{\sigma} \right)$$

and substituting $\overline{\Psi}_1 = (I_{a^+; \sigma; \eta}^{\alpha} \Psi_1)$, $\overline{\Psi}_2 = (I_{a^+; \sigma; \eta}^{\alpha} \Psi_2)$, and $\overline{\Omega} = (I_{a^+; \sigma; \eta}^{\alpha} \Omega)$ in Theorem 2.7 and Corollary 2.8, we get the result for the Erdélyi–Kober fractional integral.

Lemma 2.13 *Let (Δ, Σ, γ) be a measure space with positive σ -finite measure, let $k : \Delta \times \Delta \rightarrow \mathbb{R}$ be a nonnegative function, and let $\Omega, \Psi_1, \Psi_2 \in U(k)$ be such that Ω is positive on $[0, \infty)$ and Ψ_1, Ψ_2 are integrable on $[0, \infty)$. Then*

$$\begin{aligned} & \Theta(\xi) \overline{\Omega^2}(\xi) - (\overline{\Omega}(\xi))^2 \\ &= (\overline{\Psi}_2(\xi) - \overline{\Omega}(\xi))(\overline{\Omega}(\xi) - \overline{\Psi}_1(\xi)) - \Theta(\xi) (\overline{\Psi}_2(\xi) - \overline{\Omega}(\xi)) (\overline{\Omega}(\xi) - \overline{\Psi}_1(\xi)) \\ &+ \Theta(\xi) \overline{\Psi_1(\xi) \Omega(\xi)} - \overline{\Psi_1(\xi) \overline{\Omega}(\xi)} + \Theta(\xi) \overline{\Psi_2(\xi) \Omega(\xi)} - \overline{\Psi_2(\xi) \overline{\Omega}(\xi)} \\ &- \Theta(\xi) \overline{\Psi_1(\xi) \Psi_2(\xi)} + \overline{\Psi_1(\xi) \overline{\Psi}_2(\xi)}, \end{aligned}$$

where $\Theta(\xi)$ is defined by (2.5).

Proof Since $\gamma, \delta > 0$, we have

$$\begin{aligned} & (\Psi_2(\delta) - \Omega(\delta))(\Omega(\gamma) - \Psi_1(\gamma)) + (\Psi_2(\gamma) - \Omega(\gamma))(\Omega(\delta) - \Psi_1(\delta)) \\ & - (\Psi_2(\gamma) - \Omega(\gamma))(\Omega(\delta) - \Psi_1(\delta)) - (\Psi_2(\delta) - \Omega(\delta))(\Omega(\delta) - \Psi_1(\delta)) \\ &= \Omega^2(\gamma) + (\Omega(\delta))^2 - 2\Omega(\gamma)\Omega(\delta) + \Psi_2(\delta)\Omega(\gamma) + \Psi_1(\gamma)\Omega(\delta) \\ & - \Psi_1(\gamma)\Psi_2(\delta) + \Psi_2(\delta)\Omega(\delta) + \Psi_1(\delta)\Omega(\gamma) - \Psi_1(\delta)\Psi_2(\gamma) \\ & - \Psi_2(\delta)\Omega(\gamma) + \Psi_1(\gamma)\Psi_2(\gamma) - \Psi_1(\gamma)\Omega(\gamma) - \Psi_2(\delta)\Omega(\delta) \\ & + \Psi_1(\delta)\Psi_2(\delta) - \Psi_1(\delta)\Omega(\delta). \end{aligned}$$

Multiplying both sides by $k(\xi, \gamma)$ and integrating with respect to the variable γ over Δ , we get

$$(\Psi_2(\delta) - \Omega(\delta))(\overline{\Omega}(\xi) - \overline{\Psi}_1(\xi)) + (\overline{\Psi}_2(\xi) - \overline{\Omega}(\xi))(\Omega(\delta) - \Psi_1(\delta))$$

$$\begin{aligned}
 & - \overline{(\Psi_2(\gamma) - \Omega(\gamma))(\Omega(\delta) - \Psi_1(\delta))} - (\Psi_2(\delta) - \Omega(\delta))(\Omega(\delta) - \Psi_1(\delta))\Theta(\xi) \\
 = & \overline{\Omega^2(\gamma)} + \Theta(\xi)\Omega^2(\delta) - 2\overline{\Omega(\xi)\Omega(\delta)} + \Psi_2(\delta)\overline{\Omega(\xi)} + \overline{\Psi_1(\xi)\Omega(\delta)} \\
 & - \overline{\Psi_1(\xi)\Psi_2(\delta)} + \overline{\Psi_2(\xi)\Omega(\delta)} + \Psi_1(\delta)\overline{\Omega(\xi)} - \Psi_1(\delta)\overline{\Psi_2(\xi)} \\
 & - \overline{\Psi_2(\xi)\Omega(\xi)} + \overline{\Psi_1(\xi)\Psi_2(\xi)}\Theta(\xi) - \overline{\Psi_1(\xi)\Omega(\xi)} - \Psi_2(\delta)\Omega(\delta)\Theta(\xi) \\
 & + \Theta(\xi)\Psi_1(\delta)\Psi_2(\delta) - \Psi_1(\delta)\Omega(\delta)\Theta(\xi).
 \end{aligned}$$

Again multiplying both sides by $k(\xi, \delta)$ and integrating with respect to the variable δ over Δ , we get

$$\begin{aligned}
 & (\overline{\Psi_2(\xi)} - \overline{\Omega(\xi)})(\overline{\Omega(\xi)} - \overline{\Psi_1(\xi)}) + (\overline{\Psi_2(\xi)} - \overline{\Omega(\xi)})(\overline{\Omega(\xi)} - \overline{\Psi_1(\xi)}) \\
 & - \Theta(\xi)\overline{(\Psi_2(\xi) - \Omega(\xi))(\Omega(\xi) - \Psi_1(\xi))} - (\overline{\Psi_2(\xi)} - \overline{\Omega(\xi)})(\overline{\Omega(\xi)} - \overline{\Psi_1(\xi)})\Theta(\xi) \\
 = & \overline{\Omega^2(\xi)}\Theta(\xi) + \Theta(\xi)\Omega^2(\xi) - 2\overline{\Omega(\xi)\Omega(\xi)} + \overline{\Psi_2(\xi)\Omega(\xi)} + \overline{\Psi_1(\gamma)\Omega(\xi)} \\
 & - \overline{\Psi_1(\xi)\Psi_2(\xi)} + \overline{\Psi_2(\xi)\Omega(\xi)} + \overline{\Psi_1(\xi)\Omega(\xi)} - \overline{\Psi_1(\xi)\Psi_2(\xi)} \\
 & - \Theta(\xi)\overline{\Psi_2(\xi)\Omega(\xi)} + \overline{\Psi_1(\xi)\Psi_2(\xi)}\Theta(\xi) - \Theta(\xi)\overline{\Psi_1(\xi)\Omega(\xi)} - \overline{\Psi_2(\xi)\Omega(\xi)}\Theta(\xi) \\
 & + \Theta(\xi)\overline{\Psi_1(\xi)\Psi_2(\xi)} - \overline{\Psi_1(\xi)\Omega(\xi)}\Theta(\xi).
 \end{aligned}$$

This completes the proof. □

Corollary 2.14 *Let $m < M, k > 0$, and let Ω be a positive function on $[0, \xi]$ such that $m \leq \Omega(\xi) \leq M$. Then*

$$\begin{aligned}
 & \Theta(\xi)(\overline{\Omega^2(\xi)}) - (\overline{\Omega(\xi)})^2 \\
 = & (M\Theta(\xi) - \overline{\Omega(\xi)})(\overline{\Omega(\xi)} - m\Theta(\xi)) - \Theta(\xi)((M - \Omega(\xi))(\Omega(\xi) - m)),
 \end{aligned}$$

where $\Theta(\xi)$ is defined by (2.5).

Remark 2.15 Taking the kernel $k(x, y)$ defined by (2.6) and substituting $\overline{\Psi_1} = I_{a+g}^{\alpha,k} \Psi_1$, $\overline{\Psi_2} = I_{a+g}^{\alpha,k} \Psi_2$, $\overline{\Omega} = I_{a+g}^{\alpha,k} \Omega$, $\overline{\Psi_1\Omega} = I_{a+g}^{\alpha,k} \Psi_1\Omega$, $\overline{\Psi_2\Omega} = I_{a+g}^{\alpha,k} \Psi_2\Omega$, $\overline{\Psi_1\Psi_2} = I_{a+g}^{\alpha,k} \Psi_1\Psi_2$, and $\overline{(\Psi_2(\xi) - \Omega(\xi))(\Omega(\xi) - \Psi_1(\xi))} = I_{a+g}^{\alpha,k} (\Psi_2(\xi) - \Omega(\xi))(\Omega(\xi) - \Psi_1(\xi))$ in Theorem 2.13 and Corollary 2.14, we get Lemma 2.9 and Corollary 2.11 of [15], respectively. In particular, by taking $k = 1$ we get Theorem 2.19 and Corollary 2.11 of [8], respectively.

Remark 2.16 Taking the kernel $k(x, y)$ defined by (2.7) and substituting $\overline{\Psi_1} = I_{a^+}^{\alpha,k} \Psi_1$, $\overline{\Psi_2} = I_{a^+}^{\alpha,k} \Psi_2$, $\overline{\Omega} = I_{a^+}^{\alpha,k} \Omega$, $\overline{\Psi_1\Omega} = I_{a^+}^{\alpha,k} \Psi_1\Omega$, $\overline{\Psi_2\Omega} = I_{a^+}^{\alpha,k} \Psi_2\Omega$, $\overline{\Psi_1\Psi_2} = I_{a^+}^{\alpha,k} \Psi_1\Psi_2$, and $\overline{(\Psi_2(\xi) - \Omega(\xi))(\Omega(\xi) - \Psi_1(\xi))} = I_{a^+}^{\alpha,k} (\Psi_2(\xi) - \Omega(\xi))(\Omega(\xi) - \Psi_1(\xi))$ in Lemma 2.13, we get Corollary 2.10 of [15]. In particular, by taking $k = 1$ Lemma 2.13 and Corollary 2.14 become Lemma 7 and Corollary 8 of [17], respectively.

Remark 2.17 Taking the kernel $k(x, y)$ defined by (2.8) and substituting $\Theta(\xi) = \frac{1}{\Gamma_k(\alpha+k)} \times (\log \xi - \log a)^{\frac{\alpha}{k}}$, $\overline{\Psi_1} = J_{a^+}^{\alpha} \Psi_1$, $\overline{\Psi_2} = J_{a^+}^{\alpha} \Psi_2$, $\overline{\Omega} = J_{a^+}^{\alpha} \Omega$, and $\overline{(\Psi_2(\xi) - \Omega(\xi))(\Omega(\xi) - \Psi_1(\xi))} = J_{a^+}^{\alpha} (\Psi_2(\xi) - \Omega(\xi))(\Omega(\xi) - \Psi_1(\xi))$ in Theorem 2.7 and Corollary 2.8, we get the inequalities for Hadamard fractional integrals.

Remark 2.18 Taking the kernel $k(x, y)$ defined by (2.9) and substituting $\Theta(x) = \frac{1}{\Gamma(\alpha+1)}(1 - (\frac{x}{\xi})^\alpha)_2 F_1(\alpha, -\eta; \alpha + 1; 1 - (\frac{x}{\xi})^\alpha)$, $\overline{\Psi}_1 = I_{a+; \sigma; \eta}^\alpha \Psi_1$, $\overline{\Psi}_2 = I_{a+; \sigma; \eta}^\alpha \Psi_2$, $\overline{\Omega} = I_{a+; \sigma; \eta}^\alpha \Omega$, and $(\Psi_2(\xi) - \Omega(\xi))(\overline{\Omega}(\xi) - \overline{\Psi}_1(\xi)) = I_{a+; \sigma; \eta}^\alpha (\Psi_2(\xi) - \Omega(\xi))(\overline{\Omega}(\xi) - \overline{\Psi}_1(\xi))$ in Theorem 2.7 and Corollary 2.8, we get the result for the Erdélyi–Kober fractional integral.

Theorem 2.19 *Let (Δ, Σ, γ) be a measure spaces with positive σ -finite measure, let $k : \Delta \times \Delta \rightarrow \mathbb{R}$ be a nonnegative function, and let $\Omega, \Psi_1, \Psi_2, \varphi_1, \varphi_2, \Upsilon \in U(k)$ be integrable functions on $[0, \xi]$. If conditions (2.1) and (2.10) are satisfied, then*

$$|\Theta(\xi)\overline{\Omega(\xi)\Upsilon(\xi)} - \overline{\Omega(\xi)\Upsilon(\xi)}| \leq \sqrt{T(\Omega, \Psi_1, \Psi_2)T(\Upsilon, \varphi_1, \varphi_2)}, \tag{2.11}$$

where

$$T(\Omega, \Psi_1, \Psi_2) = (\overline{\Psi}_2(\xi) - \overline{\Omega}(\xi))(\overline{\Omega}(\xi) - \overline{\Psi}_1(\xi)) + \Theta(\xi)\overline{\Psi_1(\xi)\Omega(\xi)} - \overline{\Psi_1(\xi)\Omega(\xi)} \\ + \Theta(\xi)\overline{\Psi_2(\xi)\Omega(\xi)} - \overline{\Psi_2(\xi)\Omega(\xi)} + \overline{\Psi_1(\xi)\Psi_2(\xi)} - \Theta(\xi)\overline{\Psi_1(\xi)\Psi_2(\xi)},$$

and

$$T(\Upsilon, \varphi_1, \varphi_2) = (\overline{\varphi}_2(\xi) - \overline{\Upsilon}(\xi))(\overline{\Upsilon}(\xi) - \overline{\varphi}_1(\xi)) + \Theta(\xi)\overline{\varphi_1(\xi)\Upsilon(\xi)} - \overline{\varphi_1(\xi)\Upsilon(\xi)} \\ + \Theta(\xi)\overline{\varphi_2(\xi)\Upsilon(\xi)} - \overline{\varphi_2(\xi)\Upsilon(\xi)} + \overline{\varphi_1(\xi)\varphi_2(\xi)} - \Theta(\xi)\overline{\varphi_1(\xi)\varphi_2(\xi)},$$

with $\Theta(\xi)$ defined by (2.5).

Proof Let $\xi > 0$, $\gamma, \delta \in [0, \xi]$, let Ω, Υ be two positive functions on $[0, \infty)$ such that conditions (2.1) and (2.10) are satisfied, and let $T(\gamma, \delta)$ be defined by

$$T(\gamma, \delta) = (\Omega(\gamma) - \Omega(\delta))(\Upsilon(\gamma) - \Upsilon(\delta)). \tag{2.12}$$

Multiplying both sides (2.12) by $\frac{1}{2}k(\xi, \gamma)k(\xi, \delta)$ and integrating with respect to the variables γ and δ over Δ and Δ , we get

$$\frac{1}{2} \int_{\Delta} \int_{\Delta} k(\xi, \delta)k(\xi, \gamma)T(\gamma, \delta) d\mu(\gamma) d\mu(\delta) = \Theta(\xi)\overline{\Omega(\xi)\Upsilon(\xi)} - \overline{\Omega(\xi)\Upsilon(\xi)}. \tag{2.13}$$

Applying the Cauchy–Schwarz inequality, we get

$$\left(\frac{1}{2} \int_{\Delta} \int_{\Delta} k(\xi, \gamma)k(\xi, \delta)(\Omega(\gamma) - \Omega(\delta))(\Upsilon(\gamma) - \Upsilon(\delta)) d\mu(\gamma) d\mu(\delta)\right)^2 \\ \leq \int_{\Delta} \int_{\Delta} k(\xi, \gamma)k(\xi, \delta)(\Omega(\gamma) - \Omega(\delta))^2 d\mu(\gamma) d\mu(\delta) \\ \times \int_{\Delta} \int_{\Delta} k(\xi, \gamma)k(\xi, \delta)(\Upsilon(\gamma) - \Upsilon(\delta))^2 d\mu(\gamma) d\mu(\delta). \tag{2.14}$$

From (2.13) and (2.14) we get

$$(\Theta(\xi)\overline{\Omega(\xi)\Upsilon(\xi)} - \overline{\Omega(\xi)\Upsilon(\xi)})^2 \leq (\Theta(\xi)\overline{\Omega^2(\xi)} - \overline{\Omega(\xi)^2})(\Theta(\xi)\overline{\Upsilon^2(\xi)} - \overline{\Upsilon(\xi)^2}).$$

Since $(\Psi_2(\xi) - \Omega(\xi))(\Omega(\xi) - \Psi_1(\xi)) \geq 0$ and $(\varphi_2(\xi) - \Upsilon(\xi))(\Upsilon(\xi) - \varphi_1(\xi)) \geq 0$, we have

$$\Theta(\xi) \overline{(\Psi_2(\xi) - \Omega(\xi))(\Omega(\xi) - \Psi_1(\xi))} \geq 0$$

and

$$\Theta(\xi) \overline{(\varphi_2(\xi) - \Upsilon(\xi))(\Upsilon(\xi) - \varphi_1(\xi))} \geq 0.$$

Thus from Lemma 2.13 we have

$$\begin{aligned} & \Theta(\xi) \overline{\Omega^2(\xi)} - (\overline{\Omega(\xi)})^2 \\ & \leq (\overline{\Psi_2(\xi)} - \overline{\Omega(\xi)})(\overline{\Omega(\xi)} - \overline{\Psi_1(\xi)}) + \Theta(\xi) \overline{\Psi_1(\xi)\Omega(\xi)\Psi_1(\xi)\Omega(\xi)} \\ & \quad + \Theta(\xi) \overline{\Psi_2(\xi)\Omega(\xi)} - \overline{\Psi_2(\xi)\Omega(\xi)} + \overline{\Psi_1(\xi)\Psi_2(\xi)} - \Theta(\xi) \overline{\Psi_1(\xi)\Psi_2(\xi)} \\ & = T(\Omega, \Psi_1, \Psi_2) \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} & \Theta(\xi) \overline{\Upsilon^2(\xi)} - (\overline{\Upsilon(\xi)})^2 \\ & \leq (\overline{\varphi_2(\xi)} - \overline{\Upsilon(\xi)})(\overline{\Upsilon(\xi)} - \overline{\varphi_1(\xi)}) + \Theta(\xi) \overline{\varphi_1(\xi)\Upsilon(\xi)} - \overline{\varphi_1(\xi)\Upsilon(\xi)} \\ & \quad + \Theta(\xi) \overline{\varphi_2(\xi)\Upsilon(\xi)} - \overline{\varphi_2(\xi)\Upsilon(\xi)} + \overline{\varphi_1(\xi)\varphi_2(\xi)} - \Theta(\xi) \overline{\varphi_1(\xi)\varphi_2(\xi)} \\ & = T(\Upsilon, \varphi_1, \varphi_2). \end{aligned} \tag{2.16}$$

Therefore inequality (2.11) follows from (2.15) and (2.16). This completes the proof. \square

Corollary 2.20 *Let $m, M, n, N \in \mathfrak{N}$, $T(\Omega, \Psi_1, \Psi_2) = T(\Omega, m, M)$, and $T(\Upsilon, \varphi_1, \varphi_2) = T(\Upsilon, n, N)$. Then inequality (2.11) reduces to*

$$|\Theta(\xi) \overline{\Omega(\xi)\Upsilon(\xi)} - \overline{\Omega(\xi)\Upsilon(\xi)}| \leq (\Theta(\xi))^2 (M - m)(N - n).$$

Remark 2.21 Taking the kernel $k(x, y)$ defined by (2.6) and substituting $\overline{\Omega} = I_{a+;g}^{\alpha,k} \Omega$, $\overline{\Upsilon} = I_{a+;g}^{\alpha,k} \Upsilon$, and $\overline{\Omega\Upsilon} = I_{a+;g}^{\alpha,k} \Omega\Upsilon$ in Theorem 2.19 and Corollary 2.20, we get Theorem 2.13 and Corollary 2.14 of [15], respectively. In particular, choosing $k = 1$, Theorem 2.13 and Corollary 2.14 lead to Theorem 2.23 and Corollary 2.26 of [8], respectively.

Remark 2.22 Taking the kernel $k(x, y)$ defined by (2.7) and substituting $\overline{\Omega}_1 = I_{a+}^{\alpha,k} \Omega$, $\overline{\Upsilon} = I_{a+}^{\alpha,k} \Upsilon$, and $\overline{\Omega\Upsilon} = I_{a+}^{\alpha,k} \Omega\Upsilon$ in Theorem 2.19 and Corollary 2.20, we get the results for the Riemann–Liouville integral. In particular, taking $k = 1$, Theorem 2.19 gives Theorem 9 of [17], and Corollary 2.20 gives Remark 10 of [17].

Remark 2.23 Taking the kernel $k(x, y)$ defined by (2.8) and substituting $\Theta(\xi) = \frac{1}{\Gamma_k(\alpha+k)} \times (\log \xi - \log a)^{\frac{\alpha}{k}}$, $\overline{\Omega} = J_{a+}^{\alpha} \Omega$, $\overline{\Upsilon} = J_{a+}^{\alpha} \Upsilon$, and $\overline{\Omega\Upsilon} = J_{a+}^{\alpha} \Omega\Upsilon$ in Theorem 2.19 and Corollary 2.20, we get the inequalities for Hadamard fractional integrals.

Remark 2.24 Taking the kernel $k(x, y)$ defined by (2.9) and substituting $\Theta(x) = \frac{1}{\Gamma(\alpha+1)}(1 - (\frac{x}{x})^\alpha)_2 F_1(\alpha, -\eta; \alpha + 1; 1 - (\frac{x}{x})^\alpha)$, $\overline{\Psi}_1 = (I_{a+; \sigma; \eta}^\alpha \Psi_1)$, $\overline{\Psi}_2 = (I_{a+; \sigma; \eta}^\alpha \Psi_2)$, and $\overline{\Omega} = (I_{a+; \sigma; \eta}^\alpha \Omega)$ in Theorem 2.19 and Corollary 2.20, we get the result for the Erdélyi–Kober-type fractional integral.

Theorem 2.25 Let $k > 0$, and let Ω and Υ be positive functions defined on $[0, \infty)$. Then the following inequalities hold:

1. $q\overline{\Omega^p}(\xi) + p\overline{\Upsilon^q}(\xi) \geq pq \frac{1}{\Theta(\xi)} \overline{\Upsilon(\xi)\overline{\Omega}}(\xi),$
2. $q\overline{\Omega^p}(\xi)\overline{\Upsilon^p}(\xi) + p\overline{\Omega^q}(\xi)\overline{\Upsilon^q}(\xi) \geq pq(\overline{\Omega(\xi)\Upsilon(\xi)})^2,$
3. $q\overline{\Omega^p}(\xi)\overline{\Upsilon^p}(\xi) + p\overline{\Omega^q}(\xi)\overline{\Upsilon^q}(\xi) \geq pq\overline{\Omega(\xi)\Upsilon^{p-1}(\xi)\overline{\Omega(\xi)\Upsilon^{q-1}(\xi)}},$
4. $q\overline{\Omega^p}(\xi)\overline{\Upsilon^q}(\xi) + p\overline{\Omega^p}(\xi)\overline{\Upsilon^q}(\xi) \geq pq\overline{\Omega^{p-1}(\xi)\Upsilon^{q-1}(\xi)\overline{\Omega(\xi)\Upsilon(\xi)}},$
5. $q\overline{\Omega^p}(\xi)\overline{\Upsilon^2}(\xi) + p\overline{\Omega^2}(\xi)\overline{\Upsilon^q}(\xi) \geq pq\overline{\Omega(\xi)\Upsilon(\xi)\overline{\Omega^{\frac{2}{q}}(\xi)\Upsilon^{\frac{2}{p}}(\xi)}},$
6. $q\overline{\Omega^2}(\xi)\overline{\Upsilon^q}(\xi) + p\overline{\Omega^p}(\xi)\overline{\Upsilon^2}(\xi) \geq pq\overline{\Omega^{\frac{2}{p}}(\xi)\Upsilon^{\frac{2}{q}}(\xi)\overline{\Omega^{p-1}(\xi)\Upsilon^{q-1}(\xi)}},$
7. $q\overline{\Omega^2}(\xi)\overline{\Upsilon^q}(\xi) + p\overline{\Omega^2}(\xi)\overline{\Upsilon^p}(\xi) \geq pq \frac{1}{\Theta(\xi)} \overline{\Omega^{\frac{2}{p}}(\xi)\Upsilon^{q-1}(\xi)\overline{\Omega^{\frac{2}{q}}(\xi)\Upsilon^{p-1}(\xi)}},$

where $\Theta(\xi)$ is defined by (2.5).

Proof By Young’s inequality we have

$$\frac{a^p}{p} + \frac{a^q}{q} \geq ab \left(a, b \geq 0, p, q > 1, \frac{1}{p} + \frac{1}{q} = 1 \right).$$

Taking $a = \Omega(\gamma)$ and $b = \Upsilon(\delta)$, we have

$$\frac{(\Omega(\gamma))^p}{p} + \frac{(\Upsilon(\delta))^q}{q} \geq \Omega(\gamma)\Upsilon(\delta)$$

for all $\Omega(\gamma), \Upsilon(\delta) \geq 0$.

Multiplying by $k(\xi, \gamma)$ and integrating with respect to γ over the interval Δ , we get

$$\int_{\Delta} k(\xi, \gamma) \frac{\Omega^p(\gamma)}{p} d\mu(\gamma) + \int_{\Delta} k(\xi, \gamma) \frac{\Upsilon^q(\delta)}{q} d\mu(\gamma) \geq \int_{\Delta} k(\xi, \gamma)\Omega(\gamma)\Upsilon(\delta) d\mu(\gamma),$$

which becomes

$$\frac{1}{p}\overline{\Omega^p}(\xi) + \frac{1}{q}\overline{\Upsilon^q}(\delta)\Theta(\xi) \geq \overline{\Upsilon(\delta)\overline{\Omega}}(\xi).$$

Again multiplying by $k(\xi, \delta)$ and integrating with respect to the variable δ over the interval Δ , we get

$$\frac{1}{p}\overline{\Omega^p}(\xi)\Theta(\xi) + \frac{1}{q}\overline{\Upsilon^q}(\xi)\Theta(\xi) \geq \overline{\Upsilon(\xi)\overline{\Omega}}(\xi),$$

which implies that

$$\frac{1}{p} \overline{\Omega^p}(\xi) + \frac{1}{q} \overline{\Upsilon^q}(\delta) \geq \frac{1}{\Theta(\xi)} \overline{\Upsilon}(\xi) \overline{\Omega}(\xi).$$

This completes the proof of part (a).

The remaining inequalities can be proved using Young’s inequality in a similar manner by taking:

2. $a = \Omega(\gamma)\Upsilon(\delta)$ and $b = \Omega(\delta)\Upsilon(\gamma)$,
3. $a = \frac{\Omega(\gamma)}{\Upsilon(\gamma)}$ and $b = \frac{\Omega(\delta)}{\Upsilon(\delta)}$, $\Upsilon(\gamma)\Upsilon(\delta) \neq 0$,
4. $a = \frac{\Omega(\delta)}{\Omega(\gamma)}$ and $b = \frac{\Upsilon(\delta)}{\Upsilon(\gamma)}$, $\Omega(\gamma)\Upsilon(\delta) \neq 0$,
5. $a = \Omega(\gamma)\Upsilon^{\frac{2}{p}}(\delta)$ and $b = \Omega^{\frac{2}{q}}(\delta)\Upsilon(\gamma)$,
6. $a = \frac{\Omega^{\frac{2}{p}}(\gamma)}{\Omega(\delta)}$ and $b = \frac{\Upsilon^{\frac{2}{q}}(\gamma)}{\Upsilon(\gamma)}$, $\Omega(\delta)\Upsilon(\delta) \neq 0$,
7. $a = \frac{\Omega^{\frac{2}{p}}(\gamma)}{\Upsilon(\delta)}$ and $b = \frac{\Omega^{\frac{2}{q}}(\delta)}{\Upsilon(\gamma)}$, $\Upsilon(\gamma)\Upsilon(\delta) \neq 0$. □

Example 2.26 Let $k > 0$, and let $\varphi_2(\gamma)$ be a positive function on $[0, \infty)$, and let $m = \min_{0 \leq \gamma \leq \xi} \frac{\Omega(\gamma)}{\Upsilon(\gamma)}$ and $M = \max_{0 \leq \gamma \leq \xi} \frac{\Omega(\gamma)}{\Upsilon(\gamma)}$. Then we have

$$0 \leq \overline{\Omega^2}(\xi) \overline{\Upsilon^2}(\xi) \leq \frac{(m + M)^2}{4mM} (\overline{\Omega(\xi)\Upsilon(\xi)})^2. \tag{2.17}$$

Proof From the min and max conditions we have that

$$\left(\frac{\Omega(\gamma)}{\Upsilon(\gamma)} - m \right) \left(M - \frac{\Omega(\gamma)}{\Upsilon(\gamma)} \right) \Upsilon^2(\gamma) \geq 0$$

and

$$\Omega^2(\gamma) + mM\Upsilon^2(\gamma) \leq (m + M)\Omega(\gamma)\Upsilon(\gamma).$$

Multiplying by $k(\xi, \gamma)$ and integrating with respect to the variable γ over the interval Δ , we get

$$\begin{aligned} & \int_{\Delta} k(\xi, \gamma) \Omega^2(\gamma) d\mu(\gamma) + mM \int_{\Delta} k(\xi, \gamma) \Upsilon^2(\gamma) d\mu(\gamma) \\ & \leq (m + M) \int_{\Delta} k(\xi, \gamma) \Omega(\gamma) \Upsilon(\gamma) d\mu(\gamma). \end{aligned}$$

This implies that

$$\overline{\Omega^2}(\xi) + mM \overline{\Upsilon^2}(\xi) \leq (m + M) \overline{\Omega(\xi)\Upsilon(\xi)}. \tag{2.18}$$

Alternatively, it follows from

$$(\sqrt{\overline{\Omega^2(\xi)}} - \sqrt{mM\overline{\Upsilon^2(\xi)}})^2 \geq 0$$

that

$$2\sqrt{\overline{\Omega^2(\xi)}}\sqrt{mM\overline{\Upsilon^2(\xi)}} \leq \overline{\Omega^2(\gamma)} + mM\overline{\Upsilon^2(\gamma)}. \quad (2.19)$$

Therefore from (2.18) and (2.19) it follows that

$$4mM\overline{\Omega^2(\xi)}\overline{\Upsilon^2(\xi)} \leq (m + M)^2(\overline{\Omega(\xi)}\overline{\Upsilon(\xi)})^2,$$

and the proof is complete. \square

Remark 2.27 Applications for the discussed fractional integrals can be given, but we omit the details.

3 Concluding remarks

Recently, Rahman [14], Rashid [15], and Kacar [8] studied a broad range of Grüss-type inequalities for different kinds of fractional integrals. Although papers [14, 15], and [8] are connected in the sense that one generalizes another, we observe that there may be a great generalization that covers all possible kinds of fractional integrals mentioned in these papers. Therefore we introduced a special class of transformations that involve general kernels over σ -finite measure and prove all the results. Motivated by the above, we successfully presented certain elegant inequalities, which generalize all the previous results.

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