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Existence of nonoscillatory solutions to nonlinear higher-order neutral dynamic equations

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Abstract

We investigate the existence of different types of nonoscillatory solutions to a class of higher-order nonlinear neutral dynamic equations on a time scale. Two examples are provided to show the significance of the conclusions.

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Keywords: Nonoscillatory solution; Nonlinear neutral dynamic equation; Higher-order; Time scale

1 Introduction

The time scale theory has been introduced and developed rapidly since 1988; see, for instance, [1–4, 7, 8]. Afterwards, many scholars were concerned with the oscillation of dynamic equations on time scales and they obtained abundant achievements. Besides, some research on the existence and asymptotic behavior of nonoscillatory solutions to dynamic equations on time scales has been also improved recently, we refer the reader to [5, 6, 9–15].

Since 2007, numerous researchers have investigated the existence of nonoscillatory solutions to several classes of nonlinear neutral dynamic equations

$$[x(t) + p(t)x(g(t))]^\Delta + f(t, x(h(t))) = 0, \quad (1)$$

$$[r(t)(x(t) + p(t)x(g(t)))^\Delta]^\Delta + f(t, x(h(t))) = 0, \quad (2)$$

and

$$(r_1(t)(r_2(t)(x(t) + p(t)x(g(t)))^\Delta)^\Delta + f(t, x(h(t)))) = 0, \quad (3)$$

successively. Zhu and Wang [15] studied (1) and presented the existence of nonoscillatory solutions by using a Banach space and Krasnoselskii's fixed point theorem. Actually, the authors gave a general way to establish the existence of nonoscillatory solutions to (1). Inspired by [15], Gao and Wang [6] were concerned with (2) under the condition

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$\int_{t_0}^{\infty} 1/r(t)\Delta t < \infty$. Similar to the results in [15], there are two types of asymptotic behavior of eventually positive solutions to (2). Later on, Deng and Wang [5] considered (2) with another condition $\int_{t_0}^{\infty} 1/r(t)\Delta t = \infty$, and summarized four types of eventually positive solutions to (2). It is clear to see that the asymptotic behavior of eventually positive solutions in [5] are more complex than that of [6]. From [5, 6], we can see the fact that the existence and asymptotic behavior of nonoscillatory solutions are greatly different for various kinds of the integral convergence and divergence of the reciprocals of the coefficients r_i in equations. To find a general relationship between these factors, some researches have been performed.

For the third-order nonlinear neutral dynamic equation (3), there exist four cases for the convergence and divergence of the integrals $\int_{t_0}^{\infty} 1/r_1(t)\Delta t$ and $\int_{t_0}^{\infty} 1/r_2(t)\Delta t$:

- (B1) $\int_{t_0}^{\infty} 1/r_1(t)\Delta t = \int_{t_0}^{\infty} 1/r_2(t)\Delta t = \infty$, see Qiu [10];
- (B2) $\int_{t_0}^{\infty} 1/r_1(t)\Delta t < \infty$ and $\int_{t_0}^{\infty} 1/r_2(t)\Delta t < \infty$, see Qiu and Wang [13];
- (B3) $\int_{t_0}^{\infty} 1/r_1(t)\Delta t = \infty$ and $\int_{t_0}^{\infty} 1/r_2(t)\Delta t < \infty$, see Qiu et al. [14];
- (B4) $\int_{t_0}^{\infty} 1/r_1(t)\Delta t < \infty$ and $\int_{t_0}^{\infty} 1/r_2(t)\Delta t = \infty$, see Qiu et al. [12].

In particular, Qiu and Wang [13] considered a higher-order nonlinear neutral dynamic equation

$$R_n(t, x(t)) + f(t, x(h(t))) = 0 \quad (4)$$

under the condition $\int_{t_0}^{\infty} 1/r_i(t)\Delta t < \infty$, $i = 1, 2, \dots, n-1$, where $n \geq 3$ and

$$R_k(t, x(t)) = \begin{cases} x(t) + p(t)x(g(t)), & k = 0, \\ r_{n-k}(t)R_{k-1}^{\Delta}(t, x(t)), & 1 \leq k \leq n-1, \\ R_{n-1}^{\Delta}(t, x(t)), & k = n. \end{cases}$$

Moreover, Qiu et al. [11] studied (4) with

$$\int_{t_0}^{\infty} \frac{\Delta t}{r_1(t)} = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{\Delta t}{r_i(t)} = M_i < \infty, \quad i = 2, 3, \dots, n-1.$$

In this paper, we continue the study on the existence of nonoscillatory solutions to (4) on a time scale \mathbb{T} , where $n \geq 3$, $\sup \mathbb{T} = \infty$, and $t \in [t_0, \infty)_{\mathbb{T}}$ with $t_0 \in \mathbb{T}$. In addition, the following assumptions are supposed to hold:

- (C1) $r_i \in C_{\text{rd}}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$, $i = 1, 2, \dots, n-1$, and there exist constants $M_i > 0$, $i = 1, 2, \dots, n-2$ such that

$$\int_{t_0}^{\infty} \frac{\Delta t}{r_i(t)} = M_i < \infty, \quad i = 1, 2, \dots, n-2, \quad \text{and} \quad \int_{t_0}^{\infty} \frac{\Delta t}{r_{n-1}(t)} = \infty;$$

- (C2) $p \in C_{\text{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $\lim_{t \rightarrow \infty} p(t) = p_0$, where $|p_0| < 1$;

- (C3) $g, h \in C_{\text{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$, $g(t) \leq t$, $\lim_{t \rightarrow \infty} g(t) = \lim_{t \rightarrow \infty} h(t) = \infty$, and $\lim_{t \rightarrow \infty} R(g(t))/R(t) = \eta \in (0, 1]$, where $R(t) = 1 + \int_{t_0}^t 1/r_{n-1}(s)\Delta s$; if $p_0 \in (-1, 0]$, there exists a sequence $\{c_k\}_{k \geq 0}$ such that $\lim_{k \rightarrow \infty} c_k = \infty$ and $g(c_{k+1}) = c_k$;

- (C4) $f \in C([t_0, \infty)_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$, $f(t, x)$ is nondecreasing in x , and $xf(t, x) > 0$ for $x \neq 0$.

By employing an appropriate Banach space and Krasnoselskii's fixed point theorem, we present the existence of different kinds of nonoscillatory solutions to (4). Finally, two examples are shown to illustrate our conclusions.

2 Auxiliary results

We denote all continuous functions mapping $[T_0, \infty)_{\mathbb{T}}$ into \mathbb{R} by $C([T_0, \infty)_{\mathbb{T}}, \mathbb{R})$. Then define a Banach space

$$BC_{\lambda}[T_0, \infty)_{\mathbb{T}} = \left\{ x \in C([T_0, \infty)_{\mathbb{T}}, \mathbb{R}) : \sup_{t \in [T_0, \infty)_{\mathbb{T}}} \left| \frac{x(t)}{R^{2\lambda}(t)} \right| < \infty \right\} \quad (5)$$

with $\|x\|_{\lambda} = \sup_{t \in [T_0, \infty)_{\mathbb{T}}} |x(t)/R^{2\lambda}(t)|$ for $\lambda = 0, 1$. In the sequel, define $z(t) = x(t) + p(t)x(g(t))$ for simplicity, and we just consider the eventually positive solutions to (4). Now, a lemma is presented to show the relationship between the functions z and x . The proof is similar to the one in [5, Lemma 2.3] and so is omitted.

Lemma 2.1 *Suppose that x is an eventually positive solution to (4) and $\lim_{t \rightarrow \infty} z(t)/R^{\lambda}(t) = a$ for $\lambda = 0, 1$. Then we have*

$$\lim_{t \rightarrow \infty} \frac{x(t)}{R^{\lambda}(t)} = \frac{a}{1 + p_0 \eta^{\lambda}}$$

if a is finite, and

$$\limsup_{t \rightarrow \infty} \frac{x(t)}{R^{\lambda}(t)} = \infty$$

if a is infinite.

Next, we divide all eventually positive solutions to (4) into four groups.

Theorem 2.2 *If x is an eventually positive solution to (4), then one of the following four cases holds:*

- (A1) $x \in A(0, 0)$;
- (A2) $x \in A(b, 0)$;
- (A3) $x \in A(\infty, b)$;
- (A4) $\limsup_{t \rightarrow \infty} x(t) = \infty$ and $\lim_{t \rightarrow \infty} x(t)/R(t) = 0$.

Here,

$$A(\alpha, \beta) = \left\{ x \in S : \lim_{t \rightarrow \infty} x(t) = \alpha, \lim_{t \rightarrow \infty} \frac{x(t)}{R(t)} = \beta \right\},$$

S is the set of all eventually positive solutions of (4), and b is a positive constant.

Proof Assume that x is an eventually positive solution to (4). From (C2) and (C3), there exist a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ and a p_1 satisfying $|p_0| < p_1 < 1$ such that $x(t) > 0$, $x(g(t)) > 0$, $x(h(t)) > 0$, and $|p(t)| \leq p_1$ for $t \in [t_1, \infty)_{\mathbb{T}}$. For $t \in [t_1, \infty)_{\mathbb{T}}$, according to (4) and (C4), we have

$$R_{n-1}^{\Delta}(t, x(t)) = -f(t, x(h(t))) < 0,$$

which means that R_{n-1} is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. Moreover, it follows that

$$R_{n-2}^{\Delta}(t, x(t)) \leq \frac{r_1(t_1)R_{n-2}^{\Delta}(t_1, x(t_1))}{r_1(t)}, \quad t \in [t_1, \infty)_{\mathbb{T}}. \quad (6)$$

If there exists a $T \in [t_1, \infty)_{\mathbb{T}}$ such that $R_{n-2}^{\Delta}(T, x(T)) \leq 0$, then by (6) we know that R_{n-2}^{Δ} is eventually negative. Otherwise, we arrive at $R_{n-2}^{\Delta}(t, x(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Hence, R_{n-2} is always eventually monotonic. Letting t be replaced by s and integrating (6) from t_1 to t , $t \in [\sigma(t_1), \infty)_{\mathbb{T}}$, by (C1) we obtain

$$\begin{aligned} R_{n-2}(t, x(t)) - R_{n-2}(t_1, x(t_1)) &\leq r_1(t_1) R_{n-2}^{\Delta}(t_1, x(t_1)) \int_{t_1}^t \frac{\Delta s}{r_1(s)} \\ &< r_1(t_1) |R_{n-2}^{\Delta}(t_1, x(t_1))| \cdot M_1, \end{aligned}$$

which means that R_{n-2} is upper bounded. When $n = 3$, we see that $r_2 z^{\Delta}$ is eventually monotonic and upper bounded. When $n \geq 4$, since $r_2 R_{n-3}^{\Delta}$ is eventually monotonic, it follows that $r_2 R_{n-3}^{\Delta}$ and R_{n-3}^{Δ} are eventually positive or eventually negative. Thus, R_{n-3} is eventually monotonic.

Since R_{n-2} is upper bounded, there exist a constant c_1 and a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $R_{n-2}(t, x(t)) \leq c_1$ for $t \in [t_2, \infty)_{\mathbb{T}}$, that is,

$$R_{n-3}^{\Delta}(t, x(t)) \leq \frac{c_1}{r_2(t)}. \quad (7)$$

Substituting s for t and integrating (7) from t_2 to t , $t \in [\sigma(t_2), \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} R_{n-3}(t, x(t)) &\leq R_{n-3}(t_2, x(t_2)) + c_1 \int_{t_2}^t \frac{\Delta s}{r_2(s)} \\ &< R_{n-3}(t_2, x(t_2)) + |c_1| \cdot M_2, \end{aligned}$$

which implies that R_{n-3} is upper bounded. When $n = 4$, we see that $r_3 z^{\Delta}$ is eventually monotonic and upper bounded. By analogy, for all $n \geq 3$, it always satisfies the requirement that $r_{n-1} z^{\Delta}$ is eventually monotonic and upper bounded. Then we need to consider two cases.

Case 1. $r_{n-1} z^{\Delta}$ is eventually strictly decreasing. We can claim that

$$0 \leq \lim_{t \rightarrow \infty} r_{n-1}(t) z^{\Delta}(t) = L_1 < \infty. \quad (8)$$

Otherwise, there exist a constant $c_2 < 0$ and a $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that

$$z^{\Delta}(t) \leq \frac{c_2}{r_{n-1}(t)}, \quad t \in [t_3, \infty)_{\mathbb{T}}. \quad (9)$$

Letting t be replaced by s and integrating (9) from t_3 to t , $t \in [\sigma(t_3), \infty)_{\mathbb{T}}$, by (C1) we have

$$z(t) \leq z(t_3) + c_2 \int_{t_3}^t \frac{\Delta s}{r_{n-1}(s)} \rightarrow -\infty$$

as $t \rightarrow \infty$. Then we get $p_0 \in (-1, 0]$, and thus there exists a $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that $x(t) < -p(t)x(g(t)) < p_1 x(g(t))$ for $t \in [t_4, \infty)_{\mathbb{T}}$. In view of (C3), there exists a positive integer N satisfying $c_k \in [t_4, \infty)_{\mathbb{T}}$ for all $k \geq N$. Moreover, for any $k \geq N + 1$,

$$x(c_k) < p_1 x(c_{k-1}) < p_1^2 x(c_{k-2}) < \cdots < p_1^{k-N} x(c_N),$$

which means that $\lim_{k \rightarrow \infty} x(c_k) = 0$ and thus $\lim_{k \rightarrow \infty} z(c_k) = 0$. It is in contradiction with $\lim_{t \rightarrow \infty} z(t) = -\infty$. Therefore, (8) holds.

If $L_1 > 0$, then we have $z^\Delta(s) > L_1/r_{n-1}(s)$ for $s \in [t_2, \infty)_{\mathbb{T}}$. Integrating this inequality from t_2 to t , $t \in [\sigma(t_2), \infty)_{\mathbb{T}}$, we obtain

$$z(t) > z(t_2) + L_1 \int_{t_2}^t \frac{\Delta s}{r_{n-1}(s)} \rightarrow \infty$$

as $t \rightarrow \infty$.

If $L_1 = 0$, since $r_{n-1}z^\Delta$ and z^Δ are both eventually positive, then it follows that z is eventually strictly increasing. From the above, we know that $\lim_{t \rightarrow \infty} z(t) < 0$ does not hold. Therefore, we get

$$0 \leq \lim_{t \rightarrow \infty} z(t) = L_0 \leq \infty.$$

Case 2. $r_{n-1}z^\Delta$ is eventually strictly increasing, which means that $r_{n-1}z^\Delta$ is eventually positive or eventually negative.

If $r_{n-1}z^\Delta$ is eventually positive, since it is also upper bounded, then there exists a constant $c_3 > 0$ such that $\lim_{t \rightarrow \infty} r_{n-1}(t)z^\Delta(t) = c_3$. Hence, there exists a $t_3 \in [t_2, \infty)_{\mathbb{T}}$ satisfying that

$$z^\Delta(t) \geq \frac{c_3}{2r_{n-1}(t)}, \quad t \in [t_3, \infty)_{\mathbb{T}}. \quad (10)$$

Substituting s for t and integrating (10) from t_3 to t , $t \in [\sigma(t_3), \infty)_{\mathbb{T}}$, we have

$$z(t) \geq z(t_3) + \frac{c_3}{2} \int_{t_3}^t \frac{\Delta s}{r_{n-1}(s)} \rightarrow \infty$$

as $t \rightarrow \infty$.

If $r_{n-1}z^\Delta$ is eventually negative, then it follows that $\lim_{t \rightarrow \infty} r_{n-1}(t)z^\Delta(t) \leq 0$. From the above, it means that $\lim_{t \rightarrow \infty} r_{n-1}(t)z^\Delta(t) = 0$ and so $z^\Delta(t) < 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$. Moreover, we get

$$0 \leq \lim_{t \rightarrow \infty} z(t) = L_0 < \infty.$$

Employing the L'Hôpital's rule in [3, Theorem 1.120], we obtain

$$\lim_{t \rightarrow \infty} r_{n-1}(t)z^\Delta(t) = \lim_{t \rightarrow \infty} \frac{z(t)}{R(t)} = L_1.$$

By virtue of Lemma 2.1, it is clear that one of the cases (A1)–(A4) holds. This completes the proof. \square

3 Main results

In this section, the existence of eventually positive solutions to (4) is presented. Now, we show a sufficient and necessary condition for the type $A(\infty, b)$.

Theorem 3.1 Equation (4) has an eventually positive solution $x \in A(\infty, b)$ if and only if there exists a constant $K > 0$ such that

$$\int_{t_0}^{\infty} \int_{t_0}^{u_{n-2}} \int_{t_0}^{u_{n-3}} \cdots \int_{t_0}^{u_1} \frac{f(u_0, KR(h(u_0)))}{\prod_{i=1}^{n-2} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-2} < \infty, \quad (11)$$

where b is a positive constant.

Proof Suppose that x is an eventually positive solution to (4) satisfying $x \in A(\infty, b)$. By Lemma 2.1 we claim that

$$\lim_{t \rightarrow \infty} z(t) = \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} r_{n-1}(t) z^{\Delta}(t) = \lim_{t \rightarrow \infty} \frac{z(t)}{R(t)} = (1 + p_0 \eta) b.$$

If $\lim_{t \rightarrow \infty} z(t) < \infty$, then it will cause $\lim_{t \rightarrow \infty} x(t) < \infty$, which contradicts $x \in A(\infty, b)$. Hence, there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, and $x(h(t)) \geq bR(h(t))/2$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Substituting u_0 for t into (4) and integrating it from t_1 to u_1 , $u_1 \in [\sigma(t_1), \infty)_{\mathbb{T}}$, we have

$$R_{n-1}(u_1, x(u_1)) - R_{n-1}(t_1, x(t_1)) = - \int_{t_1}^{u_1} f(u_0, x(h(u_0))) \Delta u_0,$$

that is,

$$R_{n-2}^{\Delta}(u_1, x(u_1)) = \frac{R_{n-1}(t_1, x(t_1))}{r_1(u_1)} - \frac{1}{r_1(u_1)} \int_{t_1}^{u_1} f(u_0, x(h(u_0))) \Delta u_0. \quad (12)$$

Integrating (12) from t_1 to u_2 , $u_2 \in [\sigma(t_1), \infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} R_{n-2}(u_2, x(u_2)) - R_{n-2}(t_1, x(t_1)) &= R_{n-1}(t_1, x(t_1)) \int_{t_1}^{u_2} \frac{\Delta u_1}{r_1(u_1)} \\ &\quad - \int_{t_1}^{u_2} \int_{t_1}^{u_1} \frac{f(u_0, x(h(u_0)))}{r_1(u_1)} \Delta u_0 \Delta u_1. \end{aligned}$$

Analogously, for all $n \geq 3$, it follows that

$$\begin{aligned} & r_{n-1}(u_{n-1}) z^{\Delta}(u_{n-1}) - r_{n-1}(t_1) z^{\Delta}(t_1) \\ &= \sum_{k=2}^{n-1} R_k(t_1, x(t_1)) \int_{t_1}^{u_{n-1}} \int_{t_1}^{u_{n-2}} \cdots \int_{t_1}^{u_{n-k+1}} \frac{\Delta u_{n-k} \Delta u_{n-k+1} \cdots \Delta u_{n-2}}{\prod_{i=n-k}^{n-2} r_i(u_i)} \\ &\quad - \int_{t_1}^{u_{n-1}} \int_{t_1}^{u_{n-2}} \cdots \int_{t_1}^{u_1} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-2} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-2}. \end{aligned} \quad (13)$$

Letting $u_{n-1} \rightarrow \infty$, for all $2 \leq k \leq n-1$, by (C1) we have

$$\begin{aligned} & \int_{t_1}^{\infty} \int_{t_1}^{u_{n-2}} \int_{t_1}^{u_{n-3}} \cdots \int_{t_1}^{u_{n-k+1}} \frac{\Delta u_{n-k} \Delta u_{n-k+1} \cdots \Delta u_{n-2}}{\prod_{i=n-k}^{n-2} r_i(u_i)} \\ & < \int_{t_1}^{\infty} \int_{t_1}^{\infty} \int_{t_1}^{\infty} \cdots \int_{t_1}^{\infty} \frac{\Delta u_{n-k} \Delta u_{n-k+1} \cdots \Delta u_{n-2}}{\prod_{i=n-k}^{n-2} r_i(u_i)} \leq \prod_{i=n-k}^{n-2} M_i < \infty. \end{aligned}$$

Hence, by virtue of (C4) and (13), it is not difficult to see that

$$\begin{aligned} & \int_{t_1}^{\infty} \int_{t_1}^{u_{n-2}} \int_{t_1}^{u_{n-3}} \cdots \int_{t_1}^{u_1} \frac{f(u_0, bR(h(u_0))/2)}{\prod_{i=1}^{n-2} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-2} \\ & \leq \int_{t_1}^{\infty} \int_{t_1}^{u_{n-2}} \int_{t_1}^{u_{n-3}} \cdots \int_{t_1}^{u_1} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-2} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-2} < \infty, \end{aligned}$$

which implies that (11) holds.

On the contrary, if there exists a constant $K > 0$ such that (11) holds, then there are two cases to be considered.

Case 1. $0 \leq p_0 < 1$. When $p_0 > 0$, choose a constant p_1 satisfying $p_0 < p_1 < (1 + 4p_0)/5 < 1$. In view of (C2) and (11), there exists a $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that for $t \in [T_0, \infty)_{\mathbb{T}}$,

$$p(t) > 0, \quad \frac{5p_1 - 1}{4} \leq p(t) \leq p_1 < 1, \quad p(t) \frac{R(g(t))}{R(t)} \geq \frac{5p_1 - 1}{4} \eta,$$

and

$$\int_{T_0}^{\infty} \int_{T_0}^{u_{n-2}} \int_{T_0}^{u_{n-3}} \cdots \int_{T_0}^{u_1} \frac{f(u_0, KR(h(u_0)))}{\prod_{i=1}^{n-2} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-2} \leq \frac{(1 - p_1 \eta)K}{8}.$$

When $p_0 = 0$, choose p_1 satisfying $|p(t)| \leq p_1 \leq 1/13$ for $t \in [T_0, \infty)_{\mathbb{T}}$. By (C3), there always exists a $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $g(t) \geq T_0$ and $h(t) \geq T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$.

Define a Banach space $BC_1[T_0, \infty)_{\mathbb{T}}$ as (5),

$$\Omega_1 = \left\{ x \in BC_1[T_0, \infty)_{\mathbb{T}} : \frac{K}{2} R(t) \leq x(t) \leq KR(t) \right\},$$

and two operators U_1 and $V_1: \Omega_1 \rightarrow BC_1[T_0, \infty)_{\mathbb{T}}$ as follows:

$$\begin{aligned} (U_1 x)(t) &= \begin{cases} 3Kp_1 \eta R(t)/4 - p(T_1)x(g(T_1))R(t)/R(T_1), & t \in [T_0, T_1)_{\mathbb{T}}, \\ 3Kp_1 \eta R(t)/4 - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}, \end{cases} \\ (V_1 x)(t) &= \begin{cases} 3KR(t)/4, & t \in [T_0, T_1)_{\mathbb{T}}, \\ 3KR(t)/4 + \int_{T_1}^t \int_{u_{n-1}}^{\infty} \int_{T_1}^{u_{n-2}} \int_{T_1}^{u_{n-3}} \cdots \int_{T_1}^{u_1} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1}, & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases} \end{aligned}$$

The proof that U_1 and V_1 satisfy the conditions in Krasnoselskii's fixed point theorem (see [10, Lemma 2.2]) is similar to that in [10, Theorem 3.1] and [13, Theorem 3.1], hence it is omitted here. In terms of Krasnoselskii's fixed point theorem, there exists an $x \in \Omega_1$ such that $(U_1 + V_1)x = x$, and then, for $t \in [T_1, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} x(t) &= \frac{3(1 + p_1 \eta)KR(t)}{4} - p(t)x(g(t)) \\ &+ \int_{T_1}^t \int_{u_{n-1}}^{\infty} \int_{T_1}^{u_{n-2}} \int_{T_1}^{u_{n-3}} \cdots \int_{T_1}^{u_1} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1}. \end{aligned}$$

Since $x(h(u_0)) \leq KR(h(u_0))$ for $u_0 \in [T_1, \infty)_{\mathbb{T}}$ and

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{1}{R(t)} \int_{T_1}^t \int_{u_{n-1}}^{\infty} \int_{T_1}^{u_{n-2}} \int_{T_1}^{u_{n-3}} \cdots \int_{T_1}^{u_1} \frac{f(u_0, KR(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} \\ &= \lim_{t \rightarrow \infty} \int_t^{\infty} \int_{T_1}^{u_{n-2}} \int_{T_1}^{u_{n-3}} \cdots \int_{T_1}^{u_1} \frac{f(u_0, KR(h(u_0)))}{\prod_{i=1}^{n-2} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-2} = 0, \end{aligned}$$

combining (C4) and Lemma 2.1, we get

$$\lim_{t \rightarrow \infty} \frac{z(t)}{R(t)} = \frac{3(1+p_1\eta)K}{4}, \quad \lim_{t \rightarrow \infty} \frac{x(t)}{R(t)} = \frac{3(1+p_1\eta)K}{4(1+p_0\eta)} > 0,$$

and $\lim_{t \rightarrow \infty} x(t) = \infty$.

Case 2. $-1 < p_0 < 0$. We choose a p_1 and a $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that $-p_0 < p_1 < (1-4p_0)/5 < 1$ and

$$\frac{5p_1-1}{4} \leq -p(t) \leq p_1 < 1, \quad t \in [T_0, \infty)_{\mathbb{T}}.$$

Moreover, define U'_1 on Ω_1 as follows:

$$(U'_1 x)(t) = \begin{cases} -3Kp_1\eta R(t)/4 - p(T_1)x(g(T_1))R(t)/R(T_1), & t \in [T_0, T_1]_{\mathbb{T}}, \\ -3Kp_1\eta R(t)/4 - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

Similarly to the proof of Case 1, we deduce that

$$\lim_{t \rightarrow \infty} \frac{z(t)}{R(t)} = \frac{3(1-p_1\eta)K}{4}, \quad \lim_{t \rightarrow \infty} \frac{x(t)}{R(t)} = \frac{3(1-p_1\eta)K}{4(1+p_0\eta)} > 0,$$

and $\lim_{t \rightarrow \infty} x(t) = \infty$. The proof is complete. \square

In the following, the sufficient conditions for the types $A(b, 0)$ and $A(\infty, 0)$ are given in Theorems 3.2 and 3.3, respectively.

Theorem 3.2 *If there exists a constant $K > 0$ such that*

$$\int_{t_0}^{\infty} \int_{u_{n-1}}^{\infty} \int_{t_0}^{u_{n-2}} \int_{t_0}^{u_{n-3}} \cdots \int_{t_0}^{u_1} \frac{f(u_0, K)}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} < \infty, \quad (14)$$

then (4) has an eventually positive solution $x \in A(b, 0)$, where b is a positive constant.

Proof Suppose that there exists a constant $K > 0$ such that (14) holds. Similarly as the proof of the sufficiency in Theorem 3.1, we consider two cases.

Case 1. $0 \leq p_0 < 1$. When $p_0 > 0$, taking p_1 chosen in Theorem 3.1, then there exists a $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$p(t) > 0, \quad \frac{5p_1-1}{4} \leq p(t) \leq p_1 < 1, \quad t \in [T_0, \infty)_{\mathbb{T}},$$

and

$$\int_{T_0}^{\infty} \int_{u_{n-1}}^{\infty} \int_{T_0}^{u_{n-2}} \int_{T_0}^{u_{n-3}} \cdots \int_{T_0}^{u_1} \frac{f(u_0, K)}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} \leq \frac{(1-p_1)K}{8}.$$

When $p_0 = 0$, choose p_1 such that $|p(t)| \leq p_1 \leq 1/13$ for $t \in [T_0, \infty)_{\mathbb{T}}$. There also exists a $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $g(t) \geq T_0$ and $h(t) \geq T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$.

Define a Banach space $BC_0[T_0, \infty)_{\mathbb{T}}$ as (5),

$$\Omega_2 = \left\{ x \in BC_0[T_0, \infty)_{\mathbb{T}} : \frac{K}{2} \leq x(t) \leq K \right\},$$

and the operators U_2 and $V_2: \Omega_2 \rightarrow BC_0[T_0, \infty)_{\mathbb{T}}$ as follows:

$$(U_2 x)(t) = \begin{cases} (U_2 x)(T_1), & t \in [T_0, T_1]_{\mathbb{T}}, \\ 3Kp_1/4 - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}, \end{cases}$$

$$(V_2 x)(t) = \begin{cases} 3K/4, & t \in [T_0, T_1]_{\mathbb{T}}, \\ 3K/4 + \int_{T_1}^t \int_{u_{n-1}}^{\infty} \int_{T_1}^{u_{n-2}} \int_{T_1}^{u_{n-3}} \cdots \int_{T_1}^{u_1} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1}, & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

Similarly, there exists an $x \in \Omega_2$ such that $(U_2 + V_2)x = x$. For $t \in [T_1, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} x(t) &= \frac{3(1+p_1)K}{4} - p(t)x(g(t)) \\ &\quad + \int_{T_1}^t \int_{u_{n-1}}^{\infty} \int_{T_1}^{u_{n-2}} \int_{T_1}^{u_{n-3}} \cdots \int_{T_1}^{u_1} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1}. \end{aligned} \quad (15)$$

Since

$$\begin{aligned} 0 &< \int_{T_1}^{\infty} \int_{u_{n-1}}^{\infty} \int_{T_1}^{u_{n-2}} \int_{T_1}^{u_{n-3}} \cdots \int_{T_1}^{u_1} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} \\ &\leq \int_{T_1}^{\infty} \int_{u_{n-1}}^{\infty} \int_{T_1}^{u_{n-2}} \int_{T_1}^{u_{n-3}} \cdots \int_{T_1}^{u_1} \frac{f(u_0, K)}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} < \infty, \end{aligned}$$

letting $t \rightarrow \infty$ in (15), we obtain $0 < \lim_{t \rightarrow \infty} z(t) < \infty$. By Lemma 2.1, it follows that

$$0 < \lim_{t \rightarrow \infty} x(t) < \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{x(t)}{R(t)} = 0,$$

which means that $x \in A(b, 0)$, where b is a positive constant.

Case 2. $-1 < p_0 < 0$. Define U'_2 on Ω_2 as

$$(U'_2 x)(t) = \begin{cases} (U'_2 x)(T_1), & t \in [T_0, T_1]_{\mathbb{T}}, \\ -3Kp_1/4 - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

Similarly, we get a conclusion as in Case 1. The proof is complete. \square

Theorem 3.3 *If there exists a constant $M > 0$ such that $|p(t)R(t)| \leq M$ for $t \in [t_0, \infty)_{\mathbb{T}}$,*

$$\int_{t_0}^{\infty} \int_{t_0}^{u_{n-2}} \int_{t_0}^{u_{n-3}} \cdots \int_{t_0}^{u_1} \frac{f(u_0, R(h(u_0)))}{\prod_{i=1}^{n-2} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-2} < \infty, \quad (16)$$

$$\int_{t_0}^{\infty} \int_{u_{n-1}}^{\infty} \int_{t_0}^{u_{n-2}} \int_{t_0}^{u_{n-3}} \cdots \int_{t_0}^{u_1} \frac{f(u_0, M + 3/4)}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} = \infty, \quad (17)$$

then (4) has an eventually positive solution $x \in A(\infty, 0)$.

Proof Suppose that there exists a constant $M > 0$ such that $|p(t)R(t)| \leq M$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Then we know $\lim_{t \rightarrow \infty} p(t) = p_0 = 0$. Choose a $T_0 \in [t_0, \infty)_{\mathbb{T}}$ and a p_1 with $0 < p_1 < 1$ satisfying

$$|p(t)| \leq p_1 < 1, \quad 2M + \frac{3}{2} \leq \frac{1}{4}R(t), \quad t \in [T_0, \infty)_{\mathbb{T}},$$

and

$$\int_{T_0}^{\infty} \int_{T_0}^{u_{n-2}} \int_{T_0}^{u_{n-3}} \cdots \int_{T_0}^{u_1} \frac{f(u_0, R(h(u_0)))}{\prod_{i=1}^{n-2} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-2} \leq \frac{1-p_1}{8}.$$

There exists a $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $g(t) \geq T_0$ and $h(t) \geq T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$.

Define a Banach space $BC_1[T_0, \infty)_{\mathbb{T}}$ as (5),

$$\Omega_3 = \left\{ x \in BC_1[T_0, \infty)_{\mathbb{T}} : M + \frac{3}{4} \leq x(t) \leq R(t) \right\},$$

and the operators U_3 and $V_3: \Omega_3 \rightarrow BC_1[T_0, \infty)_{\mathbb{T}}$ as follows:

$$(U_3 x)(t) = \begin{cases} M + 3/4 - p(T_1)x(g(T_1))R(t)/R(T_1), & t \in [T_0, T_1)_{\mathbb{T}}, \\ M + 3/4 - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}, \end{cases}$$

$$(V_3 x)(t) = \begin{cases} M + 3/4, & t \in [T_0, T_1)_{\mathbb{T}}, \\ M + 3/4 + \int_{T_1}^t \int_{u_{n-1}}^{\infty} \int_{T_1}^{u_{n-2}} \int_{T_1}^{u_{n-3}} \cdots \int_{T_1}^{u_1} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1}, & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

Similarly, there exists an $x \in \Omega_3$ such that $(U_3 + V_3)x = x$. For $t \in [T_1, \infty)_{\mathbb{T}}$, we have

$$x(t) = 2M + \frac{3}{2} - p(t)x(g(t)) + \int_{T_1}^t \int_{u_{n-1}}^{\infty} \int_{T_1}^{u_{n-2}} \int_{T_1}^{u_{n-3}} \cdots \int_{T_1}^{u_1} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1}.$$

Since $M + 3/4 \leq x(h(u_0)) \leq R(h(u_0))$ for $u_0 \in [T_1, \infty)_{\mathbb{T}}$, in view of (C3), (C4), (16), and (17), we obtain

$$\lim_{t \rightarrow \infty} \int_{T_1}^t \int_{u_{n-1}}^{\infty} \int_{T_1}^{u_{n-2}} \int_{T_1}^{u_{n-3}} \cdots \int_{T_1}^{u_1} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} = \infty$$

and

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{1}{R(t)} \int_{T_1}^t \int_{u_{n-1}}^\infty \int_{T_1}^{u_{n-2}} \int_{T_1}^{u_{n-3}} \cdots \int_{T_1}^{u_1} \frac{f(u_0, R(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} \\ &= \lim_{t \rightarrow \infty} \int_t^\infty \int_{T_1}^{u_{n-2}} \int_{T_1}^{u_{n-3}} \cdots \int_{T_1}^{u_1} \frac{f(u_0, R(h(u_0)))}{\prod_{i=1}^{n-2} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-2} = 0. \end{aligned}$$

It is not difficult to see that $\lim_{t \rightarrow \infty} z(t) = \infty$ and $\lim_{t \rightarrow \infty} z(t)/R(t) = 0$. Since $|p(t)x(g(t))| \leq |p(t)R(t)| \leq M$ for $t \in [T_1, \infty)_{\mathbb{T}}$, we arrive at

$$\lim_{t \rightarrow \infty} x(t) = \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{x(t)}{R(t)} = 0$$

due to Lemma 2.1. The proof is complete. \square

Next, we present the necessary conditions for the types $A(b, 0)$ and $A(\infty, 0)$. An additional assumption

$$\int_{t_0}^\infty \int_{u_{n-1}}^\infty \int_{t_0}^{u_{n-2}} \int_{t_0}^{u_{n-3}} \cdots \int_{t_0}^{u_{n-k+1}} \frac{\Delta u_{n-k} \Delta u_{n-k+1} \cdots \Delta u_{n-1}}{\prod_{i=n-k}^{n-1} r_i(u_i)} < \infty \quad (18)$$

is required, where $2 \leq k \leq n-1$. Then we deduce Theorems 3.4 and 3.5.

Theorem 3.4 Suppose that (18) holds for all $2 \leq k \leq n-1$. If (4) has an eventually positive solution $x \in A(b, 0)$, where b is a positive constant, then there exists a constant $K > 0$ such that (14) holds.

Proof Suppose that $x \in A(b, 0)$ is an eventually positive solution to (4), where b is a positive constant. Then we have

$$\lim_{t \rightarrow \infty} z(t) = (1 + p_0)b, \quad \lim_{t \rightarrow \infty} r_{n-1}(t)z^\Delta(t) = \lim_{t \rightarrow \infty} \frac{z(t)}{R(t)} = 0,$$

and there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) \geq b/2$, $x(g(t)) \geq b/2$, and $x(h(t)) \geq b/2$, $t \in [t_1, \infty)_{\mathbb{T}}$. As by the proof in Theorem 3.1, for all $n \geq 3$, there exists a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that

$$\begin{aligned} & r_{n-1}(u_{n-1})z^\Delta(u_{n-1}) - r_{n-1}(t_2)z^\Delta(t_2) \\ &= \sum_{k=2}^{n-1} R_k(t_1, x(t_1)) \int_{t_2}^{u_{n-1}} \int_{t_1}^{u_{n-2}} \int_{t_1}^{u_{n-3}} \cdots \int_{t_1}^{u_{n-k+1}} \frac{\Delta u_{n-k} \Delta u_{n-k+1} \cdots \Delta u_{n-2}}{\prod_{i=n-k}^{n-2} r_i(u_i)} \\ &\quad - \int_{t_2}^{u_{n-1}} \int_{t_1}^{u_{n-2}} \int_{t_1}^{u_{n-3}} \cdots \int_{t_1}^{u_1} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-2} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-2}. \end{aligned}$$

Letting $u_{n-1} \rightarrow \infty$, we obtain

$$\begin{aligned} & -r_{n-1}(t_2)z^\Delta(t_2) \\ &= \sum_{k=2}^{n-1} R_k(t_1, x(t_1)) \int_{t_2}^\infty \int_{t_1}^{u_{n-2}} \int_{t_1}^{u_{n-3}} \cdots \int_{t_1}^{u_{n-k+1}} \frac{\Delta u_{n-k} \Delta u_{n-k+1} \cdots \Delta u_{n-2}}{\prod_{i=n-k}^{n-2} r_i(u_i)} \\ &\quad - \int_{t_2}^\infty \int_{t_1}^{u_{n-2}} \int_{t_1}^{u_{n-3}} \cdots \int_{t_1}^{u_1} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-2} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-2}. \end{aligned} \quad (19)$$

From (18), it follows that

$$\int_{t_2}^{\infty} \int_{t_1}^{u_{n-2}} \int_{t_1}^{u_{n-3}} \cdots \int_{t_1}^{u_1} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-2} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-2} < \infty. \quad (20)$$

Since $x(h(u_0)) \geq b/2$, $u_0 \in [t_1, \infty)_{\mathbb{T}}$, by (C4) we get

$$\int_{t_2}^{\infty} \int_{t_1}^{u_{n-2}} \int_{t_1}^{u_{n-3}} \cdots \int_{t_1}^{u_1} \frac{f(u_0, b/2)}{\prod_{i=1}^{n-2} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-2} < \infty. \quad (21)$$

Substituting u_{n-1} for t_2 into (19), we have

$$\begin{aligned} z^{\Delta}(u_{n-1}) &= - \sum_{k=2}^{n-1} R_k(t_1, x(t_1)) \int_{u_{n-1}}^{\infty} \int_{t_1}^{u_{n-2}} \int_{t_1}^{u_{n-3}} \cdots \int_{t_1}^{u_{n-k+1}} \frac{\Delta u_{n-k} \Delta u_{n-k+1} \cdots \Delta u_{n-2}}{\prod_{i=n-k}^{n-1} r_i(u_i)} \\ &\quad + \int_{u_{n-1}}^{\infty} \int_{t_1}^{u_{n-2}} \int_{t_1}^{u_{n-3}} \cdots \int_{t_1}^{u_1} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-2}. \end{aligned} \quad (22)$$

Integrating (22) from t_2 to t , $t \in [\sigma(t_2), \infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} z(t) - z(t_2) &= - \sum_{k=2}^{n-1} R_k(t_1, x(t_1)) \int_{t_2}^t \int_{u_{n-1}}^{\infty} \int_{t_1}^{u_{n-2}} \int_{t_1}^{u_{n-3}} \cdots \int_{t_1}^{u_{n-k+1}} \frac{\Delta u_{n-k} \Delta u_{n-k+1} \cdots \Delta u_{n-1}}{\prod_{i=n-k}^{n-1} r_i(u_i)} \\ &\quad + \int_{t_2}^t \int_{u_{n-1}}^{\infty} \int_{t_1}^{u_{n-2}} \int_{t_1}^{u_{n-3}} \cdots \int_{t_1}^{u_1} \frac{f(u_0, x(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1}. \end{aligned} \quad (23)$$

Letting $t \rightarrow \infty$, by (18) we deduce that

$$\int_{t_2}^{\infty} \int_{u_{n-1}}^{\infty} \int_{t_1}^{u_{n-2}} \int_{t_1}^{u_{n-3}} \cdots \int_{t_1}^{u_1} \frac{f(u_0, b/2)}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} < \infty,$$

which covers (21) and implies that (14) holds. The proof is complete. \square

Theorem 3.5 Suppose that (18) holds for all $2 \leq k \leq n-1$. If (4) has an eventually positive solution $x \in A(\infty, 0)$, then

$$\int_{t_0}^{\infty} \int_{t_0}^{u_{n-2}} \int_{t_0}^{u_{n-3}} \cdots \int_{t_0}^{u_1} \frac{f(u_0, 3/4)}{\prod_{i=1}^{n-2} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-2} < \infty \quad (24)$$

and

$$\int_{t_0}^{\infty} \int_{u_{n-1}}^{\infty} \int_{t_0}^{u_{n-2}} \int_{t_0}^{u_{n-3}} \cdots \int_{t_0}^{u_1} \frac{f(u_0, R(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} = \infty. \quad (25)$$

Proof Suppose that $x \in A(\infty, 0)$ is an eventually positive solution to (4). Then we have

$$\lim_{t \rightarrow \infty} z(t) = \infty, \quad \lim_{t \rightarrow \infty} r_{n-1}(t) z^{\Delta}(t) = \lim_{t \rightarrow \infty} \frac{z(t)}{R(t)} = 0,$$

and there exists a $T \in [t_0, \infty)_{\mathbb{T}}$ satisfying that $3/4 \leq x(t) \leq R(t)$ for $t \in [T, \infty)_{\mathbb{T}}$. From (C3), there exists a $t_1 \in [T, \infty)_{\mathbb{T}}$ such that $g(t) \geq T$ and $h(t) \geq T$, $t \in [t_1, \infty)_{\mathbb{T}}$. As the proof in Theorem 3.4, there exists a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that (20) holds. In view of (C4), we get

$$\int_{t_2}^{\infty} \int_{t_1}^{u_{n-2}} \int_{t_1}^{u_{n-3}} \cdots \int_{t_1}^{u_1} \frac{f(u_0, 3/4)}{\prod_{i=1}^{n-2} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-2} < \infty,$$

that is, (24) holds. Proceeding as in the proof of Theorem 3.4, we arrive at (23). Letting $t \rightarrow \infty$, by (C4) and (18) we obtain

$$\int_{t_2}^{\infty} \int_{u_{n-1}}^{\infty} \int_{t_1}^{u_{n-2}} \int_{t_1}^{u_{n-3}} \cdots \int_{t_1}^{u_1} \frac{f(u_0, R(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} = \infty,$$

which implies that (25) holds. The proof is complete. \square

Remark 3.6 From Theorems 3.2 and 3.4, it is clear to see that, when (18) holds for all $2 \leq k \leq n-1$, (4) has an eventually positive solution $x \in A(b, 0)$ if and only if there exists a constant $K > 0$ such that (14) holds, where b is a positive constant.

Remark 3.7 Assume that $x \in A(0, 0)$ is an eventually positive solution to (4). The asymptotic behavior of x is more complex than that in other types. Hence, it is not easy to find a sufficient and necessary condition for the type $A(0, 0)$. Some sufficient conditions were presented to ensure that (4) has an eventually positive solution $x \in A(0, 0)$, we refer the reader to [13, Theorems 3.2 and 3.3].

Remark 3.8 It is obvious that the theorems reported in this paper cover those in [12] when $n = 3$. Furthermore, they are consistent with the conclusions in [5] when $n = 2$. Besides, they also complement and improve the results in [13].

4 Examples

In this section, we present two typical examples to illustrate the applications of our results.

Example 4.1 Let $\mathbb{T} = \bigcup_{n=0}^{\infty} [4n, 4n+3]$. For $t \in [4, \infty)_{\mathbb{T}}$, consider the n th-order dynamic equation

$$R_n(t, x(t)) + \frac{x(t)}{t^2} = 0, \quad (26)$$

where $n \geq 4$,

$$R_k(t, x(t)) = \begin{cases} x(t) + (t-1)x(t-4)/(3t), & k=0, \\ t^{\alpha_{n-k}} R_{k-1}^{\Delta}(t, x(t)), & 1 \leq k \leq n-1, \\ R_{n-1}^{\Delta}(t, x(t)), & k=n, \end{cases}$$

$\alpha_1 > 2$, $\alpha_{n-2} > 3$, and $\alpha_{n-1} = -1$. If $n \geq 5$, then let $\alpha_i > 1$, $i = 2, 3, \dots, n-3$.

Here, $r_i(t) = t^{\alpha_i}$, $i = 1, 2, \dots, n-1$, $p(t) = (t-1)/(3t)$, $g(t) = t-4$, $h(t) = t$, and $f(t, x) = x/t^2$. It is not difficult to see that (C1)–(C4) are all satisfied. Since

$$R(t) = 1 + \int_4^t s \Delta s = O(t^2),$$

it follows that

$$\begin{aligned}
 & \int_4^\infty \int_4^{u_{n-2}} \int_4^{u_{n-3}} \cdots \int_4^{u_1} \frac{f(u_0, R(h(u_0)))}{\prod_{i=1}^{n-2} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-2} \\
 & < O(1) \cdot \int_4^\infty \int_4^\infty \cdots \int_4^\infty \frac{\Delta u_1 \Delta u_2 \cdots \Delta u_{n-2}}{u_1^{\alpha_1-1} \cdot \prod_{i=2}^{n-2} u_i^{\alpha_i}} \\
 & = O(1) \cdot \int_4^\infty \frac{\Delta u_1}{u_1^{\alpha_1-1}} \cdot \prod_{i=2}^{n-2} \int_4^\infty \frac{\Delta u_i}{u_i^{\alpha_i}} < \infty, \\
 & \int_4^\infty \int_{u_{n-1}}^\infty \int_4^{u_{n-2}} \int_4^{u_{n-3}} \cdots \int_4^{u_1} \frac{f(u_0, 1)}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} \\
 & = \int_4^\infty \int_{u_{n-1}}^\infty \int_4^{u_{n-2}} \int_4^{u_{n-3}} \cdots \int_4^{u_1} \frac{\Delta u_0 \Delta u_1 \cdots \Delta u_{n-1}}{u_0^2 \cdot \prod_{i=1}^{n-1} r_i(u_i)} \\
 & < O(1) \cdot \int_4^\infty \int_{u_{n-1}}^\infty \frac{u_{n-1}}{u_{n-2}^{\alpha_{n-2}}} \Delta u_{n-2} \Delta u_{n-1} \cdot \prod_{i=1}^{n-3} \int_4^\infty \frac{\Delta u_i}{u_i^{\alpha_i}} \\
 & = O(1) \cdot \int_4^\infty \frac{\Delta u_{n-1}}{u_{n-1}^{\alpha_{n-2}-2}} \cdot \prod_{i=1}^{n-3} \int_4^\infty \frac{\Delta u_i}{u_i^{\alpha_i}} < \infty,
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_4^\infty \int_{u_{n-1}}^\infty \int_4^{u_{n-2}} \int_4^{u_{n-3}} \cdots \int_4^{u_1} \frac{f(u_0, R(h(u_0)))}{\prod_{i=1}^{n-1} r_i(u_i)} \Delta u_0 \Delta u_1 \cdots \Delta u_{n-1} \\
 & = O(1) \cdot \int_4^\infty \int_{u_{n-1}}^\infty \int_4^{u_{n-2}} \int_4^{u_{n-3}} \cdots \int_4^{u_1} \frac{\Delta u_0 \Delta u_1 \cdots \Delta u_{n-1}}{\prod_{i=1}^{n-1} r_i(u_i)} \\
 & < O(1) \cdot \int_4^\infty \int_{u_{n-1}}^\infty \frac{u_{n-1}}{u_{n-2}^{\alpha_{n-2}}} \Delta u_{n-2} \Delta u_{n-1} \cdot \int_4^\infty \frac{\Delta u_1}{u_1^{\alpha_1-1}} \cdot \prod_{i=2}^{n-3} \int_4^\infty \frac{\Delta u_i}{u_i^{\alpha_i}} < \infty,
 \end{aligned}$$

where $\prod_{i=2}^{n-3} 1/u_i^{\alpha_i} \Delta u_i$ is deleted when $n = 4$. The results mean that (11) and (14) hold, but (25) is not satisfied. Therefore, by Theorems 3.1 and 3.2, we conclude that (26) has eventually positive solutions $x_1 \in A(\infty, b_1)$ and $x_2 \in A(b_2, 0)$, where b_1 and b_2 are both positive constants. However, (26) has no eventually positive solutions in $A(\infty, 0)$ due to Theorem 3.5. In addition, for $n = 3$, letting $r_1(t) = t^{\alpha_1}$ and $r_2(t) = 1/t$, where $\alpha_1 > 4$, we can get similar conclusions.

Example 4.2 Let $\mathbb{T} = [1, \infty)_{\mathbb{R}}$. For $t \in [3, \infty)_{\mathbb{T}}$, consider the fourth-order dynamic equation

$$R_4(t, x(t)) + \frac{x(\sqrt{t})}{t^\beta} = 0, \quad (27)$$

where

$$R_k(t, x(t)) = \begin{cases} x(t) - x(t/3)/t, & k = 0, \\ t^{\alpha_4-k} R_{k-1}^\Delta(t, x(t)), & 1 \leq k \leq 3, \\ R_3^\Delta(t, x(t)), & k = 4, \end{cases}$$

$\alpha_1 = \alpha_2 = 2$, $\alpha_3 = 0$, and $\beta \geq 3/2$.

Here, $r_1(t) = r_2(t) = t^2$, $r_3(t) = 1$, $p(t) = -1/t$, $g(t) = t/3$, $h(t) = \sqrt{t}$, and $f(t, x) = x/t^\beta$. Obviously, conditions (C1)–(C4) are satisfied. Since

$$R(t) = 1 + \int_3^t ds = t - 2,$$

taking $M = 5/4$, we obtain

$$|p(t)R(t)| = \frac{t-2}{t} < M,$$

$$\int_3^\infty \int_3^{u_2} \int_3^{u_1} \frac{f(u_0, R(h(u_0)))}{u_1^2 u_2^2} du_0 du_1 du_2 < \int_3^\infty \int_3^{u_2} \int_3^{u_1} \frac{du_0 du_1 du_2}{u_0^{\beta-1/2} u_1^2 u_2^2} < \infty,$$

and

$$\int_3^\infty \int_{u_3}^\infty \int_3^{u_2} \int_3^{u_1} \frac{f(u_0, M + 3/4)}{u_1^2 u_2^2} du_0 du_1 du_2 du_3$$

$$= \int_3^\infty \int_{u_3}^\infty \int_3^{u_2} \int_3^{u_1} \frac{2}{u_0^\beta u_1^2 u_2^2} du_0 du_1 du_2 du_3 = \infty.$$

Therefore, we deduce that (27) has an eventually positive solution $x \in A(\infty, 0)$ in terms of Theorem 3.3.

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Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All four authors contributed equally to this work. They all read and approved the final version of the manuscript.

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