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Existence of solutions for subquadratic convex operator equations at resonance and applications to Hamiltonian systems

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Abstract

This paper investigates the existence of solutions to subquadratic operator equations with convex nonlinearities and resonance by means of the index theory for self-adjoint linear operators developed by Dong and dual least action principle developed by Clarke and Ekeland. Applying the results to subquadratic convex Hamiltonian systems satisfying several boundary value conditions including Bolza boundary value conditions, generalized periodic boundary value conditions and Sturm–Liouville boundary value conditions yield some new theorems concerning the existence of solutions or nontrivial solutions. In particular, some famous results about solutions to subquadratic convex Hamiltonian systems by Mawhin and Willem and Ekeland are special cases of the theorems.

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1 Introduction and main results

Periodic solutions of Hamiltonian systems are important in applications. In recent years, the existence of periodic solutions for Hamiltonian systems has been studied extensively by means of critical point theory and the least action principle, and many interesting results have been obtained. Different solvability hypotheses on the potential are given, such as: the convexity conditions (see [2, 5, 7–9, 15, 16, 20]); the subquadratic conditions (see [5, 7, 10–13, 15–18, 20]) and the sublinear conditions (see [19]). In particular, Mawhin and Willem [15, 16] and Ekeland [7] investigated the existence of periodic solutions to convex Hamiltonian systems under subquadratic growth assumptions on the Hamiltonian. In this paper, we will generalize their results to abstract operator equations. Let X be a real infinite-dimensional separable Hilbert space with norm $\| \cdot \|$ and inner product $(\cdot, \cdot)$. Let $A : D(A) \subset X \to X$ be an unbounded self-adjoint operator with $\sigma(A) = \sigma_d(A) = \{\lambda \in \mathbf{R} | \lambda$ belongs to the point spectrum of $A\}$. We consider the existence of solutions of the following equation:

$$Ax - B_1 x \in \partial \Phi(x), \tag{1.1}$$

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where $B_1 \in \mathcal{L}_s(X)$ and $\mathcal{L}_s(X)$ denotes the usual space consisting of bounded symmetric operators in X, $\nu_A(B_1) \neq 0$ (see the Appendix for the notation $\nu_A(B_1)$), $\Phi : X \to \mathbf{R}$ is continuous and convex, and $\partial\Phi$ denotes the subdifferential of Φ. Set $\Phi^*(y) = \sup_{x \in X}\{(y, x) - \Phi(x)\}$ and $\Phi_\epsilon(x) = \Phi(x) + \frac{\epsilon}{2}\|x\|^2$ for $\epsilon > 0$. Now we use the index $(i_A(B), \nu_A(B)) \in \mathbf{Z} \times \mathbf{N}$ defined in [5, 6] (see the Appendix) for all $B \in \mathcal{L}_s(X)$ to formulate our main results.

Theorem 1.1 *Assume that $\Phi \in C(X, \mathbf{R})$ and satisfies*

(Φ_1) Φ *is convex;*
(Φ_2) *there exists $l_0 \in X$ such that for all $x \in X$ one has*

$$\Phi(x) \geq (x, l_0); \tag{1.2}$$

(Φ_3) *there exist $B_2 \in \mathcal{L}_s(X)$, $\epsilon_0 > 0$ and $c > 0$, such that $B_2 \geq B_1$ and $B_2 > B_1$ with respect to* $\ker(A - B_1)$ *(see the Appendix for the notation $B_2 \geq B_1$ and $B_2 > B_1$), $\nu_A(B_2 + \epsilon_0) = 0$, $i_A(B_2 + \epsilon_0) = i_A(B_1) + \nu_A(B_1)$, and for all $x \in X$ one has*

$$\Phi(x) \leq \frac{1}{2}\big((B_2 - B_1)x, x\big) + c; \tag{1.3}$$

(Φ_4) $\Phi(x) \to +\infty$ *as $\|x\| \to \infty$ and $x \in \ker(A - B_1)$.*
Then (1.1) *has at least one solution x such that $y = (A - B_1)x$ minimizes the dual action*

$$\psi_1 : \quad R(A - B_1) \to \mathbf{R}, \qquad y \to -\frac{1}{2}\big(\Lambda^{-1}y, y\big) + \Phi^*(y)$$

provided $\Phi_\epsilon^* \in C^1(X, \mathbf{R})$, where $\Lambda x = (A - B_1)x$ for all $x \in D(A) \cap R(A - B_1)$.

Theorem 1.2 *Under the hypotheses in Theorem 1.1, assume, in addition, that*

(Φ_5) $\nabla\Phi(\theta) = \theta$, $\Phi(\theta) = 0$ *and there exist $B_{01} \in \mathcal{L}_s(X)$ and $\epsilon_{01} > 0$ such that $B_{01} - \epsilon_{01} \geq B_1$, $i_A(B_{01}) > i_A(B_1) + \nu_A(B_1)$, and*

$$\Phi(x) \geq \frac{1}{2}\big((B_{01} - B_1)x, x\big), \quad \forall x \in X \text{ with small } \|x\|. \tag{1.4}$$

Then (1.1) *has at least one nontrivial solution.*

Remark 1 In [5], Theorem 3.1.7, one only assumed that (Φ_1) and (Φ_3) hold. The following BVP can serve as a counterexample:

$$\begin{cases} \ddot{x}(t) + \pi^2 x(t) = 1, & t \in (0, 1), \\ x(0) = 0 = x(1). \end{cases} \tag{1.5}$$

In fact, it is easy to check that Eq. (1.5) satisfies the conditions of Theorem 3.1.7 in [5], but it has no solution, so Theorem 3.1.7 is also incorrect. These show that our results improve Theorem 3.1.7 in [5].

The paper is organized as follows. The proofs of Theorems 1.1–1.2 are given in Sect. 2 and in Sect. 3 we investigate their applications to convex Hamiltonian systems satisfying

several boundary value conditions including Bolza boundary value conditions, generalized periodic boundary value conditions and Sturm–Liouville boundary value conditions, and we obtain some new theorems on the existence of solutions or nontrivial solutions. In addition, we give some remarks to illustrate that some famous results about solutions to subquadratic convex Hamiltonian systems by Mawhin and Willem [16] and Ekeland [7] are special cases of these results. In the Appendix we recall some useful results concerning the index theory for unbounded linear self-adjoint operator equations from [5, 6] used in other sections.

2 Proofs of the theorems

In order to prove Theorems 1.1–1.2 we need some lemmas from [8, 16].

Lemma 2.1 ([16], Theorem 1.1) *If $\varphi : X \to \mathbf{R}$ is weakly lower semi-continuous on a Hilbert space X and has a bounded minimizing sequence, then φ has a minimum on X.*

Lemma 2.2 ([16], Theorem 1.2) *If X is a Hilbert space and $\varphi : X \to \mathbf{R}$ is lower semi-continuous and convex, then φ is weakly lower semi-continuous.*

Let X be a Hilbert space, we shall denote by $\Gamma(X)$ the set of all convex lower semi-continuous functions $F : X \to \mathbf{R}$. The conjugate function F^* of F is the function $F^* : X \to \mathbf{R}$ defined by

$$F^*(\nu) = \sup_{u \in X}\big\{(\nu, u) - F(u)\big\}.$$

We define the subdifferential of a function $F \in \Gamma(X)$ at a point $u \in X$ to be the set

$$\partial F(u) = \big\{\nu \in X : F(w) \geq F(u) + (\nu, w - u) \text{ for all } w \in X\big\},$$

and say that F is subdifferentiable at u if $\partial F(u) \neq \emptyset$.

Lemma 2.3 ([8], Proposition 5.1 and Corollary 5.2) *If $F \in \Gamma(X)$, the following statements are equivalent*:

   (i) $\nu \in \partial F(u)$;
  (ii) $F(u) + F^*(\nu) = (\nu, u)$;
 (iii) $u \in \partial F^*(\nu)$.

Lemma 2.4 ([8], Proposition 5.3) *If $F : X \to \mathbf{R}$ is convex and differentiable at u, then $\partial F(u) = \{\nabla F(u)\}$.*

Proposition 2.5 *For any $B_1 \in \mathcal{L}_s(X)$, if $x \in R(A - B_1) \cap D(A) \subset X$, then there exists $\tilde{\lambda}_0 > 0$ such that*

$$\|x\|^2 \leq \frac{1}{\tilde{\lambda}_0^2}\big\|(A - B_1)x\big\|^2.$$

Proof By Kato–Rellich's theorem ([1], Theorem 6.5.10), $A - B_1$ is unbounded self-adjoint. $\sigma(A) = \sigma_d(A)$ shows that there is a basis $\{e_i\}_{i=-\infty}^{\infty}$ of X and $\cdots \leq \lambda_{-2} \leq \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq$

$\cdots$ such that $Ae_j = \lambda_j e_j$, $D(A) = \{\sum_{j=-\infty}^{\infty} c_j e_j | \sum_{j=-\infty}^{\infty} \lambda_j^2 c_j^2 < \infty\}$. Let $\|x\|_A^2 = \|x\|^2 + \|Ax\|^2 = \sum_{j=-\infty}^{\infty}(\lambda_j^2 + 1)c_j^2$ for all $x \in D(A)$. If $\{x_j\} \subset D(A)$ such that $\{\|x_j\|_A\}$ is bounded, then $\{x_j\}$ has a convergent subsequence in X and so does $\{B_1 x_j\}$. This implies that $B_1 : (D(A), \|\cdot\|_A) \to X$ is compact and B_1 is A-compact. By Weyl's theorem ([1], Theorem 6.5.21), we can see that $A - B_1$ is also an unbounded linear self-adjoint operator in X and $\sigma_{\mathrm{ess}}(A - B_1) = \sigma_{\mathrm{ess}}(A) = \emptyset$, i.e., $\sigma(A - B_1) = \sigma_d(A - B_1)$ is unbounded. From Lemma 3.2.1 in [5], we know that $A - B_1 : D(A) \to X$ is continuous, $\ker(A - B_1)$ is finite-dimensional, $R(A - B_1)$ is closed and $X = \ker(A - B_1) \oplus R(A - B_1)$. Note that $\nu_A(B_1) \neq 0$, we can assume that there is a basis $\{\tilde{e}_j\}_{j=-\infty}^{\infty}$ of X and $\cdots \leq \tilde{\lambda}_{-2} \leq \tilde{\lambda}_{-1} \leq 0 \leq \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots$ such that $(A - B_1)\tilde{e}_j = \tilde{\lambda}_j \tilde{e}_j$. Thus, for all $x = \sum_{\tilde{\lambda}_j \neq 0} c_j \tilde{e}_j \in R(A - B_1) \cap D(A)$, one has

$$\|x\|^2 = \sum_{\tilde{\lambda}_j \neq 0} c_j^2 \leq \frac{1}{\tilde{\lambda}_0^2} \sum_{\tilde{\lambda}_j \neq 0} \tilde{\lambda}_j^2 c_j^2 = \frac{1}{\tilde{\lambda}_0^2} \left\| (A - B_1)x \right\|^2,$$

where $\tilde{\lambda}_0 = \min\{|\tilde{\lambda}_j| \neq 0\}$. □

Consider the dual functional $\psi : R(A - B_1) \to \mathbf{R}$ defined by

$$\psi(x) = -\frac{1}{2}(\Lambda^{-1}x, x) + F^*(x), \tag{2.1}$$

where $F^*(x) = \sup_{y \in X}\{(x, y) - F(y)\}$ for every $F \in \Gamma(X)$, $\Lambda = (A - B_1)|_{R(A - B_1)} : R(A - B_1) \cap D(A) \to R(A - B_1)$ and for the existence of Λ^{-1}, we refer to the Appendix.

Proposition 2.6 *Let $F : X \to \mathbf{R}$ be convex and continuous and assume $F^* \in C^1(X, \mathbf{R})$. Then the dual functional $\psi : R(A - B_1) \to \mathbf{R}$ defined by (2.1) is continuously differentiable and, if $y \in R(A - B_1)$ is a critical point of ψ, the element x defined by $x = \Lambda^{-1}y + \xi$ for some $\xi \in \ker(A - B_1)$ satisfies*

$$(A - B_1)x \in \partial F(x).$$

Proof If $y \in R(A - B_1)$ is a critical point of ψ, Lemma 2.4 implies that, for all $h \in R(A - B_1)$, one has

$$0 = -(\Lambda^{-1}y, h) + (\nabla F^*(y), h). \tag{2.2}$$

It is then easy to see that $-\Lambda^{-1}y + \nabla F^*(y) \in \ker(A - B_1)$ via $X = \ker(A - B_1) \oplus R(A - B_1)$, and there is $\xi \in \ker(A - B_1)$ such that

$$-\Lambda^{-1}y + \nabla F^*(y) = \xi.$$

Setting $x = \Lambda^{-1}y + \xi = \nabla F^*(y)$, we have $x \in X$, $(A - B_1)x = \Lambda x = y$ and, by duality, $y \in \partial F(x)$. Thus

$$(A - B_1)x \in \partial F(x).$$

The proof is complete. □

Proof of Theorem 1.1 We divide the proof into three steps.

   Step 1. Existence of a solution for a perturbed problem. Let

$$\Phi_\epsilon: \quad X \to \mathbf{R}, \qquad x \to \frac{\epsilon}{2}\|x\|^2 + \Phi(x), \tag{2.3}$$

where $\epsilon \in (0, \epsilon_0)$. From (Φ_2) and (Φ_3), we can obtain

$$\frac{\epsilon}{2}\|x\|^2 - \|l_0\|\|x\| \le \Phi_\epsilon(x) \le \frac{\epsilon_0}{2}\|x\|^2 + \frac{1}{2}\big((B_2 - B_1)x, x\big) + c, \tag{2.4}$$

hence

$$\frac{\epsilon\|x\|^2}{4} - \frac{\|l_0\|^2}{\epsilon} \le \Phi_\epsilon(x) \le \frac{\epsilon_0 + \|B_2 - B_1\|}{2}\|x\|^2 + c.$$

So that by Proposition 2.6, the perturbed dual functional

$$\psi_\epsilon(y) = -\frac{1}{2}\big(\Lambda^{-1}y, y\big) + \Phi_\epsilon^*(y)$$

is continuously differentiable on $R(A - B_1)$ and if $y_\epsilon \in R(A - B_1)$ is a critical point of ψ_ϵ, then the element x_ϵ defined by

$$x_\epsilon = \Lambda^{-1}y_\epsilon + \xi$$

for some $\xi \in \ker(A - B_1)$ is a solution of $(A - B_1)x = \nabla\Phi_\epsilon(x)$, i.e.,

$$(A - B_1)x_\epsilon - \epsilon x_\epsilon \in \partial\Phi(x_\epsilon), \tag{2.5}$$

and the relation

$$(A - B_1)x_\epsilon = y_\epsilon$$

holds. Again by (2.4) and (2.1), we can obtain

$$\Phi_\epsilon^*(y) \ge \frac{1}{2}\big(y, (B_2 + \epsilon_0 - B_1)^{-1}y\big) - c$$

and

$$\psi_\epsilon(y) \ge -\frac{1}{2}\big(y, \Lambda^{-1}y\big) + \frac{1}{2}\big(y, (B_2 + \epsilon_0 - B_1)^{-1}y\big) - c$$

$$= \frac{1}{2}q_{A,(B_2+\epsilon_0)|B_1}(y, y) - c, \quad \forall y \in R(A - B_1). \tag{2.6}$$

From Proposition A.4, Lemma A.6 and (Φ_3), we have $\nu_A(B_1) = i_A(B_2 + \epsilon_0) - i_A(B_1) = I_A(B_1, B_2 + \epsilon_0) = i_A((B_2 + \epsilon_0)|B_1) + \nu_A(B_1)$, which implies that $i_A((B_2 + \epsilon_0)|B_1) = 0$ and $E_A^-((B_2 + \epsilon_0)|B_1) = \{\theta\}$. By (1) of Lemma A.6, we have $\nu_A((B_2 + \epsilon_0)|B_1) = \nu_A(B_2 + \epsilon_0) = 0$ and $E_A^0((B_2 + \epsilon_0)|B_1) = \{\theta\}$. By (A.2) of the Appendix, we have

$$R(A - B_1) = E_A^-\big((B_2 + \epsilon_0)|B_1\big) \oplus E_A^0\big((B_2 + \epsilon_0)|B_1\big) \oplus E_A^+\big((B_2 + \epsilon_0)|B_1\big).$$

Thus we can see that $R(A - B_1) = E_A^+((B_2 + \epsilon_0)|B_1)$, which implies that $\psi_\epsilon(u) \to +\infty$ as $\|u\| \to +\infty$ and $\psi_\epsilon(u)$ is bounded from below. Assume that $\psi_\epsilon(y_j) \to \inf \psi_\epsilon$; then $\psi_\epsilon(y_j)$ is bounded and $\|y_j\|$ is also bounded. Now $\psi_{\epsilon,1}(y) = \Phi_\epsilon^*(y)$ is weakly lower semi-continuous on $R(A - B_1)$ by Lemma 2.2 and $\psi_{\epsilon,2}(y) = -\frac{1}{2}(y, \Lambda^{-1}y)$ is weakly continuous on $R(A - B_1)$ by the compactness of Λ^{-1}. Thus $\psi_\epsilon = \psi_{\epsilon,1} + \psi_{\epsilon,2}$ is weakly lower semi-continuous, and by Lemma 2.1, has a minimum at some point $y_\epsilon \in R(A - B_1)$.

Step 2. A posteriori estimates on x_ϵ. From the obvious inequality $\Phi(x) \leq \Phi_\epsilon(x)$ we deduce $\Phi_\epsilon^*(y) \leq \Phi^*(y)$. Thus, we can obtain

$$\frac{1}{2}q_{A,(B_2+\epsilon_0)|B_1}(y_\epsilon, y_\epsilon) - c \leq \psi_\epsilon(y_\epsilon) \leq \psi_\epsilon(y) \leq -\frac{1}{2}(\Lambda^{-1}y, y) + \Phi^*(y) = \psi_1(y),$$

for all $y \in R(A - B_1)$, which implies that $\psi_\epsilon(y_\epsilon) \leq \psi_1(y_0) = c_1 < +\infty$ and $\|y_\epsilon\| \leq c_2$ with c_2 independent of ϵ. By $y_\epsilon = \Lambda x_\epsilon$, we know that

$$\|\Lambda \tilde{x}_\epsilon\| = \|\Lambda x_\epsilon\| \leq c_2,$$

where $\tilde{x}_\epsilon = x_\epsilon - \bar{x}_\epsilon$, $\bar{x}_\epsilon \in \ker(A - B_1)$. Proposition 2.5 implies that $\|\tilde{x}_\epsilon\| \leq c_3$. From the convexity of Φ, hypothesis (Φ_3) and (2.5) we have

$$\begin{aligned}
\Phi\left(\frac{\bar{x}_\epsilon}{2}\right) = \Phi\left(\frac{x_\epsilon}{2} - \frac{\tilde{x}_\epsilon}{2}\right) &\leq \frac{1}{2}\Phi(x_\epsilon) + \frac{1}{2}\Phi(-\tilde{x}_\epsilon) \\
&\leq \frac{1}{2}\left(\partial\Phi(x_\epsilon), x_\epsilon\right) + \frac{1}{2}\Phi(\theta) + \frac{1}{4}\|B_2 - B_1\| \cdot \|\tilde{x}_\epsilon\|^2 + \frac{c}{2} \\
&\leq \frac{1}{2}\left((A - B_1)x_\epsilon, x_\epsilon\right) - \frac{\epsilon}{2}\|x_\epsilon\|^2 + \frac{1}{4}\|B_2 - B_1\| \cdot \|\tilde{x}_\epsilon\|^2 + c \\
&\leq \frac{1}{2}\|\Lambda^{-1}\| \cdot \|\Lambda x_\epsilon\|^2 + \frac{1}{4}\|B_2 - B_1\| \cdot \|\tilde{x}_\epsilon\|^2 + c \\
&\leq c_4
\end{aligned}$$

due to the boundedness of B_1, B_2 and Λ^{-1}. By assumption (Φ_4), $\|\bar{x}_\epsilon\| \leq c_5$. Then

$$\|x_\epsilon\| \leq \|\tilde{x}_\epsilon\| + \|\bar{x}_\epsilon\| \leq c_3 + c_5 = c_6.$$

Step 3. Existence of a solution for the original problem. Since $\|\Lambda \tilde{x}_\epsilon\| \leq c_2$ and $\|\bar{x}_\epsilon\| \leq c_5$, there is a sequence $\{\varepsilon_n\}$ in $(0, \epsilon_0)$ tending to 0 and some $\tilde{x} \in R(A - B_1)$, $\bar{x} \in \ker(A - B_1)$ such that $\Lambda \tilde{x}_{\epsilon_n} \rightharpoonup \Lambda \tilde{x}$ and $\bar{x}_{\epsilon_n} \to \bar{x}$ in X. Moreover, by the compactness of Λ^{-1}, we have

$$\tilde{x}_{\epsilon_n} \to \tilde{x}, \qquad x_{\epsilon_n} = \tilde{x}_{\epsilon_n} + \bar{x}_{\epsilon_n} \to x = \tilde{x} + \bar{x} \in X.$$

Again note that $\Lambda x_\epsilon = y_\epsilon$ and $\|\Lambda x_\epsilon\| \leq c_2$, we have $\Lambda^{-1}y_\epsilon = x_\epsilon - \bar{x}_\epsilon$ and $y_{\epsilon_n} \rightharpoonup y$ in $R(A - B_1)$, so that $\Lambda^{-1}y_{\epsilon_n}$ converges to

$$\Lambda^{-1}y = x - \bar{x}. \tag{2.7}$$

From Proposition 2.6, we can see that

$$\Lambda x_{\epsilon_n} - \epsilon_n x_{\epsilon_n} \in \partial\Phi(x_{\epsilon_n}). \tag{2.8}$$

By definition,

$$\Phi(u) \geq \Phi(x_{\epsilon_n}) + (\Lambda x_{\epsilon_n} - \epsilon_n x_{\epsilon_n}, u - x_{\epsilon_n}), \quad \forall u \in X.$$

Taking the limit yields

$$\Lambda x \in \partial \Phi(x),$$

which shows that $x \in X$ is a solution of (1.1).

Finally, as $\Phi_\epsilon^*(y) \leq \Phi^*(y)$, we have, for all $h \in R(A - B_1)$

$$\psi_{\epsilon_n}(y_{\epsilon_n}) \leq \psi_{\epsilon_n}(h) \leq \psi(h).$$

Now, by the duality between x_{ϵ_n} and y_{ϵ_n}, we have

$$
\begin{aligned}
\psi_{\epsilon_n}(y_{\epsilon_n}) &= -\frac{1}{2}\left(\Lambda^{-1} y_{\epsilon_n}, y_{\epsilon_n}\right) + \Phi_{\epsilon_n}^*(y_{\epsilon_n}) \\
&= -\frac{1}{2}\left(\Lambda^{-1} y_{\epsilon_n}, y_{\epsilon_n}\right) + (x_{\epsilon_n}, y_{\epsilon_n}) - \Phi_{\epsilon_n}(x_{\epsilon_n}) \\
&= -\frac{1}{2}\left(\Lambda^{-1} y_{\epsilon_n}, y_{\epsilon_n}\right) + (x_{\epsilon_n}, y_{\epsilon_n}) - \Phi(x_{\epsilon_n}) - \frac{\epsilon_n}{2}\|x_{\epsilon_n}\|^2.
\end{aligned}
\tag{2.9}
$$

It follows from (1.1) and (2.7) that

$$y = (A - B_1)x \in \partial \Phi(x). \tag{2.10}$$

Letting $n \to \infty$ in (2.9), by (2.10) and Lemma 2.3, we obtain

$$
\begin{aligned}
\lim_{n \to \infty} \psi_{\epsilon_n}(y_{\epsilon_n}) &= -\frac{1}{2}\left(\Lambda^{-1} y, y\right) + (x, y) - \Phi(x) \\
&= -\frac{1}{2}\left(\Lambda^{-1} y, y\right) + \Phi^*(y) = \psi(y).
\end{aligned}
$$

Thus $\psi(y) \leq \psi(h)$ for all $h \in R(A - B_1)$ and the proof is complete. □

Proof of Theorem 1.2 From Theorem 1.1 we know that there is $y_0 \in R(A - B_1)$ such that $x = \Lambda^{-1} y_0 + \bar{x}$ for some $\bar{x} \in \ker(A - B_1)$ is a solution of (1.1), and

$$\psi(y_0) \leq \psi(h), \quad \forall h \in R(A - B_1).$$

If $\theta \in \partial \Phi(\theta)$, then θ is a solution of (1.1). We can prove that $y_0 \neq \theta$ under assumption (Φ_5). In fact, by (1.4), we can obtain

$$
\begin{aligned}
\psi(h) &\leq -\frac{1}{2}\left(h, \Lambda^{-1} h\right) + \frac{1}{2}\left(h, (B_{01} - B_1)^{-1} h\right) \\
&= \frac{1}{2} q_{A, B_{01}|B_1}(h, h) \quad \text{as } h \to \theta.
\end{aligned}
$$

From Proposition A.4, Lemma A.6 and (Φ_5), we have $I_A(B_1, B_{01}) = i_A(B_{01}|B_1) + \nu_A(B_1) = i_A(B_{01}) - i_A(B_1) > \nu_A(B_1)$, which shows that $i_A(B_{01}|B_1) > 0$ and $\dim(E_A^-(B_{01}|B_1)) \geq 1$ by (A.2)

in the Appendix. So, for any $h \in E_A^-(B_{01}|B_1) \backslash \{\theta\}$ small enough, we have $\psi(h) < 0 = \psi(\theta)$. Hence $y_0 \neq \theta$ and $x = \Lambda^{-1} y_0 + \bar{x} \neq \theta$. This completes the proof. $\qquad \square$

3 Applications to convex Hamiltonian systems

In this section, we consider the applications of the main results to convex Hamiltonian systems. For systematic research of Hamiltonian systems, we refer to the excellent books [7, 14].

3.1 First order Hamiltonian systems satisfying Bolza boundary value conditions

As a first example, we now consider a BVP for a the nonlinear Hamiltonian systems (3.1)–(3.3):

$$-J\dot{x} - \bar{B}_1(t)x - H'(t,x) = 0, \quad t \in [0,1]; \tag{3.1}$$

$$x_1(0)\cos\alpha + x_2(0)\sin\alpha = 0, \tag{3.2}$$

$$x_1(1)\cos\beta + x_2(1)\sin\beta = 0, \tag{3.3}$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, $\bar{B}_1 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n})) = \{B(t) = (b_{jk})_{2n \times 2n} | b_{jk}(t) = b_{kj}(t), t \in [0,1],$ $b_{jk}(t) \in L^\infty([0,1])\}$ with $v_{\alpha,\beta}^f(\bar{B}_1) \neq 0$ and I_n is the identity matrix on $\mathbf{R}^n$, $0 \leq \alpha < \pi$, $0 < \beta \leq \pi$, $x = (x_1, x_2) \in \mathbf{R}^n \times \mathbf{R}^n$, $k \in \mathbf{Z}$, $H : [0,1] \times \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ is differentiable and $H'(t,x)$ is the gradient of H with respect to x.

From Theorems 1.1–1.2, we have the following theorem.

Theorem 3.1 *Assume that $H \in C^1(\mathbf{R}^{2n}, \mathbf{R})$ and satisfies*

(H_1) *$H(t,x)$ is convex in $x \in \mathbf{R}^{2n}$ for almost every $t \in [0,1]$;*

(H_2) *there exists $l_0 \in L^2([0,1], \mathbf{R}^{2n})$ such that for all $x \in \mathbf{R}^{2n}$ one has*

$$H(t,x) \geq (x, l_0(t)); \tag{3.4}$$

(H_3) *there exist $\bar{B}_2 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$ and $\gamma \in L^1([0,1], \mathbf{R}^+)$, such that $\bar{B}_2 \geq \bar{B}_1$, $v_{\alpha,\beta}^f(\bar{B}_2) = 0$, $i_{\alpha,\beta}^f(\bar{B}_2) = i_{\alpha,\beta}^f(\bar{B}_1) + v_{\alpha,\beta}^f(\bar{B}_1)$, and for every $x \in \mathbf{R}^{2n}$ and a.e. $t \in [0,1]$ one has*

$$H(t,x) \leq \frac{1}{2}((\bar{B}_2(t) - \bar{B}_1(t))x, x) + \gamma(t); \tag{3.5}$$

(H_4) *$\int_0^1 H(t,x)\,dt \to +\infty$ as $\|x\|_{L^2} \to \infty$ and $x \in \ker(A_1 - \bar{B}_1)$.*

Then (3.1)–(3.3) has at least one solution.

Proof Let $X = L^2([0,1], \mathbf{R}^{2n})$, $D(A_1) = \{x \in H^1([0,1], \mathbf{R}^{2n}) | x$ satisfies (3.2)–(3.3)$\}$. Define $A_1 : D(A_1) \to X$ by $(A_1 x)(t) = -J\dot{x}(t)$. It is easy to see that A_1 is self-adjoint in X and $\sigma(A_1) = \sigma_d(A_1) = \{\beta - \alpha + k\pi | k \in \mathbf{Z}\}$.

Set $H_\epsilon(t,x) = \frac{\epsilon}{2}|x|^2 + H(t,x)$, $\Phi(x) = \int_0^1 H(t, x(t))\,dt$, and $\Phi_\epsilon(x) = \int_0^1 H_\epsilon(t, x(t))\,dt$ for $x \in X$. Φ and Φ_ϵ are convex because of the convexity of H with respect to x. We have the following proposition.

Proposition 3.2 ([16], Proposition 2.2) *Let $F \in C^1(\mathbf{R}^n, \mathbf{R})$ be a convex function such that, for some $\alpha > 0$, $b_1 \geq 0$, $b_2 \geq 0$, one has*

$$-b_1 \leq F(u) \leq \frac{\alpha}{2}|u|^2 + b_2$$

whenever $u \in \mathbf{R}^n$. Then

$$\left|F'(u)\right| \leq 2\alpha\left[|u| + b_1 + b_2\right]$$

for $u \in \mathbf{R}^n$.

By (H_2)–(H_3),

$$\frac{\epsilon}{4}|x|^2 - \frac{1}{\epsilon}\left|l_0(t)\right|^2 \leq H_\epsilon(t,x) \leq \frac{1}{2}(c+\epsilon)|x|^2 + \left|\gamma(t)\right|$$

for all $x \in \mathbf{R}^{2n}$ and a.e. $t \in [0,1]$. So $\Phi_\epsilon \in C^1(X, \mathbf{R})$ via Proposition 3.2. Moreover,

$$\frac{1}{2(c+\epsilon)}|y|^2 - \left|\gamma(t)\right| \leq H_\epsilon^*(t,y) \leq \frac{1}{\epsilon}\left(|y|^2 + \left|l_0(t)\right|^2\right)$$

for all $y \in \mathbf{R}^{2n}$ and a.e. $t \in [0,1]$. By ([16], Proposition 2.4), $H_\epsilon^*(t, \cdot) \in C^1(\mathbf{R}^{2n}, \mathbf{R})$ for a.e. $t \in [0,1]$. And from ([7], pages 93–96),

$$\Phi_\epsilon^*(y) = \int_0^1 H_\epsilon^*\left(t, y(t)\right) dt$$

for $y \in X$. Hence, $\Phi_\epsilon^* \in C^1(X, \mathbf{R})$ via Proposition 3.2 again.

(Φ_1) follows from (H_1) and obviously, (Φ_2) follows from (H_2), (Φ_4) follows from (H_4). We need only to show that (Φ_3) follows from (H_3). In fact, for non-zero $x \in \ker(A_1 - \bar{B}_1)$, $x = x(t)$ is a non-zero solution of $-J\dot{x} - \bar{B}_1(t)x = 0$ and (3.2)–(3.3). So $x = x(t)$ has at most finitely many zeros on $[0,1]$, and $(\bar{B}_2 x, x) > (\bar{B}_1 x, x)$. Moreover, from Proposition 3.3, condition (H_3) and Proposition A.8 (see the Appendix), we know that there exists $\epsilon_0 > 0$ such that $v_{\alpha,\beta}^f(B_2 + \epsilon_0 I_{2n}) = 0$, $i_{\alpha,\beta}^f(B_2 + \epsilon_0 I_{2n}) = i_{\alpha,\beta}^f(B_1) + v_{\alpha,\beta}^f(B_1)$, which implies that (Φ_3) holds. The proof is complete. $\qquad\square$

Proposition 3.3 ([3], Proposition 2.13) *For any $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$ we have*

$$v_{\alpha,\beta}^f(B + \epsilon I_{2n}) = 0 = v_{\alpha,\beta}^f(B - \epsilon I_{2n}),$$

$$i_{\alpha,\beta}^f(B - \epsilon I_{2n}) = i_{\alpha,\beta}^f(B),$$

$$i_{\alpha,\beta}^f(B + \epsilon I_{2n}) = i_{\alpha,\beta}^f(B) + v_{\alpha,\beta}^f(B).$$

In particular, if $v_{\alpha,\beta}^f(B) = 0$, we have $i_{\alpha,\beta}^f(B + \epsilon I_{2n}) = i_{\alpha,\beta}^f(B)$ for all $\epsilon \in (0, \epsilon_0]$.

Theorem 3.4 *Under the conditions in Theorem 3.1, assume, in addition, that*

(H_5)　$H'(t,\theta) = \theta$, $H(t,\theta) = 0$ *and there exists* $B_{01} \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$ *such that* $B_{01} \geq \bar{B}_1$,
　　　$i^f_{\alpha,\beta}(B_{01}) > i^f_{\alpha,\beta}(\bar{B}_1) + v^f_{\alpha,\beta}(\bar{B}_1)$, *and*

$$H(t,x) \geq \frac{1}{2}\big((B_{01} - \bar{B}_1)x,x\big), \quad \forall x \in \mathbf{R}^{2n} \text{ with small } |x|. \tag{3.6}$$

Then (3.1)–(3.3) *has at least one nontrivial solution.*

Proof It is obvious that (H_5) implies (Φ_5) and it remains to apply Theorems 1.1–1.2.　　□

3.2 First order Hamiltonian systems satisfying generalized periodic boundary value conditions

As a second example, consider the nonlinear Hamiltonian systems:

$$-J\dot{x} - \bar{B}_1 x - H'(t,x) = 0, \quad t \in [0,1]; \tag{3.7}$$

$$x(1) = Px(0), \tag{3.8}$$

where $\bar{B}_1 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$ with $v^f_P(B_1) \neq 0$, P is symplectic, i.e. $P^T J P = J$, $H : [0,1] \times \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ is differentiable and $H'(t,x)$ is the gradient of H with respect to x.

Theorem 3.5 *Assume that* $H \in C^1(\mathbf{R}^{2n}, \mathbf{R})$ *and satisfies* (H_1), (H_2) *and*

($H_{3,1}$)　*there exist* $\bar{B}_2 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$ *and* $\gamma \in L^1([0,1], \mathbf{R}^+)$, *such that* $\bar{B}_2 \geq \bar{B}_1$,
　　　$v^f_P(\bar{B}_2) = 0$, $i^f_P(\bar{B}_2) = i^f_P(\bar{B}_1) + v^f_P(\bar{B}_1)$, *and for every* $x \in \mathbf{R}^{2n}$ *and a.e.* $t \in [0,1]$ *one has* (3.5) *holds.*
($H_{4,1}$)　$\int_0^1 H(t,x)\,dt \to +\infty$ *as* $\|x\|_{L^2} \to \infty$ *and* $x \in \ker(A_2 - \bar{B}_1)$.
　　　Then (3.7)–(3.8) *has at least one solution.*

Proof Let $X = L^2([0,1], \mathbf{R}^{2n})$, $D(A_2) = \{x \in H^1([0,1], \mathbf{R}^{2n}) | x \text{ satisfies } (3.8)\}$. Define $A_2 : D(A_2) \to X$ by $(A_2 x)(t) = -J\dot{x}(t)$. It is easy to check that A_2 is self-adjoint in X and $\sigma(A_2) = \sigma_d(A_2) = \{\mu_i + 2j\pi \,|\, i = 1, 2, \ldots, m; j \in \mathbf{Z}\}$, where $\{\mu_i\}_{i=1}^m \subseteq (0, 2\pi]$ are such that, for some non-zero $\xi_i \in \mathbf{R}^{2n}$, $(e^{-J\mu_i}P - I_{2n})\xi_i = 0$ and $(\xi_i, \xi_j)_{\mathbf{R}^{2n}} = 0$ if $\mu_i = \mu_j$ for $i \neq j$.

Set $\Phi(x) = \int_0^1 H(t,x)\,dt$ for $x \in X$. Because of Proposition 3.2 and Proposition 3.6 and Proposition A.10 the rest of the proof of Theorem 3.5 is similar to the one of Theorem 3.1.　　　　　　　　　　　　　　　　　　　　　　　　　　　　　□

Proposition 3.6 ([4], Proposition 2.13) *For any* $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$, *there exists* $\epsilon_0 > 0$ *such that for any* $\epsilon \in (0, \epsilon_0]$ *we have*

$$v^f_P(B + \epsilon I_{2n}) = 0 = v^f_P(B - \epsilon I_{2n}),$$

$$i^f_P(B - \epsilon I_{2n}) = i^f_P(B),$$

$$i^f_P(B + \epsilon I_{2n}) = i^f_P(B) + v^f_P(B).$$

In particular, if $v^f_P(B) = 0$, *we have* $i^f_P(B + \epsilon I_{2n}) = i^f_P(B)$ *for all* $\epsilon \in (0, \epsilon_0]$.

Remark 2　Note that $L^4([0,1], \mathbf{R}^{2n}) \subset L^2([0,1], \mathbf{R}^{2n})$, $L^2([0,1], \mathbf{R}^+) \subset L^1([0,1], \mathbf{R}^+)$. If $P = I_{2n}$, $T = 1$, $\bar{B}_1 = 0I_{2n}$, $\bar{B}_2 = \alpha_0 I_{2n} < 2\pi I_{2n}$, $l_0 \in L^4([0,1], \mathbf{R}^{2n})$ and $\gamma \in L^2([0,1], \mathbf{R}^+)$, then Theorem 3.5 reduces to ([16], Theorem 3.1; [7], Proposition III.2.2). Moreover, since $P = I_{2n}$,

$T = 1$, $\bar{B}_2 = \bar{B}_1 + \epsilon I_{2n}$, where $\epsilon > 0$ is small enough, Theorem 3.5 reduces to ([7], Proposition III.2.3). Clearly, our result generalizes Theorem 3.1 in [16] and Propositions III.2.2–III.2.3 in [7].

Corollary 3.7 *Assume that $H \in C^1(\mathbf{R}^{2n}, \mathbf{R})$ and there exists $\bar{B}_2 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$ with $\bar{B}_2 \geq \bar{B}_1$, such that $H(t,x) + \frac{1}{2}(\bar{B}_1(t)x, x)$ is $(\bar{B}_1, \bar{B}_2)$-subquadratic at infinity, i.e., (H_1) holds, and there exists $c \in \mathbf{R}$ such that*

$$H(t,x) + \frac{1}{2}(\bar{B}_1(t)x, x) \leq \frac{1}{2}(\bar{B}_2(t)x, x) + c$$

for all (t,x), and $n(\|x\|_{L^2}) \leq H(t,x)$ with $\frac{n(s)}{s} \to +\infty$ as $s \to +\infty$. Let $\gamma_i(t)$ with $0 \leq \gamma_1(t) \leq \cdots \leq \gamma_{2n}(t)$ be the eigenvalues of $\bar{B}_2(t) - \bar{B}_1(t)$, and let λ_j be the eigenvalues of $A_2 - \bar{B}_1$, and set

$$\gamma = \max\{\gamma_{2n}(t) | t \in [0,1]\}, \qquad \lambda = \min\{\lambda_j > 0\}.$$

Then (3.7)–(3.8) has at least one solution if $\lambda > \gamma$.

Proof Clearly, since $H(t,x) + \frac{1}{2}(\bar{B}_1(t)x, x)$ is $(\bar{B}_1, \bar{B}_2)$-subquadratic at infinity, we can see that the conditions (H_2), $(H_{4,1})$, and (3.5) (cf. $(H_{3,1})$) hold. It remains to prove that $v_P^f(\bar{B}_2) = 0$ and $i_P^f(\bar{B}_2) = i_P^f(\bar{B}_1) + v_P^f(\bar{B}_1)$. In fact, by the definition of γ we have $\bar{B}_2(t) \leq \bar{B}_1(t) + \gamma I_{2n}$. Since $\lambda > \gamma$, $A_2 x - \bar{B}_1 x - \mu x = 0$ has no nontrivial solutions for any $\mu \in (0, \gamma]$. From Definition A.9 and (2) of Proposition A.4, we know that $i_P^f(\bar{B}_1 + \gamma I_{2n}) = i_P^f(\bar{B}_1) + v_P^f(\bar{B}_1)$. Since $\bar{B}_1(t) < \bar{B}_2(t) \leq \bar{B}_1(t) + \gamma I_{2n}$, we have $i_P^f(\bar{B}_1) + v_P^f(\bar{B}_1) \leq i_P^f(\bar{B}_2)$, $i_P^f(\bar{B}_2) + v_P^f(\bar{B}_2) \leq i_P^f(\bar{B}_1 + \gamma I_{2n})$. So $v_P^f(\bar{B}_2) = 0$ and $i_P^f(\bar{B}_2) = i_P^f(\bar{B}_1) + v_P^f(\bar{B}_1)$. $\qquad \square$

Remark 3 Noticing that Theorem III.2.1 in [7] considers the eigenvalue of operator $-A_2 + \bar{B}_1$, it is easy see that Corollary 3.7 reduces to ([7], Theorem III.2.1) as $P = I_{2n}$, $T = 1$.

Similarly to Theorem 3.4, we have the following.

Theorem 3.8 *Under the hypotheses of Theorem 3.5, assume, in addition, that*

$(H_{5,1})$ $H'(t,\theta) = \theta$, $H(t,\theta) = 0$ *and there exists $B_{01} \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$ such that $B_{01} \geq \bar{B}_1$, $i_p^f(B_{01}) > i_P^f(\bar{B}_1) + v_P^f(\bar{B}_1)$, and (3.6) holds.*

Then (3.7)–(3.8) has at least one nontrivial solution.

3.3 Second order Hamiltonian systems satisfying Sturm–Liouville boundary value conditions

As a third example, consider the BVP (3.9)–(3.11):

$$-\big(\Lambda(t)x'\big)' - \bar{B}_1 x - V'(t,x) = 0, \tag{3.9}$$

$$x(0)\cos\alpha - \Lambda(0)x'(0)\sin\alpha = 0, \tag{3.10}$$

$$x(1)\cos\beta - \Lambda(1)x'(1)\sin\beta = 0, \tag{3.11}$$

where $\Lambda \in C([0,1], \mathcal{L}_s(\mathbf{R}^n))$, $\Lambda(t)$ is positive definite for $t \in [0,1]$, and $\bar{B}_1 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ with $v_{\Lambda,\alpha,\beta}^s(\bar{B}_1) \neq 0$, $0 \leq \alpha < \pi$, $0 < \beta \leq \pi$.

Theorem 3.9 *Assume that $V \in C^1(\mathbf{R}^n, \mathbf{R})$ and satisfies*

(V_1) *$H(t,x)$ is convex in $x \in \mathbf{R}^n$ for almost every $t \in [0,1]$;*

(V_2) *there exists $l_0 \in L^2([0,1], \mathbf{R}^n)$ such that for all $x \in \mathbf{R}^n$ one has*

$$V(t,x) \geq (x, l_0(t)); \tag{3.12}$$

(V_3) *there exist $\bar{B}_2 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ and $\gamma \in L^1([0,1], \mathbf{R}^+)$ such that $\bar{B}_2 \geq \bar{B}_1$, $\nu_{\Lambda,\alpha,\beta}^s(\bar{B}_2) = 0$, $i_{\Lambda,\alpha,\beta}^s(\bar{B}_2) = i_{\Lambda,\alpha,\beta}^s(\bar{B}_1) + \nu_{\Lambda,\alpha,\beta}^s(\bar{B}_1)$, and for every $x \in \mathbf{R}^n$ and a.e. $t \in [0,1]$ one has*

$$V(t,x) \leq \frac{1}{2}\big((\bar{B}_2 - \bar{B}_1)x, x\big) + \gamma(t); \tag{3.13}$$

(V_4) *$\int_0^1 V(t,x)\, dt \to +\infty$ as $\|x\|_{L^2} \to \infty$ and $x \in \ker(A_3 - \bar{B}_1)$.*

Then (3.9)–(3.11) *has at least one solution.*

Proof Let $X = L^2([0,1], \mathbf{R}^n)$,

$$D(A_3) = \big\{x \in C^2([0,1], \mathbf{R}^n) \,|\, (\Lambda(t)x')' \in L^2([0,1], \mathbf{R}^n), x \text{ satisfies } (3.10)–(3.11)\big\}.$$

Define $A_3 : D(A_3) \to X$ by $(A_3 x)(t) = -(\Lambda(t)x'(t))'$. Using the results presented in [5, Sect. 3.3], it is possible to show that A_3 is self-adjoint and $\sigma(A_3) = \sigma_d(A_3)$. In particular, if $\Lambda(t) = I_n$, $\alpha = 0$, $\beta = \pi$, we have $\sigma(A_3) = \sigma_d(A_3) = \{k^2\pi^2 | k \in \mathbf{Z}\}$.

Set $\Phi(x) = \int_0^1 V(t,x)\, dt$ for $x \in X$. Because of Proposition 3.2 and Proposition 3.10 and Proposition A.12 the rest of the proof of Theorem 3.9 is similar to Theorem 3.1. $\qquad\square$

Proposition 3.10 *For any $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$ we have*

$$\nu_{\Lambda,\alpha,\beta}^s(B + \epsilon I_n) = 0 = \nu_{\Lambda,\alpha,\beta}^s(B - \epsilon I_n),$$
$$i_{\Lambda,\alpha,\beta}^s(B - \epsilon I_n) = i_{\Lambda,\alpha,\beta}^s(B),$$
$$i_{\Lambda,\alpha,\beta}^s(B + \epsilon I_n) = i_{\Lambda,\alpha,\beta}^s(B) + \nu_{\Lambda,\alpha,\beta}^s(B).$$

In particular, if $\nu_{\Lambda,\alpha,\beta}^s(B) = 0$, we have $i_{\Lambda,\alpha,\beta}^s(B + \epsilon I_n) = i_{\Lambda,\alpha,\beta}^s(B)$ for all $\epsilon \in (0, \epsilon_0]$.

Similarly to Theorem 3.4, we have the following.

Theorem 3.11 *Under the hypotheses of Theorem 3.9, assume, in addition, that*

(V_5) *$V'(t,\theta) = \theta$, $V(t,\theta) = 0$ and there exists $B_{01} \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ such that $B_{01} \geq \bar{B}_1$, $i_{\Lambda,\alpha,\beta}^s(B_{01}) > i_{\Lambda,\alpha,\beta}^s(\bar{B}_1) + \nu_{\Lambda,\alpha,\beta}^s(\bar{B}_1)$, and*

$$V(t,x) \geq \frac{1}{2}\big((B_{01} - \bar{B}_1)x, x\big) \quad \forall x \in \mathbf{R}^n \text{ with small } |x|. \tag{3.14}$$

Then (3.9)–(3.11) *has at least one nontrivial solution.*

3.4 Second order Hamiltonian systems satisfying generalized periodic boundary value conditions

Finally, we consider the BVP (3.15)–(3.17):

$$-x'' - \bar{B}_1 x - V'(t, x) = 0, \tag{3.15}$$

$$x(1) = Mx(0), \tag{3.16}$$

$$x'(1) = Nx'(0), \tag{3.17}$$

where $\bar{B}_1 \in L^\infty([0, 1], \mathcal{L}_s(\mathbf{R}^n))$ with $\nu_M^s(\bar{B}_1) \neq 0$, $M, N \in GL(n)$ and $MN^T = I_n$.

Theorem 3.12 *Assume that $V \in C^1(\mathbf{R}^n, \mathbf{R})$ and satisfies (V_1), (V_2) and*

$(V_{3,1})$ *there exist $\bar{B}_2 \in L^\infty([0, 1], \mathcal{L}_s(\mathbf{R}^n))$ and $\gamma \in L^1([0, 1], \mathbf{R}^+)$ such that $\bar{B}_2 \geq \bar{B}_1$,*
$\nu_M^s(\bar{B}_2) = 0$, $i_M^s(\bar{B}_2) = i_M^s(\bar{B}_1) + \nu_M^s(\bar{B}_1)$, *and for every $x \in \mathbf{R}^n$ and a.e. $t \in [0, 1]$ (3.13)*
holds;

$(V_{4,1})$ $\int_0^1 V(t, x)\, dt \to +\infty$ *as $\|x\|_{L^2} \to \infty$ and $x \in \ker(A_4 - \bar{B}_1)$.*

Then (3.15)–(3.17) has at least one solution.

Proof Let $X = L^2([0, 1], \mathbf{R}^n)$,

$$D(A_4) = \left\{ x \in C^2([0, 1], \mathbf{R}^n) \,|\, x'' \in L^2([0, 1], \mathbf{R}^n), x \text{ satisfies (3.16)–(3.17)} \right\}.$$

Define $A_4 : D(A_4) \to X$ by $(A_4 x)(t) = -x''(t)$. Using the results presented in [6, Sect. 7.1], it is possible to show that A_4 is self-adjoint in X and $\sigma(A_4) = \sigma_d(A_4) \subset [0, +\infty)$. In particular, if $M = N = I_n$, we have $\sigma(A_4) = \sigma_d(A_4) = \{(2k)^2\pi^2 | k \in \mathbf{Z}\}$.

Set $\Phi(x) = \int_0^1 V(t, x)\, dt$ for $x \in X$. Because of Proposition 3.2 and Proposition 3.13 the rest of the proof of Theorem 3.12 is similar to Theorem 3.1. $\square$

Proposition 3.13 *For any $B \in L^\infty([0, 1], \mathcal{L}_s(\mathbf{R}^n))$, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$ we have*

$$\nu_M^s(B + \epsilon I_n) = 0 = \nu_M^s(B - \epsilon I_n),$$

$$i_M^s(B - \epsilon I_n) = i_M^s(B),$$

$$i_M^s(B + \epsilon I_n) = i_M^s(B) + \nu_M^s(B).$$

In particular, if $\nu_M^s(B) = 0$, we have $i_M^s(B + \epsilon I_n) = i_M^s(B)$ for all $\epsilon \in (0, \epsilon_0]$.

Remark 4 Note that $L^4([0, 1], \mathbf{R}^n) \subset L^2([0, 1], \mathbf{R}^n)$, $L^2([0, 1], \mathbf{R}^+) \subset L^1([0, 1], \mathbf{R}^+)$. If $M = N = I_n$, $T = 1$, $\bar{B}_1 = 0 I_n < \bar{B}_2 = \alpha^2 I_n < (2\pi)^2 I_n$, $l_0 \in L^4([0, 1], \mathbf{R}^{2n})$ and $\gamma \in L^2([0, 1], \mathbf{R}^+)$, then Theorem 3.12 reduces to ([16], Theorem 3.6). Moreover, as $\bar{B}_1 = (2k\pi)^2 I_n < \bar{B}_2 = \alpha\pi^2 I_n < (2(k+1)\pi)^2 I_n$ where $k \in \mathbf{N}$, then Theorem 3.12 reduces to ([16], Theorem 3.7). Clearly, our result generalizes Theorems 3.6–3.7 in [16].

Similarly to Theorem 3.4, we have the following.

Theorem 3.14 *Under the hypotheses of Theorem 3.12, assume, in addition, that*

$(V_{5,1})$ $V'(t,\theta) = \theta$, $V(t,\theta) = 0$ *and there exists* $B_{01} \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ *such that* $B_{01} \geq \bar{B}_1$,
$\quad i^s_M(B_{01}) > i^s_M(\bar{B}_1) + v^s_M(\bar{B}_1)$, *and* (3.14) *holds.*
Then (3.15)–(3.17) *has at least one nontrivial solution.*

Appendix

In this section we recall some results from [5, 6] related to index theory for unbounded linear self-adjoint operator equations. Let X be a real infinite-dimensional separable Hilbert space with norm $\| \cdot \|$ and inner product $(\cdot, \cdot)$. Let $A : D(A) \to X$ be unbounded self-adjoint and $\sigma(A) = \sigma_d(A) = \{\lambda \in \mathbf{R} | \lambda$ belongs to the point spectrum of $A\}$. For any $B_1, B_2 \in \mathcal{L}_s(X)$, we write $B_1 < B_2$ with respect to X_1 (a subspace of X) if and only if $(B_1 x, x) < (B_2 x, x)$ for all $x \in X_1 \setminus \{\theta\}$; and write $B_1 \leq B_2$ w.r.t. X_1 if and only if $(B_1 x, x) \leq (B_2 x, x)$ for all $x \in X_1$. If $X = X_1$ we just write $B_1 < B_2$ or $B_1 \leq B_2$.

Definition A.1 ([5], Definition 3.1.1) For any $B \in \mathcal{L}_s(X)$, define

$$v_A(B) = \dim \ker(A - B).$$

$v_A(B)$ is called the nullity of B, where $\dim \ker(A - B) = $ dimension of $\ker(A - B)$ and $\ker(A - B) = \{x \in X | Ax = Bx\}$.

It was proved in [5] that the nullity $v_A(B)$ is finite.

Definition A.2 ([5], Definition 3.1.2) For any $B_1, B_2 \in \mathcal{L}_s(X)$ with $B_1 < B_2$, define

$$I_A(B_1, B_2) = \sum_{\lambda \in [0,1)} v_A\big((1-\lambda)B_1 + \lambda B_2\big);$$

and, for any $B_1, B_2 \in \mathcal{L}_s(X)$, define

$$I_A(B_1, B_2) = I_A(B_1, k\mathrm{Id}) - I_A(B_2, k\mathrm{Id}),$$

where $\mathrm{Id} : X \to X$ is the identity map and $k\mathrm{Id} > B_1$, $k\mathrm{Id} > B_2$ for some real number $k > 0$.

Let $B_0 \in \mathcal{L}_s(X)$ be fixed and let $i_A(B_0)$ be a prescribed integer associated with B_0.

Definition A.3 ([5], Definition 3.1.3) For any $B \in \mathcal{L}_s(X)$, define

$$i_A(B) = i_A(B_0) + I(B_0, B).$$

$i_A(B)$ is called the index of B and $i_A(B_0)$ is called initial index. Generally, the initial index can be any prescribed integer and the index $i_A(B)$ depends on B_0 and the initial index.

Proposition A.4 ([5], Proposition 3.1.5)
  (1) *For any* $B, B_1, B_2 \in \mathcal{L}_s(X), I_A(B_1, B_2)$ *and* $i_A(B)$ *are well-defined and finite*;
  (2) *for any* $B_1, B_2 \in \mathcal{L}_s(X)$, $I_A(B_1, B_2) = i_A(B_2) - i_A(B_1)$;
  (3) *for any* $B_1, B_2, B_3 \in \mathcal{L}_s(X)$, $I_A(B_1, B_2) + I_A(B_2, B_3) = I_A(B_1, B_3)$;
  (4) *for any* $B_1, B_2 \in \mathcal{L}_s(X)$, *if* $B_1 < B_2$, *then* $v_A(B_1) + i_A(B_1) \leq i_A(B_2)$.

Let $B_0 \in \mathcal{L}_s(X)$ be given. From Lemma 3.2.1 in [5], we know that $A - B_0 : Y \to X$ is continuous, $\ker(A - B_0)$ is finite-dimensional, $R(A - B_0)$ is closed and $X = \ker(A - B_0) \oplus R(A - B_0)$, and the operator $\Lambda_0 := (A - B_0)|_{R(A-B_0)} : R(A - B_0) \cap Y \to R(A - B_0)$ is invertible and the inverse $\Lambda_0^{-1} : R(A - B_0) \to R(A - B_0) \cap Y \hookrightarrow R(A - B_0)$ is compact and self-adjoint. For any $B \in \mathcal{L}_s(X)$ with $B - B_0 \geq \epsilon$ for some constant $\epsilon > 0$, we define a bilinear form:

$$q_{A,B|B_0}(x,y) = -\left(\Lambda_0^{-1}x, y\right) + \left((B - B_0)^{-1}x, y\right), \quad \forall x, y \in R(A - B_0). \tag{A.1}$$

By Proposition 3.2.2 in [5] we have

$$R(A - B_0) = E_A^+(B|B_0) \oplus E_A^0(B|B_0) \oplus E_A^-(B|B_0), \tag{A.2}$$

such that $q_{A,B|B_0}$ is positive definite, null and negative definite on $E_A^+(B|B_0)$, $E_A^0(B|B_0)$ and $E_A^-(B|B_0)$, respectively. Moreover, $E_A^0(B|B_0)$ and $E_A^-(B|B_0)$ are finite-dimensional.

Definition A.5 ([5], Definition 3.2.3) For any $B \in \mathcal{L}_s(X)$ with $B - B_0 \geq \epsilon$ for some constant $\epsilon > 0$, we define

$$i_A(B|B_0) = \dim E_A^-(B|B_0), \nu_A(B|B_0) = \dim E_A^0(B|B_0).$$

This relative index is a kind of Morse index. It plays an important role in the relationship between Morse index and the index $(i_A(B), \nu_A(B))$.

Lemma A.6 ([5], Theorem 3.2.4) *For any $B \in \mathcal{L}_s(X)$ with $B - B_0 \geq \epsilon$ for some constant $\epsilon > 0$, we have*
  (1) $\nu_A(B|B_0) = \nu_A(B),$
  (2) $i_A(B|B_0) = I_A(B_0, B) - \nu_A(B_0).$

As a first example, consider the following first order Hamiltonian system:

$$-J\dot{x} = B(t)x, \tag{A.3}$$

$$x_1(0)\cos\alpha + x_2(0)\sin\alpha = 0, \tag{A.4}$$

$$x_1(1)\cos\beta + x_2(1)\sin\beta = 0, \tag{A.5}$$

where $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$, $0 \leq \alpha < \pi$ and $0 < \beta \leq \pi$, $x = (x_1, x_2) \in \mathbf{R}^n \times \mathbf{R}^n$, $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. Let $X = L^2([0,1], \mathbf{R}^{2n})$, $D(A) = \{x \in H^1([0,1], \mathbf{R}^{2n}) | x \text{ satisfies (A.4)–(A.5)}\}$. Define $A : D(A) \to X$ by $(Ax)(t) = -J\dot{x}(t)$. It is easy to check that A is self-adjoint and $\sigma(A) = \sigma_d(A) = \{\beta - \alpha + k\pi | k \in \mathbf{Z}\}$. For any $B_1, B_2 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$, we write $B_1 \leq B_2$ if $B_1(t) \leq B_2(t)$, for a.e. $t \in [0,1]$; and we write $B_1 < B_2$ if $B_1(t) \leq B_2(t)$ and $B_1(t) < B_2(t)$ on a subset of $[0,1]$ with positive measure.

Definition A.7 ([5], Definition 3.4.1) For any $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$, we define

$$\nu_{\alpha,\beta}^f(B) = \dim\ker(A - B),$$

$$i_{\alpha,\beta}^f\left(\mathrm{diag}\{0, I_n\}\right) = i_{I_n, \alpha, \beta}^s(0),$$

$$i^f_{\alpha,\beta}(B) = i^f_{\alpha,\beta}\big(\mathrm{diag}\{0, I_n\}\big) + I^f_{\alpha,\beta}\big(\mathrm{diag}\{0, I_n\}, B\big),$$

and

$$I^f_{\alpha,\beta}(B_1, B_2) = \sum_{\lambda \in [0,1)} v^f_{\alpha,\beta}\big((1-\lambda)B_1 + \lambda B_2\big) \quad \text{as } B_1 < B_2,$$

$$I^f_{\alpha,\beta}(B_1, B_2) = I^f_{\alpha,\beta}(B_1, kId) - I^f_{\alpha,\beta}(B_2, kId),$$

for every B_1, B_2 with $kId > B_1$, $kId > B_2$.

Proposition A.8 ([5], Proposition 3.4.2; [6], Proposition 8.1.1)

(1) *For any* $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$, $v^f_{\alpha,\beta}$ *is the dimension of the solution subspace of* (A.3)–(A.5) *and*

$$\big(i^f_{\alpha,\beta}(B), v^f_{\alpha,\beta}(B)\big) \in \mathbf{Z} \times \{0, 1, 2, \dots, n\}.$$

(2) *For any* $B_1, B_2 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$, *if* $B_1 \le B_2$, *then* $i^f_{\alpha,\beta}(B_1) \le i^f_{\alpha,\beta}(B_2)$ *and* $i^f_{\alpha,\beta}(B_1) + v^f_{\alpha,\beta}(B_1) \le i^f_{\alpha,\beta}(B_2) + v^f_{\alpha,\beta}(B_2)$; *if* $B_1 < B_2$, *then* $i^f_{\alpha,\beta}(B_1) + v^f_{\alpha,\beta}(B_1) \le i^f_{\alpha,\beta}(B_2)$.

(3) *For any* $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$,

$$\big(v^s_{\alpha,\beta}(B), i^s_{\alpha,\beta}(B)\big) = \big(v^f_{\alpha,\beta}(\mathrm{diag}\{B, I_n\}), i^f_{\alpha,\beta}(\mathrm{diag}\{B, I_n\})\big).$$

As a second example, consider the following first order Hamiltonian system:

$$-J\dot{x} = B(t)x, \tag{A.6}$$

$$x(1) = Px(0), \tag{A.7}$$

where P is symplectic, i.e., $P^T JP = J$. Let $X = L^2([0,1], \mathbf{R}^{2n})$, $D(A) = \{x \in H^1([0,1], \mathbf{R}^{2n}) |$ x satisfies (A.7)}. Define $A : D(A) \to X$ by $(Ax)(t) = -J\dot{x}(t)$. It is easy to check that A is self-adjoint and $\sigma(A) = \sigma_d(A) = \{\mu_i + 2j\pi \,|\, i = 1, 2, \dots, m; j \in \mathbf{Z}\}$, where $\{\mu_i\}^m_{i=1} \subseteq (0, 2\pi]$ such that, for some non-zero $\xi_i \in \mathbf{R}^{2n}$, $(e^{-J\mu_i}P - I_{2n})\xi_i = 0$ and $(\xi_i, \xi_j)_{\mathbf{R}^{2n}} = 0$ if $\mu_i = \mu_j$ for $i \ne j$.

Definition A.9 ([5], Definition 3.5.1) For any $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$, we define

$$v^f_P(B) = \dim \ker(A - B),$$

$$i^f_P(B) = i^f_P(0) + I^f_P(0, B),$$

and

$$I^f_P(B_1, B_2) = \sum_{\lambda \in [0,1)} v^f_P\big((1-\lambda)B_1 + \lambda B_2\big) \quad \text{as } B_1 < B_2,$$

$$I^f_P(B_1, B_2) = I^f_P(B_1, kId) - I^f_P(B_2, kId),$$

for every B_1, B_2 with $kId > B_1$, $kId > B_2$.

Proposition A.10 ([5], Proposition 3.5.2)

(1) *For any* $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$, *we have* $v^f_P(B) \in \{0, 1, 2, \dots, 2n\}$.

(2) *For any $B_1, B_2 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$, if $B_1 \leq B_2$, then $i_P^f(B_1) \leq i_P^f(B_2)$ and*
   $i_P^f(B_1) + v_P^f(B_1) \leq i_P^f(B_2) + v_P^f(B_2)$; *if $B_1 < B_2$, then $i_P^f(B_1) + v_P^f(B_1) \leq i_P^f(B_2)$.*

As a third example, consider the following second order Hamiltonian system:

$$-\left(\Lambda(t)x'\right)' = B(t)x, \tag{A.8}$$

$$x(0)\cos\alpha - \Lambda(0)x'(0)\sin\alpha = 0, \tag{A.9}$$

$$x(1)\cos\beta - \Lambda(1)x'(1)\sin\beta = 0, \tag{A.10}$$

where $\Lambda \in C([0,1], \mathcal{L}_s(\mathbf{R}^n))$, $\Lambda(t)$ is positive definite for $t \in [0,1]$, $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, $0 \leq \alpha < \pi$ and $0 < \beta \leq \pi$. Let $X = L^2([0,1], \mathbf{R}^n)$, $D(A) = \{x \in C^2([0,1], \mathbf{R}^n) | (\Lambda(t)x')' \in L^2([0,1], \mathbf{R}^n), x$ satisfies (A.9)–(A.10)$\}$. Define $A : D(A) \to X$ by $(Ax)(t) = -(\Lambda(t)x'(t))'$. From Sect. 3.3 in [5], we can check that A is self-adjoint and $\sigma(A) = \sigma_d(A)$. In particular, if $\Lambda(t) = I_n, \alpha = 0, \beta = \pi$, we have $\sigma(A) = \sigma_d(A) = \{k^2\pi^2 | k \in \mathbf{Z}\}$.

Definition A.11 ([5], Sect. 3.3) For any $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, we define

$$v_{\Lambda,\alpha,\beta}^s(B) = \dim\ker(A - B),$$
$$i_{\Lambda,\alpha,\beta}^s(B) = \sum_{\lambda>0} v_{\Lambda,\alpha,\beta}^s(B + \lambda I_n).$$

Proposition A.12 ([5], Propositions 2.3.3 and 3.1.5)
   (1) *For any $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, we have $v_{\Lambda,\alpha,\beta}^s \in \{0,1,2,\dots,n\}$.*
   (2) *For any $B_1, B_2 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$, if $B_1 \leq B_2$, then $i_{\Lambda,\alpha,\beta}^s(B_1) \leq i_{\Lambda,\alpha,\beta}^s(B_2)$ and*
   $i_{\Lambda,\alpha,\beta}^s(B_1) + v_{\Lambda,\alpha,\beta}^s(B_1) \leq i_{\Lambda,\alpha,\beta}^s(B_2) + v_{\Lambda,\alpha,\beta}^s(B_2)$; *if $B_1 < B_2$, then*
   $i_{\Lambda,\alpha,\beta}^s(B_1) + v_{\Lambda,\alpha,\beta}^s(B_1) \leq i_{\Lambda,\alpha,\beta}^s(B_2)$.

Availability of data and materials
Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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