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Oscillatory behavior of second-order nonlinear neutral differential equations

Jun Liu[1*], Xi Liu[2] and Yuanhong Yu[3]

*Correspondence:
liujunxei@126.com
[1]Institute of Applied Mathematics, Qujing Normal University, Qujing, P.R. China
Full list of author information is available at the end of the article

Abstract

We shall consider a class of second-order nonlinear neutral differential equations. Some new oscillation criteria are established by using the Riccati transformation technique. One example is given to show the applicability of the main results.

MSC: 34K11

Keywords: Oscillation; Neutral differential equation; Riccati transformation

1 Introduction

In this paper, we study the oscillation of a class of second-order nonlinear differential equations,

$$\big(r(t)\big(z'(t)\big)^{\alpha}\big)' + f\big(t, x\big(\sigma(t)\big)\big) = 0, \quad t \geq t_0 > 0, \tag{1}$$

where $z(t) = x(t) - p(t)x(\tau(t))$, $\alpha > 0$, and α is the ratio of two odd integers. The following assumptions are satisfied:

(H_1) $r, p \in C([t_0, \infty), R), r(t) > 0, 0 \leq p(t) \leq p_0 < 1$.

(H_2) $\tau \in C([t_0, \infty), R), \tau(t) \leq t, \lim_{t\to\infty} \tau(t) = \infty$.

(H_3) $\sigma \in C^1([t_0, \infty), R), \sigma(t) \leq t, \sigma'(t) > 0, \lim_{t\to\infty} \sigma(t) = \infty$.

(H_4) $f \in C(R, R)$, $uf(t, u) > 0$ for all $u \neq 0$, and there exists a function $q(t) \in C([t_0, \infty]$, $[0, \infty))$ such that $|f(t, u)| \geq q(t)|u^{\alpha}|$.

Second-order and third-order differential equations are widely used in population dynamics, physics, technology and other fields. Many scholars have studied the oscillation of second-order differential equations [1–10]. Similarly, many scholars have studied the oscillation of third-order differential equations [11–14]. On this basis, this paper studies the second-order neutral differential Eq. (1), Some new oscillation criteria are established by using the Riccati transformation technique.

2 Lemmas

In order to establish the oscillation criterion of Eq. (1), we will give three lemmas.

Lemma 2.1 *Assume that*

$$\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(t)\,dt = \infty \tag{2}$$

and $x(t)$ *is an eventually positive solution of Eq.* (1). *Then* $z(t)$ *has the following two possible cases*:

(i) $z(t) > 0$, $z'(t) > 0$, $(r(t)(z'(t))^{\alpha})' \le 0$;

(ii) $z(t) < 0$, $z'(t) > 0$, $(r(t)(z'(t))^{\alpha})' \le 0$.

Proof Since $x(t)$ is an eventually positive solution of (1), there exists a $t_1 \ge t_0$ such that $x(t) > 0$, for $t \ge t_1$. From (1), we have

$$\bigl(r(t)\bigl(z'(t)\bigr)^{\alpha}\bigr)' \le 0$$

hence $r(t)(z'(t))^{\alpha}$ is decreasing function and of one sign, therefore $z'(t)$ is also of one sign, that is, there exists a $t_2 \ge t_1$ such that, for $t \ge t_2$, $z'(t) > 0$ or $z'(t) < 0$.

If $z'(t) > 0$, we have (i) or (ii). Now, we prove that $z'(t) < 0$ will not happen.

If $z'(t) < 0$, we have

$$r(t)\bigl(-z'(t)\bigr)^{\alpha} \ge r(t_2)\bigl(-z'(t_2)\bigr)^{\alpha} = K \ge 0,$$

where $K = r(t_2)(-z'(t_2))^{\alpha} \ge 0$, that is,

$$z'(t) \le -k^{\frac{1}{\alpha}} r^{-\frac{1}{\alpha}}(t).$$

Integrating this inequality from t_2 to t, we have

$$z(t) \le z(t_2) - k^{\frac{1}{\alpha}} \int_{t_2}^{t} r^{-\frac{1}{\alpha}}(s)\,ds$$

by condition (2), $\lim_{t\to\infty} z(t) = -\infty$. We will consider the following two cases.

Case 1. If $x(t)$ is unbounded, then there exists a sequence $\{t_m\}$, such that $\lim_{m\to\infty} t_m = \infty$ and $\lim_{m\to\infty} x(t_m) = \infty$, here $x(t_m) = \max\{x(s) : t_0 \le s \le t_m\}$. Hence, we have

$$\begin{aligned} x\bigl(\tau(t_m)\bigr) &= \max\bigl\{x(s) : t_0 \le s \le \tau(t_m)\bigr\} \\ &\le \max\bigl\{x(s) : t_0 \le s \le t_m\bigr\} = x(t_m). \end{aligned}$$

We get

$$z(t_m) = x(t_m) - p(t_m)x\bigl(\tau(t_m)\bigr) \ge \bigl[1 - p(t_m)\bigr]x(t_m) > 0.$$

This contradicts $\lim_{t\to\infty} z(t) = -\infty$.

Case 2. If $x(t)$ is bounded, then $z(t)$ is bounded, this contradicts $\lim_{t\to\infty} z(t) = -\infty$.

Hence, $z(t)$ satisfies one of the cases (i) and (ii). □

Lemma 2.2 *Assume that $x(t)$ is a positive solution of Eq.* (1) *and $z(t)$ satisfies case* (i) *of Lemma* 2.1, *then*

$$z(t) \geq R(t) r^{\frac{1}{\alpha}}(t) z'(t), \qquad \left(\frac{z(t)}{R(t)}\right)' \leq 0,$$

where $R(t) = \int_T^t r^{-\frac{1}{\alpha}}(s)\,ds$, $T \geq t_0$.

Proof For $t > T \geq t_0$, we have

$$z(t) = z(T) + \int_T^t \frac{r^{\frac{1}{\alpha}}(s) z'(s)}{r^{\frac{1}{\alpha}}(s)}\,ds \geq r^{\frac{1}{\alpha}}(t) z'(t) \int_T^t r^{-\frac{1}{\alpha}}(s)\,ds = R(t) r^{\frac{1}{\alpha}}(t) z'(t).$$

Thus, we conclude that

$$\left(\frac{z(t)}{R(t)}\right)' = \frac{z'(t)R(t) - z(t)R'(t)}{R^2(t)} \leq \frac{z'(t)R(t) - R(t) r^{\frac{1}{\alpha}}(t) z'(t) r^{-\frac{1}{\alpha}}(t)}{R^2(t)} = 0. \qquad \square$$

Lemma 2.3 *Assume that $x(t)$ is an eventually positive solution of* (1) *and*

$$\limsup_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^t \left(\frac{1}{r(s)} \int_s^t q(u)\,du\right)^{\frac{1}{\alpha}} ds > p_0. \tag{3}$$

Then the impossibility for $z(t)$ satisfies case (*ii*) *of Lemma* 2.1.

Proof Assume that $z(t)$ satisfies case (ii) of Lemma 2.1, we have

$$-z(t) = -x(t) + p(t)x\big(\tau(t)\big) < p(t)x\big(\tau(t)\big) \leq p_0 x\big(\tau(t)\big).$$

That is,

$$x\big(\tau(t)\big) \geq -\frac{1}{p_0} z(t).$$

We deduce that

$$x(t) \geq -\frac{1}{p_0} z\big(\tau^{-1}(t)\big), \qquad x\big(\sigma(t)\big) \geq -\frac{1}{p_0} z\big(\tau^{-1}\big(\sigma(t)\big)\big).$$

From (1) and (H_4), we have

$$\big(r(t)\big(z'(t)\big)^\alpha\big)' + q(t)\big(x\big(\sigma(t)\big)\big)^\alpha \leq 0.$$

We get

$$\big(r(t)\big(z'(t)\big)^\alpha\big)' + q(t)\left(-\frac{1}{p_0}\right)^\alpha z^\alpha\big(\tau^{-1}\big(\sigma(t)\big)\big) \leq 0.$$

Integrating this inequality from s to t, we conclude that

$$r(t)\big(z'(t)\big)^\alpha - r(s)\big(z'(s)\big)^\alpha - \frac{1}{p_0^\alpha} \int_s^t q(u) z^\alpha\big(\tau^{-1}\big(\sigma(u)\big)\big)\,du \leq 0.$$

That is,

$$-z'(s) \le \frac{1}{p_0}\left(\frac{1}{r(s)}\int_s^t q(u)z^\alpha\bigl(\tau^{-1}\bigl(\sigma(u)\bigr)\bigr)\,du\right)^{\frac{1}{\alpha}}.$$

Integrating this inequality from $\tau^{-1}(\sigma(t))$ to t, we get

$$z\bigl(\tau^{-1}\bigl(\sigma(t)\bigr)\bigr) - z(t) \le \frac{1}{p_0} z\bigl(\tau^{-1}\bigl(\sigma(t)\bigr)\bigr)\int_{\tau^{-1}(\sigma(t))}^t \left(\frac{1}{r(s)}\int_s^t q(u)\,du\right)^{\frac{1}{\alpha}} ds.$$

Since $z(t) < 0$, we have

$$\int_{\tau^{-1}(\sigma(t))}^t \left(\frac{1}{r(s)}\int_s^t q(u)\,du\right)^{\frac{1}{\alpha}} ds \le p_0.$$

This contradicts (3). Thus the impossibility for $z(t)$ satisfies case (ii) of Lemma 2.1. □

3 Oscillation results

Theorem 3.1 *Assume that* (2) *and* (3) *be satisfied. If there exists a positive function* $\rho \in C^1([t_0,\infty),(0,\infty))$, *such that, for all sufficiently large* $T \ge t_0$,

$$\int_{t_0}^\infty \left[\rho(t)\bar{Q}(t) - \frac{r(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^\alpha(t)}\right] dt = \infty, \tag{4}$$

where $\bar{Q}(t) = Q(t)\frac{R^\alpha(\sigma(t))}{R^\alpha(t)})$, $Q(t) = q(t)[1+\bar{p}(\sigma(t))]^\alpha$, $\bar{p}(t) = p(t)\frac{R(\tau(t))}{R(t)}$, *then Eq.* (1) *is oscillatory.*

Proof Assume that $x(t) > 0$. From Lemma 2.1, $z(t)$ satisfies one of the cases (i) and (ii).

Case (i). Suppose that case (i) holds, from Lemma 2.2, we have

$$\frac{z(t)}{R(t)} \le \frac{z(\tau(t))}{R(\tau(t))}.$$

That is,

$$z\bigl(\tau(t)\bigr) \ge R\bigl(\tau(t)\bigr)\frac{z(t)}{R(t)}.$$

We get

$$z(t) = x(t) - p(t)x\bigl(\tau(t)\bigr) \le x(t) - p(t)z\bigl(\tau(t)\bigr) \le x(t) - p(t)R\bigl(\tau(t)\bigr)\frac{z(t)}{R(t)}.$$

That is,

$$x(t) \ge \left[1 + p(t)\frac{R(\tau(t))}{R(t)}\right]z(t) = \bigl[1+\bar{p}(t)\bigr]z(t),$$

where $\bar{p}(t) = p(t)\frac{R(\tau(t))}{R(t)}$.

From (1), we conclude that

$$\bigl(r(t)\bigl(z'(t)\bigr)^{\alpha}\bigr)' + q(t)x^{\alpha}\bigl(\sigma(t)\bigr) \le 0.$$

Then we have

$$\bigl(r(t)\bigl(z'(t)\bigr)^{\alpha}\bigr)' + q(t)\bigl[1+\bar{p}\bigl(\sigma(t)\bigr)\bigr]^{\alpha} z^{\alpha}\bigl(\sigma(t)\bigr) \le 0.$$

That is,

$$\bigl(r(t)\bigl(z'(t)\bigr)^{\alpha}\bigr)' \le -Q(t)z^{\alpha}\bigl(\sigma(t)\bigr), \tag{5}$$

where $Q(t) = q(t)[1+\bar{p}(\sigma(t))]^{\alpha}$.

We define a function $w(t)$ of the generalized Riccati transformation by

$$w(t) = \frac{\rho(t)r(t)(z'(t))^{\alpha}}{z^{\alpha}(t)}.$$

Then $w(t) > 0$, from Lemma 2.2, we have $\frac{z(\sigma(t))}{R(\sigma(t))} \ge \frac{z(t)}{R(t)}$, that is, $\frac{z(\sigma(t))}{z(t)} \ge \frac{R(\sigma(t))}{R(t)}$.

Using the inequality [2]

$$Bu - Au^{\frac{\theta+1}{\theta}} \le \frac{\theta^{\theta}}{(\theta+1)^{\theta+1}}\frac{B^{\theta+1}}{A^{\theta}}, \quad \theta > 0, A > 0, B \in R,$$

we have

$$\begin{aligned} w'(t) &= \rho'(t)\frac{r(t)(z'(t))^{\alpha}}{z^{\alpha}(t)} + \rho(t)\frac{(r(t)(z'(t))^{\alpha})'}{z^{\alpha}(t)} - \rho(t)\frac{\alpha r(t)(z'(t))^{\alpha+1}}{z^{\alpha+1}(t)} \\ &\le \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)Q(t)\frac{z^{\alpha}(\sigma(t))}{z^{\alpha}(t)} - \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}w^{\frac{\alpha+1}{\alpha}}(t) \\ &\le \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)Q(t)\frac{R^{\alpha}(\sigma(t))}{R^{\alpha}(t)} - \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}w^{\frac{\alpha+1}{\alpha}}(t) \\ &\le -\rho(t)\bar{Q}(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\alpha}{(\rho(t)r(t))^{1/\alpha}}w^{\frac{\alpha+1}{\alpha}}(t) \\ &= -\rho(t)\bar{Q}(t) + \frac{r(t)(\rho'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(t)}, \end{aligned} \tag{6}$$

where $\bar{Q}(t) = Q(t)\frac{R^{\alpha}(\sigma(t))}{R^{\alpha}(t)})$.

Integrating this inequality from T to t, we have

$$w(t) \le w(T) - \int_T^t \biggl(\rho(s)\bar{Q}(s) - \frac{r(s)(\rho'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(s)}\biggr)\,ds.$$

From (4), we get $\lim w(t)_{t\to\infty} = -\infty$, this contradicts $w(t) > 0$.

Case (ii). If $z(t)$ satisfies (ii), then due to Lemma 2.3, Eq. (1) is oscillatory. □

Theorem 3.2 *Assume that* (2) *and* (3) *are satisfied. If there exists a positive function* $\varphi \in C^1([t_0,\infty),(0,\infty))$ *such that, for all sufficiently large* $T \geq t_0$,

$$\int_{t_0}^{\infty}\left[\bar{Q}(t)-\frac{\varphi^{\alpha+1}(t)}{r^{1/\alpha}(t)}\right]\exp\left[(\alpha+1)\int_T^t\frac{\varphi(s)}{r^{1/\alpha}(s)}\,ds\right]=\infty, \tag{7}$$

then Eq. (1) *is oscillatory.*

Proof We use the counter-evidence method, suppose we have a non-oscillatory solution $x(t)$ of Eq. (1), as above, suppose that $x(t)$ is a positive solution of (1), by using Lemma 2.1, $z(t)$ satisfies one of (i) and (ii), we discuss each of the two cases separately.

Case (i). Assume that $z(t)$ has property (i), we obtain (5). We define a function $V(t)$ of a generalized Riccati transformation by

$$V(t)=\frac{r(t)(z'(t))^{\alpha}}{z^{\alpha}(t)}.$$

Then $V(t)>0$, using the Yang inequality $\frac{1}{p}a^p+\frac{1}{q}b^q\geq ab$, $\frac{1}{p}+\frac{1}{q}=1$, similar to (6), we have

$$\begin{aligned}V'(t)&=\frac{(r(t)(z'(t))^{\alpha})'}{z^{\alpha}(t)}-\frac{\alpha r(t)(z'(t))^{\alpha+1}}{z^{\alpha+1}(t)}\\&\leq-\bar{Q}(t)-\frac{\alpha}{r^{1/\alpha}(t)}V^{\frac{\alpha+1}{\alpha}}(t)\\&=-\left[\bar{Q}(t)-r^{-\frac{1}{\alpha}}(t)\varphi^{\alpha+1}(t)\right]-(\alpha+1)r^{-\frac{1}{\alpha}}(t)\left[\frac{1}{\alpha+1}\varphi^{\alpha+1}(t)+\frac{\alpha}{\alpha+1}V^{\frac{\alpha+1}{\alpha}}(t)\right]\\&=-\left[\bar{Q}(t)-r^{-\frac{1}{\alpha}}(t)\varphi^{\alpha+1}(t)\right]-(\alpha+1)r^{-\frac{1}{\alpha}}(t)\varphi(t)V(t).\end{aligned}$$

That is,

$$V'(t)+(\alpha+1)r^{-\frac{1}{\alpha}}(t)\varphi(t)V(t)\leq-\left[\bar{Q}(t)-r^{-\frac{1}{\alpha}}(t)\varphi^{\alpha+1}(t)\right].$$

We get

$$\begin{aligned}&\left[V'(t)+(\alpha+1)r^{-\frac{1}{\alpha}}(t)\varphi(t)V(t)\right]\exp[(\alpha+1)\int_T^t\frac{\varphi(s)}{r^{1/\alpha}(s)}\,ds\\&\quad\leq-\left[\bar{Q}(t)-r^{-\frac{1}{\alpha}}(t)\varphi^{\alpha+1}(t)\right]\exp[(\alpha+1)\int_T^t\frac{\varphi(s)}{r^{1/\alpha}(s)}\,ds.\end{aligned}$$

That is,

$$\begin{aligned}&\left(V(t)\cdot\exp\left[(\alpha+1)\int_T^t r^{-\frac{1}{\alpha}}(s)\varphi(s)\,ds\right]\right)'\\&\quad\leq-\left[\bar{Q}(t)-r^{-\frac{1}{\alpha}}(t)\varphi^{\alpha+1}(t)\right]\exp[(\alpha+1)\int_T^t\frac{\varphi(s)}{r^{1/\alpha}(s)}\,ds.\end{aligned}$$

Integrating this inequality from T to t, we get

$$0\leq V(t)\cdot\exp\left[(\alpha+1)\int_T^t r^{-\frac{1}{\alpha}}(s)\varphi(s)\,ds\right]$$

$$\leq V(T) - \int_T^t \left(\left[\bar{Q}(t) - r^{-\frac{1}{\alpha}}(t)\varphi^{\alpha+1}(t) \right] \exp[(\alpha+1) \int_T^t \frac{\varphi(s)}{r^{1/\alpha}(s)}\, ds \right) dt.$$

This contradicts (7).

Case (ii). If $z(t)$ satisfies (ii), then due to Lemma 2.3, Eq. (1) is oscillatory. □

Example Consider the following equation:

$$\left(\left(x(t) - px(t-1)' \right)^{\frac{1}{3}} \right)' + q_0 x^{\frac{1}{3}}(t-2) = 0. \tag{8}$$

Comparing Eq. (8) with Eq. (1), let $r(t) = 1$, $\alpha = \frac{1}{3}$, $\tau(t) = t-1$, $\sigma(t) = t-2$, $q(t) = q_0 > 0$, $p(t) = p < 1$ is a positive constant. Choose $\rho(t) = t$, $\varphi(t) = 1$, we now verify (3):

$$\limsup_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^t \left(\frac{1}{r(s)} \int_s^t q(u)\, du \right)^{1/\alpha} ds = \limsup_{t\to\infty} \int_{t-1}^t q_0 (t-s)^3\, ds = \frac{q_0}{4} > p_0.$$

Therefore, if $\frac{q_0}{4} > p_0$, obviously, the conditions of Theorem 3.1 and Theorem 3.2 are satisfied, then Eq. (8) is oscillatory.

Then the conditions of Theorem 3.1 and Theorem 3.2 are satisfied.

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Authors' contributions
All three authors contributed equally to this work. All authors read and approved the final manuscript.

Authors' information
[1]Institute of Applied Mathematics, Qujing Normal University, Qujing, Yunnan 655011, P.R. China, professor. [2]School of Information Science and Engineering, Yunnan University, Kunming, Yunnan 650091, P.R. China, doctor. [3]Academy of Mathematics and systems Science, China Academy Science, Beijing, 100190, P.R. China, researcher.

Author details
[1]Institute of Applied Mathematics, Qujing Normal University, Qujing, P.R. China. [2]School of Information Science and Engineering, Yunnan University, Kunming, P.R. China. [3]Academy of Mathematics and systems Science, China Academy Science, Beijing, P.R. China.

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References

1. Li, Q., Wang, R., Chen, F., Li, T.X.: Oscillation of second-order nonlinear delay differential equations with nonpositive neutral coefficients. Adv. Differ. Equ. **2015**, 35 (2015)
2. Candan, T.: Oscillation behavior of second-order nonlinear neutral differential equations with distributed deviating arguments. Appl. Math. Comput. **262**, 199–203 (2015)
3. Liu, J., Luo, H.Y., Liu, X.: Oscillation criteria for half-linear functional differential equation with damping. Therm. Sci. **18**, 1537–1542 (2014)

4. Luo, H.Y., Liu, J., Liu, X.: Oscillation behavior of a class of new generalized Emden–Fowler equations. Therm. Sci. **18**, 1567–1572 (2014)
5. Dzurina, J., Stavroulakis, L.P.: Oscillatory criteria for second order delay differential equations. Appl. Math. Comput. **140**, 445–453 (2003)
6. Li, W.T.: Interval oscillation of second-order half-linear functional differential equations. Appl. Math. Comput. **155**, 451–468 (2004)
7. Liu, H., Meng, H., Liu, P.: Oscillation and asymptotic analysis on a new generalized Emden–Fowler equation. Appl. Math. Comput. **219**, 2739–2748 (2012)
8. Manojlovic, J., Shoukaku, Y., Tanigawa, T., Yoshida, N.: Oscillation criteria for second order differential equations with positive and negative coefficients. Appl. Math. Comput. **181**, 853–863 (2006)
9. Weng, A., Sun, J.: Oscillation of second order delay differential equations. Appl. Math. Comput. **198**, 930–935 (2008)
10. Thandapani, E., Muthulakshmi, V., Graef, J.R.: Oscillation criteria for second order nonlinear neutral delay differential equations with positive and negative coefficients. Int. J. Pure Appl. Math. **70**, 261–274 (2011)
11. Jiang, C.M., Jiang, Y., Li, T.X.: Asymptotic behavior of third-order differential equations with nonpositive neutral coefficients and distributed deviating arguments. Adv. Differ. Equ. **2016**, 105 (2016)
12. Zhang, Q.X., Gao, L., Yu, Y.H.: Oscillation criteria for third-order neutral differential equations with continuously distributed delay. Appl. Math. Lett. **25**, 1514–1519 (2012)
13. Aktas, M.F., Cakmak, D., Tiryaki, A.: On the qualitative behaviors of solutions of third-order nonlinear functional differential equations. Appl. Math. Lett. **24**, 1849–1855 (2011)
14. Candan, T.: Oscillation criteria and asymptotic properties of solutions of third-order nonlinear neutral differential equations. Math. Methods Appl. Sci. **38**, 1379–1392 (2015)