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On subclasses of analytic functions based on a quantum symmetric conformable differential operator with application

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Abstract

Quantum calculus (the calculus without limit) appeared for the first time in fluid mechanics, noncommutative geometry and combinatorics studies. Recently, it has been included into the field of geometric function theory to extend differential operators, integral operators, and classes of analytic functions, especially the classes that are generated by convolution product (Hadamard product). In this effort, we aim to introduce a quantum symmetric conformable differential operator (Q-SCDO). This operator generalized some well-know differential operators such as Sàlàgean differential operator. By employing the Q-SCDO, we present subclasses of analytic functions to study some of its geometric solutions of q -Painlevé differential equation (type III).

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1 Introduction

The conception of q -calculus model is a creative method for designs of the q -special functions. The procedure of q -calculus improves various kinds of orthogonal polynomials, operators, and special functions, which realize the form of their typical complements. The idea of q -calculus was principally realized by Carmichael [1], Jackson [2], Mason [3], and Trjitzinsky [4]. An analysis of this calculus for the early mechanism was offered by Ismail et al. [5]. Numerous integral and derivative features were formulated by using the convolution concept; for example, the Sàlàgean derivative [6], Al-Oboudi derivative (generalization of the Sàlàgean derivative) [7], and the symmetric Sàlàgean derivative [8]. It is significant to notify that the procedure of convolution finds its uses in different research, analysis, and study of the geometric properties of regular functions (see [9–11]). Here, we aim to study some geometric properties of a new quantum symmetric conformable differential operator (Q-SCDO). The classes of analytic functions are suggested by using the convolution product. The consequences are generalized classes in the open unit disk.

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2 Methodology

This section provides the mathematical information that is used in this paper. Let \wedge be the category of smooth functions given as follows:

$$\Upsilon(\xi) = \xi + \sum_{n=2}^{\infty} \Upsilon_n \xi^n, \quad \xi \in U, \tag{2.1}$$

where $U = \{\xi \in \mathbb{C} : |\xi| < 1\}$.

Definition 1 Two functions Υ_1 and Υ_2 in \wedge are said to be subordinate, denoted by $\Upsilon_1 \prec \Upsilon_2$, if we can find a Schwarz function Υ with $\Upsilon(0) = 0$ and $|\Upsilon(\xi)| < 1$ such that $\Upsilon_1(\xi) = \Upsilon_2(\Upsilon(\xi))$, $\xi \in U$ (the details can be found in [12]). Obviously, $\Upsilon_1(\xi) \prec \Upsilon_2(\xi)$ implies $\Upsilon_1(0) = \Upsilon_2(0)$ and $\Upsilon_1(U) \subset \Upsilon_2(U)$. In addition, the subordinate $\Upsilon_1(\xi) \prec_r \Upsilon_1(\xi)$, $\xi \in U(r)$ is written by

$$\Upsilon_1(r\xi) \prec \Upsilon_2(r\xi), \quad r < 1.$$

Definition 2 For two functions Υ_1 and Υ_2 in \wedge , the Hadamard or convolution product is defined as

$$\begin{aligned} \Upsilon_1(\xi) * \Upsilon_2(\xi) &= \left(\xi + \sum_{n=2}^{\infty} \Upsilon_n \xi^n \right) * \left(\xi + \sum_{n=2}^{\infty} \Upsilon_n \xi^n \right) \\ &= \left(\xi + \sum_{n=2}^{\infty} \Upsilon_n \Upsilon_n \xi^n \right), \quad \xi \in U. \end{aligned} \tag{2.2}$$

Definition 3 For each nonnegative integer n , the value of q -integer number, denoted by $[n]_q$, is defined by $[n]_q = \frac{1-q^n}{1-q}$, where $[0]_q = 0$, $[1]_q = 1$ and $\lim_{q \rightarrow 1^-} [n]_q = n$.

Example 2.1 $[1]_{0.5} = 1$, $[2]_{0.5} = 1.5$, $[3]_{0.5} = 1.75$, $[2]_{0.75} = 1.75$, $[3]_{0.5} = 2.312$, $[2]_{0.99} = 1.99$, $[3]_{0.99} = 2.97$, $[3]_1 = 3$.

Definition 4 The q -difference operator of Υ is written by the formula

$$\Delta_q \Upsilon(\xi) = \frac{\Upsilon(q\xi) - \Upsilon(\xi)}{q\xi - \xi}, \quad \xi \in U. \tag{2.3}$$

Clearly, we have $\Delta_q \xi^n = [n]_q \xi^{n-1}$. Consequently, for $\Upsilon \in \wedge$, we have

$$\Delta_q \Upsilon(\xi) = \sum_{n=1}^{\infty} \Upsilon_n [n]_q \xi^{n-1}, \quad \xi \in U, \Upsilon_1 = 1. \tag{2.4}$$

For $\Upsilon \in \wedge$, the Sàlàgean q -derivative factor [13] is formulated as follows:

$$\begin{aligned} S_q^0 \Upsilon(\xi) &= \Upsilon(\xi), \\ S_q^1 \Upsilon(\xi) &= \xi \Delta_q \Upsilon(\xi), \\ &\dots \\ S_q^k \Upsilon(\xi) &= \xi \Delta_q (S_q^{k-1} \Upsilon(\xi)), \end{aligned} \tag{2.5}$$

where k is a positive integer.

A computation based on the definition of Δ_q implies that

$$\begin{aligned} S_q^k \gamma(\xi) &= \xi + \sum_{n=2}^{\infty} [n]_q^k \gamma_n \xi^n \\ &= \left(\xi + \sum_{n=2}^{\infty} \gamma_n \xi^n \right) * \left(\xi + \sum_{n=2}^{\infty} [n]_q^k \xi^n \right) \\ &:= \gamma(\xi) * \Psi_q^k(\xi). \end{aligned}$$

Obviously,

$$\lim_{q \rightarrow 1^-} S_q^k \gamma(\xi) = \xi + \sum_{n=2}^{\infty} n^k \gamma_n \xi^n, \tag{2.6}$$

the Sàlăgean derivative factor [6].

Definition 5 Let $\gamma(\xi) \in \Lambda$, and let $\nu \in [0, 1]$ be a constant. Then Q-SCDO has the following operations:

$$\begin{aligned} [S_\nu^0]_q \gamma(\xi) &= \gamma(\xi), \\ [S_\nu^1]_q \gamma(\xi) &= \left(\frac{\kappa_1(\nu, \xi)}{\kappa_1(\nu, \xi) + \kappa_0(\nu, \xi)} \right) \xi \Delta_q \gamma(\xi) - \left(\frac{\kappa_0(\nu, \xi)}{\kappa_1(\nu, \xi) + \kappa_0(\nu, \xi)} \right) \xi \Delta_q \gamma(-\xi) \\ &= \left(\frac{\kappa_1(\nu, \xi)}{\kappa_1(\nu, \xi) + \kappa_0(\nu, \xi)} \right) \left(\xi + \sum_{n=2}^{\infty} [n]_q \gamma_n \xi^n \right) \\ &\quad - \left(\frac{\kappa_0(\nu, \xi)}{\kappa_1(\nu, \xi) + \kappa_0(\nu, \xi)} \right) \left(-\xi + \sum_{n=2}^{\infty} [n]_q (-1)^n \gamma_n \xi^n \right) \\ &= \xi + \sum_{n=2}^{\infty} [n]_q \left(\frac{\kappa_1(\nu, \xi) + (-1)^{n+1} \kappa_0(\nu, \xi)}{\kappa_1(\nu, \xi) + \kappa_0(\nu, \xi)} \right) \gamma_n \xi^n, \tag{2.7} \\ S_\nu^2 \gamma(\xi) &= S_\nu^1 [S_\nu^1 \gamma(\xi)] \\ &= \xi + \sum_{n=2}^{\infty} [n]_q^2 \left(\frac{\kappa_1(\nu, \xi) + (-1)^{n+1} \kappa_0(\nu, \xi)}{\kappa_1(\nu, \xi) + \kappa_0(\nu, \xi)} \right)^2 \gamma_n \xi^n, \\ &\vdots \\ [S_\nu^k]_q \gamma(\xi) &= S_\nu^1 [S_\nu^{k-1} \gamma(\xi)] \\ &= \xi + \sum_{n=2}^{\infty} [n]_q^k \left(\frac{\kappa_1(\nu, \xi) + (-1)^{n+1} \kappa_0(\nu, \xi)}{\kappa_1(\nu, \xi) + \kappa_0(\nu, \xi)} \right)^k \gamma_n \xi^n, \end{aligned}$$

so that $\kappa_1(\nu, \xi) \neq -\kappa_0(\nu, \xi)$,

$$\lim_{\nu \rightarrow 0} \kappa_1(\nu, \xi) = 1, \quad \lim_{\nu \rightarrow 1} \kappa_1(\nu, \xi) = 0, \quad \kappa_1(\nu, \xi) \neq 0, \forall \xi \in \cup, \nu \in (0, 1),$$

and

$$\lim_{\nu \rightarrow 0} \kappa_0(\nu, \xi) = 0, \quad \lim_{\nu \rightarrow 1} \kappa_0(\nu, \xi) = 1, \quad \kappa_0(\nu, \xi) \neq 0, \forall \xi \in \cup \nu \in (0, 1).$$

The value $\nu = 0$ indicates the Sàlàgean derivative

$$\lim_{q \rightarrow 1^-} \mathcal{S}^k \Upsilon(\xi) = \xi + \sum_{n=2}^{\infty} n^k \Upsilon_n \xi^n.$$

Moreover, the following operator can be located in [14], where

$$\lim_{q \rightarrow 1^-} [\mathcal{S}_\nu^k]_q \Upsilon(\xi) = \mathcal{S}_\nu^k \Upsilon(\xi).$$

3 Convolution classes

Based on the definition (2.7), we introduce the following classes. Denote the following functions:

$$\Psi_q^k(\xi) := \xi + \sum_{n=2}^{\infty} [n]_q^k \xi^n; \tag{3.1}$$

$$\Phi_\nu^k(\xi) := \xi + \sum_{n=2}^{\infty} \left(\frac{\kappa_1(\nu, \xi) + (-1)^{n+1} \kappa_0(\nu, \xi)}{\kappa_1(\nu, \xi) + \kappa_0(\nu, \xi)} \right)^k \xi^n. \tag{3.2}$$

Thus, in terms of the convolution product, the factor (2.7) is formulated as follows:

$$[\mathcal{S}_\nu^k]_q \Upsilon(\xi) = \Psi_q^k(\xi) * \Phi_\nu^k(\xi) * \Upsilon(\xi), \quad \forall \Upsilon \in \bigwedge. \tag{3.3}$$

Let Υ be a function from \bigwedge and $\sigma(\xi)$ be a convex univalent function in \cup such that $\sigma(0) = 1$. The class $\mathcal{E}_{q_1, q_2}^k(\sigma)$ is defined by

$$\mathcal{E}_{q_1, q_2}^{\nu, k}(\sigma) = \left\{ \Upsilon \in \bigwedge : \frac{[\mathcal{S}_\nu^k]_{q_1} \Upsilon(\xi)}{[\mathcal{S}_\nu^k]_{q_2} \Upsilon(\xi)} = \frac{\Psi_{q_1}^k(\xi) * \Phi_\nu^k(\xi) * \Upsilon(\xi)}{\Psi_{q_2}^k(\xi) * \Phi_\nu^k(\xi) * \Upsilon(\xi)} \prec \sigma(\xi), \sigma(0) = 1 \right\}. \tag{3.4}$$

Also, we define a special class involving the above functions when $\nu \rightarrow 0$, as follows:

$$\mathcal{E}_{q_1, q_2}^{0, k}(\sigma) = \left\{ \Upsilon \in \bigwedge : \frac{[\mathcal{S}_\nu^k]_{q_1} \Upsilon(\xi)}{[\mathcal{S}_\nu^k]_{q_2} \Upsilon(\xi)} = \frac{\Psi_{q_1}^k(\xi) * \Upsilon(\xi)}{\Psi_{q_2}^k(\xi) * \Upsilon(\xi)} \prec \sigma(\xi), \sigma(0) = 1 \right\}. \tag{3.5}$$

When $k = 0$, we have Dziok subclass [15].

We denote by $\mathcal{S}^*(\sigma)$ the class of all functions given by

$$\mathcal{S}^*(\sigma) = \left\{ \Upsilon \in \bigwedge : \frac{\xi \left(\frac{\xi}{1-\xi} \right)^\nu * \Upsilon(\xi)}{\left(\frac{\xi}{1-\xi} \right) * \Upsilon(\xi)} \prec \sigma(\xi), \sigma(0) = 1 \right\}, \tag{3.6}$$

and by $\mathcal{C}^*(\sigma)$ the class of all functions

$$\mathcal{C}(\sigma) = \left\{ \Upsilon \in \bigwedge : \frac{\xi \left(\frac{\xi}{(1-\xi)^2} \right)^\nu * \Upsilon(\xi)}{\left(\frac{\xi}{(1-\xi)^2} \right) * \Upsilon(\xi)} \prec \sigma(\xi), \sigma(0) = 1 \right\}. \tag{3.7}$$

The following preliminary result can be found in [16, 17].

Lemma 3.1 *If K is smooth (analytic) in \cup , $\gamma \in \mathcal{C}(\frac{1+\xi}{1-\xi})$ is convex and $g \in S^*(\frac{1+\xi}{1-\xi})$ is starlike then*

$$\frac{\gamma * (Kg)}{\gamma * g}(\cup) \subseteq \overline{\text{co}}(K(\cup)), \tag{3.8}$$

where $\overline{\text{co}}(K(\cup))$ is the closed convex hull of $K(\cup)$.

Lemma 3.2 *For analytic functions $h, \bar{h} \in \cup$, the subordination $h \prec \bar{h}$ implies that*

$$\int_0^{2\pi} |h(\xi)|^p d\theta \leq \int_0^{2\pi} |\bar{h}(\xi)|^p d\theta, \tag{3.9}$$

where $\xi = re^{i\theta}$, $0 < r < 1$, and p is a positive number.

Some of the few studies in q -calculus are realized by comparison between two different values of calculus. Class $\mathcal{E}_{q_1, q_2}^{v, k}(\sigma)$ shows the relation between the q_1 - and q_2 -calculus depending on the operator (2.7).

4 Inclusions

This section deals with the geometric representations of the class $\mathcal{E}_{q_1, q_2}^{v, k}(\sigma)$, $q_1 \neq q_2$ and their consequences.

Theorem 4.1 *Let $\gamma \in \Lambda$ and let the function $g := \Psi_{q_2}^k * \gamma \in S^*(\frac{1+\xi}{1-\xi})$, $\xi \in \cup$. If $\gamma \in \mathcal{E}_{q_1, q_2}^{0, k}(\sigma)$, $q_1 \neq q_2$ and the function $\Phi^k(\xi) \in \mathcal{C}(\frac{1+\xi}{1-\xi})$ then $\gamma \in \mathcal{E}_{q_1, q_2}^{v, k}(\sigma)$, $\sigma(0) = 1$.*

Proof Suppose that $\gamma \in \mathcal{E}_{q_1, q_2}^{0, k}(\sigma)$. This implies that there is a Schwarz function v with $v(0) = 0$ and $|v(\xi)| < 1$ satisfying the following relation:

$$\frac{\Psi_{q_1}^k(\xi) * \gamma(\xi)}{\Psi_{q_2}^k(\xi) * \gamma(\xi)} = \sigma(v(\xi)) \quad (\xi \in \cup). \tag{4.1}$$

This leads to

$$\Psi_{q_1}^k(\xi) * \gamma(\xi) = (\Psi_{q_2}^k(\xi) * \gamma(\xi))\sigma(v(\xi)) = g(\xi)\sigma(v(\xi)). \tag{4.2}$$

By employing the convolution’s properties, we arrive at

$$\begin{aligned} \frac{(\Psi_{q_1}^k * \Phi_v^k * \gamma)(\xi)}{(\Psi_{q_2}^k * \Phi_v^k * \gamma)(\xi)} &= \frac{\Phi_v^k(\xi) * (\Psi_{q_1}^k * \gamma)(\xi)}{\Phi_v^k(\xi) * (\Psi_{q_2}^k * \gamma)(\xi)} \\ &= \frac{\Phi_v^k(\xi) * [g(\xi)\sigma(v(\xi))]}{\Phi_v^k(\xi) * g(\xi)}. \end{aligned} \tag{4.3}$$

Accordingly, by virtue of Lemma 3.1, we obtain

$$\frac{(\Psi_{q_1}^k * \Phi_v^k * \gamma)(\xi)}{(\Psi_{q_2}^k * \Phi_v^k * \gamma)(\xi)} \in \overline{\text{co}}(\sigma(v(\cup))) \subset \overline{\text{co}}(\sigma(\cup)). \tag{4.4}$$

Since $\sigma(\xi)$ is a convex univalent function in \cup with $\sigma(0) = 1$, by the concept of subordination, we conclude that

$$\frac{\Psi_{q_1}^k(z) * \Phi_v^k(\xi) * \Upsilon(\xi)}{\Psi_{q_2}^k(\xi) * \Phi_v^k(\xi) * \Upsilon(\xi)} < \sigma(\xi), \tag{4.5}$$

which means that $\Upsilon \in \mathcal{E}_{q_1, q_2}^{v, k}(\sigma)$. This completes the proof. □

In this place, we note that the conclusion of Theorem 4.1 yields the following consequence:

Corollary 4.2 *Let Υ be a function from \wedge and $\sigma(\xi)$ be a convex univalent function in \cup such that $\sigma(0) = 1$. Then*

$$\mathcal{E}_{q_1, q_2}^{0, k}(\sigma) \subset \mathcal{E}_{q_1, q_2}^{v, k}(\sigma).$$

In general, we have the following result:

Theorem 4.3 *Let $\Upsilon \in \wedge$ and let the function $G := \Psi_{q_2}^k * \Phi_v^k * \Upsilon \in S^*(\frac{1+\xi}{1-\xi})$, $\xi \in \cup$. If $\rho_1 := \Psi_{q_1}^k * \Phi_v^k \prec_r \rho_2 := \Psi_{q_2}^k * \Phi_v^k$ for some $r < 1$ and the function $\rho_2 \in \mathcal{C}(\frac{1+\xi}{1-\xi})$ then*

$$\mathcal{E}_{q_1, q_2}^{v, k}(\sigma) \subset \mathcal{E}_{q_1, q_2}^{v, k+1}(\sigma). \tag{4.6}$$

Proof Suppose that $\Upsilon \in \mathcal{E}_{q_1, q_2}^{v, k}(\sigma)$. Then there is a Schwarz transform ω with $\omega(0) = 0$ and $|\omega(\xi)| < 1$ such that

$$\frac{(\Psi_{q_1}^k * \Phi_v^k * \Upsilon)(\xi)}{(\Psi_{q_2}^k * \Phi_v^k * \Upsilon)(\xi)} = \sigma(\omega(\xi)), \quad \xi \in \cup. \tag{4.7}$$

This yields the following equality:

$$(\Psi_{q_1}^k * \Phi_v^k * \Upsilon)(\xi) = (\Psi_{q_2}^k * \Phi_v^k * \Upsilon)(\xi) \sigma(\omega(\xi)) = G(\xi) \sigma(\omega(\xi)). \tag{4.8}$$

By considering the convolution's properties, we obtain

$$\frac{(\Psi_{q_1}^{k+1} * \Phi_v^{k+1} * \Upsilon)(\xi)}{(\Psi_{q_2}^{k+1} * \Phi_v^{k+1} * \Upsilon)(\xi)} = \frac{\rho_1(\xi) * (G(\xi) \sigma(\xi))}{\rho_2(\xi) * G(\xi)}. \tag{4.9}$$

Since $\rho_1 \prec_r \rho_2$, by letting $r \rightarrow 1$, we obtain $\rho_1(\xi) = \rho_2(\xi)$. As a result, by Lemma 3.1, we deduce that

$$\frac{(\Psi_{q_1}^{k+1} * \Phi_v^{k+1} * \Upsilon)(\xi)}{(\Psi_{q_2}^{k+1} * \Phi_v^{k+1} * \Upsilon)(\xi)} = \frac{\rho_1(\xi) * (G(\xi) \sigma(\xi))}{\rho_2(\xi) * G(\xi)} \in \overline{\text{co}}(\sigma(\omega(\cup))) \subset \overline{\text{co}}(\sigma(\cup)). \tag{4.10}$$

Since $\sigma(\xi)$ is a convex univalent function in \cup with $\sigma(0) = 1$, then by the definition of subordination, we obtain

$$\frac{(\Psi_{q_1}^{k+1} * \Phi_v^{k+1} * \Upsilon)(\xi)}{(\Psi_{q_2}^{k+1} * \Phi_v^{k+1} * \Upsilon)(\xi)} < \sigma(\xi) \implies \Upsilon \in \mathcal{E}_{q_1, q_2}^{v, k+1}(\sigma), \tag{4.11}$$

which completes the proof. □

We note that if we replace the condition of Theorem 4.3 by $\rho_2 \prec_r \rho_1$ such that $\rho_1 \in \mathcal{C}(\frac{1+\xi}{1-\xi})$ then we obtain the same conclusion.

Theorem 4.4 *Let $\gamma \in \Lambda$ and let the function $H := \Psi_{q_2}^k * \Phi_{v_1}^k * \gamma \in S^*(\frac{1+\xi}{1-\xi})$, $\xi \in \cup$. If $\Phi_{v_1}^k \prec_r \Phi_{v_2}^k$ for some $r < 1$ then*

$$\mathcal{E}_{q_1, q_2}^{v_1, k}(\sigma) \subset \mathcal{E}_{q_1, q_2}^{v_2, k}(\sigma). \tag{4.12}$$

Proof Suppose that $\gamma \in \mathcal{E}_{q_1, q_2}^{v_1, k}(\sigma)$. Consequently, a Schwarz function ϑ exists with $\vartheta(0) = 0$ and $|\vartheta(z)| < 1$ such that

$$\frac{(\Psi_{q_1}^k * \Phi_{v_1}^k * \psi)(z)}{(\Psi_{q_2}^k * \Phi_{v_1}^k * \gamma)(\xi)} = \sigma(\vartheta(\xi)), \quad \xi \in \cup. \tag{4.13}$$

This yields

$$(\Psi_{q_1}^k * \Phi_{v_1}^k * \gamma)(\xi) = (\Psi_{q_2}^k * \Phi_{v_1}^k * \gamma)(\xi)\sigma(\omega(\xi)) = H(\xi)\sigma(\vartheta(\xi)). \tag{4.14}$$

But the condition $\Phi_{v_1}^k \prec_r \Phi_{v_2}^k$ implies that $\Phi_{v_1}^k(r\xi) = \Phi_{v_2}^k(r\xi)$ (for some r). It is clear that $\eta(\xi) = \xi \in \mathcal{C}(\frac{1+\xi}{1-\xi})$; therefore, by the convolution’s properties, we attain

$$\frac{(\Psi_{q_1}^k * \Phi_{v_2}^k * \gamma)(\xi)}{(\Psi_{q_2}^k * \Phi_{v_2}^k * \gamma)(\xi)} = \frac{\eta(\xi) * (H(\xi)\vartheta(\xi))}{\eta(\xi) * H(\xi)}, \quad \xi \in \cup. \tag{4.15}$$

Thus, in view of Lemma 3.1, we get

$$\frac{(\Psi_{q_1}^k * \Phi_{v_2}^k * \gamma)(\xi)}{(\Psi_{q_2}^k * \Phi_{v_2}^k * \gamma)(\xi)} \in \overline{\text{co}}(\sigma(\vartheta(\cup))) \subset \overline{\text{co}}(\sigma(\cup)). \tag{4.16}$$

Since $\sigma(\xi)$ is a convex univalent function in \cup with $\sigma(0) = 1$, then by the definition of subordination, we obtain

$$\frac{(\Psi_{q_1}^k * \Phi_{v_2}^k * \gamma)(\xi)}{(\Psi_{q_2}^k * \Phi_{v_2}^k * \gamma)(\xi)} \prec \sigma(\xi) \implies \gamma \in \mathcal{E}_{q_1, q_2}^{v_2, k}(\sigma), \tag{4.17}$$

which completes the proof. □

We record that if we change the condition of Theorem 4.4 by $\Phi_{v_2}^k \prec_r \Phi_{v_1}^k$, we have

$$\mathcal{E}_{q_1, q_2}^{v_2, k}(\sigma) \subset \mathcal{E}_{q_1, q_2}^{v_1, k}(\sigma). \tag{4.18}$$

5 Integral inequalities

The following section deals with some inequalities containing the operator (2.7). For two functions $h(\xi) = \sum a_n \xi^n$ and $\tilde{h}(\xi) = \sum b_n \xi^n$, we have $h \ll \tilde{h}$ if and only if $|a_n| \leq |b_n|, \forall n$. This inequality is known as the majorization of two analytic functions.

Theorem 5.1 Consider the operator $[\mathcal{S}_v^k]_q \Upsilon (\xi)$, $\Upsilon \in \Lambda$. If the coefficients of Υ satisfy the inequality $|\Upsilon_n| \leq (\frac{1}{nv})^k$, $v \in (0, 1)$ then

$$\int_0^{2\pi} \left| \frac{[\mathcal{S}_v^k]_q \Upsilon (\xi)}{\xi} \right|^p d\theta \leq \int_0^{2\pi} \left| \left(\frac{1+\xi}{1-\xi} \right)^\delta \right|^p d\theta, \quad p > 0. \tag{5.1}$$

Proof Let

$$\sigma(\xi, \delta) = \left(\frac{1+\xi}{1-\xi} \right)^\delta, \quad \xi \in \cup, \delta \geq 1. \tag{5.2}$$

Then, a straightforward computation implies that

$$\begin{aligned} \sigma(\xi, 1) &= 1 + \sum_{n=1} (2n)\xi^n, \\ \sigma(\xi, 2) &= 1 + \sum_{n=1} (4n)\xi^n = 1 + 4\xi + 8\xi^2 + 12\xi^3 + 16\xi^4 + 20\xi^5 + \dots, \\ \sigma(\xi, 3) &= 1 + \sum_{n=1} (2 + 4n^2)\xi^n = 1 + 6\xi + 18\xi^2 + 38\xi^3 + \dots, \\ \sigma(\xi, 4) &= 1 + \sum_{n=1} \frac{1}{3}(8n(2 + n^2))\xi^n = 1 + 8\xi + 16\xi^2 + 24\xi^3 + \dots, \\ &\vdots \end{aligned} \tag{5.3}$$

Comparing Eq. (5.3) and the coefficients of $[\mathcal{S}_v^k]_q \Upsilon (\xi)$, which are satisfying

$$\lim_{q \rightarrow 1^-} \left| [n]_q^k \left(\frac{\kappa_1(v, \xi) + (-1)^{n+1} \kappa_0(v, \xi)}{\kappa_1(v, \xi) + \kappa_0(v, \xi)} \right)^k \Upsilon_n \right| \leq 1, \tag{5.4}$$

we conclude that $[\mathcal{S}_v^k]_q \Upsilon (\xi)$ is majorized by the function $\sigma(\xi, \delta)$ for all $\delta \geq 1$. By the properties of majorization [18], we have

$$\frac{[\mathcal{S}_v^k]_q \Upsilon (\xi)}{\xi} < \sigma(\xi, \delta), \quad \xi \in \cup. \tag{5.5}$$

Thus, according to Lemma 3.2, we conclude that

$$\int_0^{2\pi} \left| \frac{[\mathcal{S}_v^k]_q \Upsilon (\xi)}{\xi} \right|^p d\theta \leq \int_0^{2\pi} \left| \left(\frac{1+\xi}{1-\xi} \right)^\delta \right|^p d\theta, \quad p > 0. \tag{5.6}$$

□

In the same manner as in the proof of Theorem 5.1, one can get the next result:

Theorem 5.2 Consider the operator $[\mathcal{S}_v^k]_q \Upsilon (\xi)$, $\Upsilon \in \Lambda$. If the coefficients of Υ satisfy the inequality $|\Upsilon_n| \leq (\frac{1}{nv})^k$, $v \in (0, 1)$ then

$$\int_0^{2\pi} |([\mathcal{S}_v^k]_q \Upsilon (\xi))'|^p d\theta \leq \int_0^{2\pi} \left| \left(\frac{1+\xi}{1-\xi} \right)^\delta \right|^p d\theta, \quad p > 0.$$

Moreover, the inequality in Theorem 5.1 can be studied in the following result:

Theorem 5.3 *Consider the operator $S_q^{\kappa, k} \psi(z)$, $\psi \in \Lambda$. If the coefficients of ψ satisfy the inequality $|\vartheta_n| \leq (\frac{1}{nk})^k$, $\kappa \in (0, \infty)$ then there is a probability measure μ on $(\partial U)^2$, for all $\delta > 1$.*

Proof Let $\epsilon, \epsilon \in \partial U$. Then we have

$$\begin{aligned} \left(\frac{1 + \epsilon\xi}{1 + \epsilon\xi}\right)^\delta &= \frac{(1 + \epsilon\xi)^\delta}{1 + \epsilon\xi} \cdot \frac{1}{(1 + \epsilon\xi)^{\delta-1}} \\ &\ll \frac{(1 + \xi)^\delta}{1 - \xi} \cdot \frac{1}{(1 - \xi)^{\delta-1}} \\ &= \left(\frac{1 + \xi}{1 - \xi}\right)^\delta, \quad \delta > 1. \end{aligned} \tag{5.7}$$

By virtue of Theorem 1.11 in [19], the functional $(\frac{1 + \epsilon\xi}{1 + \epsilon\xi})^\delta$ defines a probability measure μ in $(\partial U)^2$ fulfilling

$$\chi(\xi) = \int_{(\partial U)^2} \left(\frac{1 + \epsilon\xi}{1 + \epsilon\xi}\right)^\delta d\mu(\epsilon, \epsilon), \quad \xi \in U. \tag{5.8}$$

Then there is a constant c (diffusion constant) such that

$$\int_{(\partial U)^2} \left(\frac{1 + \epsilon\xi}{1 + \epsilon\xi}\right)^\delta d\mu(\epsilon, \epsilon) = c \int_{(\partial U)^2} \left(\frac{[S_q^\kappa]_q \Upsilon(\xi\epsilon)}{\epsilon\xi}\right)^\delta d\mu(\epsilon, \epsilon), \quad \xi \in U. \tag{5.9}$$

This completes the proof. □

6 A class of differential equations

This section deals with an application of the operator (2.7) in a class of differential equations (for recent work see [20]). The class of quantum III-Painlevé differential equations has been studied recently in [21–23]. This class takes the formula

$$\xi \Upsilon(\xi) \frac{d^2 \Upsilon(\xi)}{d\xi^2} = \xi \left(\frac{d \Upsilon(\xi)}{d\xi}\right)^2 - \Upsilon(\xi) \frac{d \Upsilon(\xi)}{d\xi}, \quad \xi \in U, \Upsilon \in \Lambda. \tag{6.1}$$

Rearranging Eq. (6.1), we have

$$\left(1 + \frac{\xi \Upsilon''(\xi)}{\Upsilon'(\xi)}\right) - \frac{\xi \Upsilon'(\xi)}{\Upsilon(\xi)} = 0, \quad \xi \in U, \tag{6.2}$$

subjected to the boundary conditions

$$\Upsilon(\xi) = \xi + \Upsilon_2 \xi^2 + O(\xi^3), \quad |\Upsilon_n| \leq \frac{1}{[Q_n]_q^k}, \quad n \geq 2, \xi \in U = \{\xi \in \mathbb{C} : |\xi| < 1\}, \tag{6.3}$$

where

$$[Q_n]_q^k := [n]_q^k \left(\frac{\kappa_1(\nu, \xi) + (-1)^{n+1} \kappa_0(\nu, \xi)}{\kappa_1(\nu, \xi) + \kappa_0(\nu, \xi)}\right)^k.$$

Now by employing the operator (2.7), Eq. (6.2) becomes (called q -Painlevé differential equation of type III)

$$\left(1 + \frac{\xi([\mathcal{S}_v^k]_q \Upsilon(\xi))''}{([\mathcal{S}_v^k]_q \Upsilon(\xi))'}\right) - \frac{\xi([\mathcal{S}_v^k]_q \Upsilon(\xi))'}{([\mathcal{S}_v^k]_q \Upsilon(\xi))} = 0, \quad \xi \in \cup, \tag{6.4}$$

subjected to (6.3). Our aim is to study the geometric solution of (6.4) satisfying the boundary condition (6.3). For this purpose, we define the following analytic class:

Definition 6 For a function $\Upsilon \in \Lambda$ and a convex function $\psi \in \cup$ with $\psi(0) = 0$, the function Υ is said to be in the class $\mathbf{V}_q(\psi)$ if and only if

$$P(\xi) := \left(1 + \frac{\xi([\mathcal{S}_v^k]_q \Upsilon(\xi))''}{([\mathcal{S}_v^k]_q \Upsilon(\xi))'}\right) - \frac{\xi([\mathcal{S}_v^k]_q \Upsilon(\xi))'}{([\mathcal{S}_v^k]_q \Upsilon(\xi))} < \psi(\xi), \quad \xi \in \cup, \tag{6.5}$$

where $\psi(\xi) \in \Lambda$.

For the functions in the class $\mathbf{V}_q(\psi)$, the following result holds.

Theorem 6.1 *If the function $\Upsilon \in \mathbf{V}_q(\psi)$ is given by (2.1), then*

$$|\Upsilon_2| \leq \frac{1}{[Q_2]_q^k}, \quad |\Upsilon_3| \leq \frac{1}{[Q_3]_q^k}. \tag{6.6}$$

Proof Let $\Upsilon \in \mathbf{V}_q(\psi)$ have the expansion

$$\Upsilon(\xi) = \xi + \Upsilon_2 \xi^2 + \Upsilon_3 \xi^3 + \dots, \quad \xi \in \cup.$$

Moreover, we let

$$[Q_n]_q^k := [n]_q^k \left(\frac{\kappa_1(\nu, \xi) + (-1)^{n+1} \kappa_0(\nu, \xi)}{\kappa_1(\nu, \xi) + \kappa_0(\nu, \xi)} \right)^k.$$

Then by the definition of subordination, there is a Schwarz function \mathbb{T} with $\mathbb{T}(0) = 0$ and $|\mathbb{T}(\xi)| < 1$ satisfying $P(\xi) = \psi(\mathbb{T}(\xi))$, $\xi \in \cup$. Furthermore, if we assume that $|\mathbb{T}(\xi)| = |\xi| < 1$, then, in view of Schwarz lemma, there is a complex number τ with $|\tau| = 1$ satisfying $\mathbb{T}(\xi) = \tau \xi$. Consequently, we obtain

$$\begin{aligned} P(\xi) &= \psi(\mathbb{T}(\xi)) \\ \implies &\left(1 + \frac{\xi([\mathcal{S}_v^k]_q \Upsilon(\xi))''}{([\mathcal{S}_v^k]_q \Upsilon(\xi))'}\right) - \frac{\xi([\mathcal{S}_v^k]_q \Upsilon(\xi))'}{([\mathcal{S}_v^k]_q \Upsilon(\xi))} = \psi(\tau(\xi)) \\ \implies &1 + (2 \Upsilon_2 [Q_2]_q^k \xi + (6 \Upsilon_3 [Q_3]_q^k - 4(\Upsilon_2 [Q_2]_q^k)^2) \xi^2 + \dots) \\ &\quad - (1 + \Upsilon_2 [Q_2]_q^k \xi + (2 \Upsilon_3 [Q_3]_q^k - (\Upsilon_2 [Q_2]_q^k)^2) \xi^2 + \dots) \\ &= \tau(\xi + \psi_2 \xi^2 + \psi_3 \xi^3 + \dots) \\ \implies &\Upsilon_2 [Q_2]_q^k \xi + (4 \Upsilon_3 [Q_3]_q^k - 3(\Upsilon_2 [Q_2]_q^k)^2) \xi^2 + \dots \\ &= \tau \xi + \tau \psi_2 \xi^2 + \tau \psi_3 \xi^3 + \dots \end{aligned}$$

It follows that

$$|\Upsilon_2[Q_2]_q^k| = |\tau| = 1, \quad |\Upsilon_3[Q_3]_q^k| \leq \frac{1}{4}(|\psi_2| + 3).$$

Since ψ is convex univalent in \cup , $|\psi_n| \leq 1, \forall n$; this implies that

$$|\Upsilon_2| \leq \frac{1}{[Q_2]_q^k}, \quad |\Upsilon_3| \leq \frac{1}{[Q_3]_q^k}.$$

Hence, the proof is complete. □

We need the following fact, which can be located in [12].

Lemma 6.2 Consider functions $f_1, f_2, f_3 : \cup \rightarrow \mathbb{C}$ such that $\Re(f_1) \geq a \geq 0$. If $f \in \mathbb{H}[1, n]$ (the set of analytic functions having the expansion $f(\xi) = 1 + \varphi_1\xi + \dots$) and

$$\Re(a\xi^2 f''(\xi) + f_1(\xi)\xi f'(\xi) + f_2(\xi)f(\xi) + f_3(\xi)) > 0, \quad a \geq 0, \xi \in \cup,$$

then $\Re(f(\xi)) > 0$.

Lemma 6.3 Let b be convex in \cup and suppose $f_1, f_2, f_3 : \cup \rightarrow \mathbb{C}$ are analytic functions such that $\Re(f_1) \geq a \geq 0$. If $g \in \mathbb{H}[0, m]$ (the set of analytic functions with the expansion $g(\xi) = g_1\xi^m + \dots$), $m \geq 1$ and

$$a\xi^2 g''(\xi) + f_1(\xi)\xi g'(\xi) + f_2(\xi)g(\xi) + f_3(\xi) < b(\xi), \quad a \geq 0, \xi \in \cup,$$

then $g(\xi) < b(\xi)$.

Lemma 6.4 Let $a, b, c \in \mathbb{R}$ be such that $a \geq 0, b \geq -a, c \geq -b$. If $q \in \mathbb{H}[0, 1]$, where $q(\xi) = q_1\xi + \dots$ and

$$a\xi^2 q''(\xi) + b\xi q'(\xi) + cq(\xi) < \xi, \quad a \geq 0, \xi \in \cup,$$

then $q(\xi) < \frac{\xi}{b+c}$, which is the best dominant.

Theorem 6.5 Let $\Upsilon \in \mathbf{V}_q(\xi)$ and $F(\xi) = \frac{\xi([S_v^k]_q \Upsilon(\xi))'}{([S_v^k]_q \Upsilon(\xi))}$. If $\Re(\xi F(\xi)) > -1, \xi \in \cup$, then $[S_v^k]_q \Upsilon \in \mathfrak{S}^*$ (starlike with respect to the origin).

Proof Let $F(\xi) = \frac{\xi([S_v^k]_q \Upsilon(\xi))'}{([S_v^k]_q \Upsilon(\xi))}$. Then a straightforward computation implies that

$$\begin{aligned} \xi F'(\xi) &= \xi \left(\frac{\xi([S_v^k]_q \Upsilon(\xi))'}{([S_v^k]_q \Upsilon(\xi))} \right)' \\ &= \frac{\xi([S_v^k]_q \Upsilon(\xi))''}{([S_v^k]_q \Upsilon(\xi))'} \frac{\xi([S_v^k]_q \Upsilon(\xi))'}{[S_v^k]_q \Upsilon(\xi)} + \frac{\xi([S_v^k]_q \Upsilon(\xi))'}{[S_v^k]_q \Upsilon(\xi)} - \left(\frac{\xi([S_v^k]_q \Upsilon(\xi))'}{[S_v^k]_q \Upsilon(\xi)} \right)^2 \\ &= \frac{\xi([S_v^k]_q \Upsilon(\xi))'}{\Upsilon(\xi)} \left(1 + \frac{\xi([S_v^k]_q \Upsilon(\xi))''}{([S_v^k]_q \Upsilon(\xi))'} - \frac{\xi([S_v^k]_q \Upsilon(\xi))'}{[S_v^k]_q \Upsilon(\xi)} \right) \\ &= F(\xi)P(\xi). \end{aligned}$$

Hence, we obtain

$$P(\xi) = \frac{\xi F'(\xi)}{F(\xi)} < \xi, \quad \psi(\xi) := \xi.$$

It is clear that $F \in \mathbb{H}[1, 1]$ and

$$\mathfrak{N}\left(\frac{\xi F'(\xi)}{F(\xi)}\right) = \mathfrak{N}(\xi) \Rightarrow 1 + \mathfrak{N}(\xi F'(\xi)) = 1 + \mathfrak{N}(\xi F(\xi)) > 0.$$

Then in view of Lemma 6.2, with $a = 0, f_1 = 1, f_2 = 0$ and $f_3 = 1$, we have

$$\mathfrak{N}(F(\xi)) = \mathfrak{N}\left(\frac{\xi ([\mathcal{S}_v^k]_q \Upsilon(\xi))'}{[\mathcal{S}_v^k]_q \Upsilon(\xi)}\right) > 0;$$

that is, $[\mathcal{S}_v^k]_q \Upsilon \in \mathfrak{S}^*$ with respect to the origin. □

Theorem 6.6 Let $\Upsilon \in \wedge$ and $F(\xi) = \frac{\xi ([\mathcal{S}_v^k]_q \Upsilon(\xi))'}{[\mathcal{S}_v^k]_q \Upsilon(\xi)}$. If

$$\frac{\xi F'(\xi)}{F(\xi)} \left(2 + \frac{\xi F''(\xi)}{F'(\xi)} - \frac{\xi F'(\xi)}{F(\xi)} \right) < \psi(\xi),$$

where ψ is convex in \cup , then $\Upsilon \in \mathbf{V}_q(\psi)$.

Proof Let $\Upsilon \in \wedge$ and $F(\xi) = \frac{\xi ([\mathcal{S}_v^k]_q \Upsilon(\xi))'}{[\mathcal{S}_v^k]_q \Upsilon(\xi)}$. As in the proof of Theorem 6.5, we have $P(\xi) = \frac{\xi F'(\xi)}{F(\xi)}$. Then a calculation gives

$$\begin{aligned} \xi P'(\xi) + P(\xi) &= \xi \left(\frac{\xi F'(\xi)}{F(\xi)} \right)' + \left(\frac{\xi F'(\xi)}{F(\xi)} \right) \\ &= \frac{\xi F'(\xi)}{F(\xi)} \left(2 + \frac{\xi F''(\xi)}{F'(\xi)} - \frac{\xi F'(\xi)}{F(\xi)} \right) \\ &< \psi(\xi). \end{aligned}$$

Obviously, $P \in \mathbb{H}[0, m], m = 1$, and, by letting $a = 0, f_1(\xi) = 1$, and $f_2(\xi) = 1$, in view of Lemma 6.3, we have

$$P(\xi) = 1 + \frac{([\mathcal{S}_v^k]_q \Upsilon(\xi))''}{([\mathcal{S}_v^k]_q \Upsilon(\xi))'} - \frac{\xi ([\mathcal{S}_v^k]_q \Upsilon(\xi))'}{[\mathcal{S}_v^k]_q \Upsilon(\xi)} < \psi(\xi).$$

Consequently, we get $\Upsilon \in \mathbf{V}_q(\psi)$. □

Theorem 6.7 Let $\Upsilon \in \wedge$ and $F(\xi) = \frac{\xi ([\mathcal{S}_v^k]_q \Upsilon(\xi))'}{[\mathcal{S}_v^k]_q \Upsilon(\xi)}$. If

$$\frac{\xi F'(\xi)}{F(\xi)} \left(1 + \frac{\xi F''(\xi)}{F'(\xi)} - \frac{\xi F'(\xi)}{F(\xi)} \right) < \xi,$$

where ψ is convex in \cup , then $\Upsilon \in \mathbf{V}_q(\xi)$.

Proof Let $\Upsilon \in \Lambda$ and $F(\xi) = \frac{\xi([\mathcal{S}_v^k]_q \Upsilon(\xi))'}{[\mathcal{S}_v^k]_q \Upsilon(\xi)}$. As in the proof of Theorem 6.5, we have $P(\xi) = \frac{\xi F'(\xi)}{F(\xi)}$. Then a straightforward calculation gives

$$\begin{aligned} \xi P'(\xi) &= \xi \left(\frac{\xi F'(\xi)}{F(\xi)} \right)' \\ &= \frac{\xi F'(\xi)}{F(\xi)} \left(1 + \frac{\xi F''(\xi)}{F'(\xi)} - \frac{\xi F'(\xi)}{F(\xi)} \right) \\ &< \xi. \end{aligned}$$

Obviously, $P \in \mathbb{H}[0, 1]$ and, by letting $a = 0$, $b = 1$, and $c = 0$, where $c \geq -b$, in view of Lemma 6.4, we have

$$P(\xi) = 1 + \frac{([\mathcal{S}_v^k]_q \Upsilon(\xi))''}{([\mathcal{S}_v^k]_q \Upsilon(\xi))'} - \frac{\xi([\mathcal{S}_v^k]_q \Upsilon(\xi))'}{[\mathcal{S}_v^k]_q \Upsilon(\xi)} < \xi.$$

Consequently, we obtain $\Upsilon \in \mathbf{V}_q(\xi)$. \square

7 Conclusion

In this paper, we presented different types of integral inequalities based on q -calculus and conformable differential operator. These inequalities described the relations between the quantum conformable differential operators for different orders.

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