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Existence of multiple solutions for nonlinear multi-point boundary value problems

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Abstract

In this paper, we study some nonlinear second order multi-point boundary value problems. We first give a lemma about the characteristic values of the corresponding linear operator. Then, by fixed point theorems in the recent existing literature, we obtain the existence of multiple solutions for these nonlinear second order multi-point boundary value problems, including two positive solutions, two negative solutions, and one sign-changing solution.

MSC: 34B15; 34B18; 47H11

Keywords: Fixed point theorem; Characteristic value; Sign-changing solution; Multi-point boundary value problem

1 Introduction

In this paper, we study the following nonlinear multi-point boundary value problem:

$$\begin{cases} -u''(t) = g(t, u(t)), & 0 \le t \le 1, \\ u'(0) = 0, & u(1) = \sum_{i=1}^{m-2} \alpha_i u(\beta_i), \end{cases} \tag{1.1}$$

where $g : [0,1] \times (-\infty, +\infty) \to (-\infty, +\infty)$ is continuous; $0 < \beta_1 < \beta_2 < \cdots < \beta_{m-2} < 1$; $\alpha_i > 0$ $(i = 1, 2, \ldots, m-2)$ with $\sum_{i=1}^{m-2} \alpha_i < 1$.

The multi-point boundary value problem is an important part of boundary value problems for ordinary differential equations, which arise in physics and applied mathematics (see [1]). In 1992, Gupta considered nonlinear three-point boundary value problems (see [2]). Since then, many authors have considered the existence of nontrivial solutions for nonlinear multi-point boundary value problems and obtained many great results. We can refer to [3–18] and the references therein. For example, in [4], Xu has studied the following multi-point boundary value problem:

$$\begin{cases} -x''(t) = f(x), & t \in [0,1], \\ x(0) = 0, & x(1) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i), \end{cases} \tag{1.2}$$

where $f : (-\infty, +\infty) \to (-\infty, +\infty)$ is continuous; $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$; $\alpha_i > 0$ $(i = 1, 2, \ldots, m-2)$. In [4], the author gave the result about the characteristic values of the relevant linear operator. In addition, the author obtained the existence of multiple positive solutions and sign-changing solutions for BVP (1.2) by applying Leray–Schauder degree methods.

In [5], Zhang and Sun studied the following nonlinear multi-point boundary value problem:

$$\begin{cases} -(Ly)(t) = h(t)f(y(t)), & t \in (0,1), \\ y'(0) = 0, & y(1) = \sum_{i=1}^{m-2} a_i y(\xi_i), \end{cases} \tag{1.3}$$

where $(Ly)(t) = (p(t)y'(t))' + q(t)y(t)$, $f : [0,+\infty) \to [0,+\infty)$ is continuous; $h : (0,1) \to [0,+\infty)$ is continuous, and h is singular at $t = 0$ and $t = 1$; $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$; $a_i > 0$ $(i = 1,2,\dots,m-2)$. The authors proved the existence of the first eigenvalue of the relevant linear operator, and they considered the existence of positive solutions for BVP (1.3). The method they used is the fixed point index theory.

In [15], Li considered the following second order three-point boundary value problem:

$$\begin{cases} -y''(t) = f(t,u(t)), & t \in [0,1], \\ y'(0) = 0, & y(1) = \alpha y(\eta), \end{cases} \tag{1.4}$$

where $\eta \in (0,1), \alpha \in (0,1); f : [0,1] \times (-\infty,+\infty) \to (-\infty,+\infty)$ is continuous. The author also studied the characteristic values of the relevant linear operator about BVP (1.4). By means of fixed point theorems with lattice structure, the author has obtained the existence results of negative and sign-changing solution for BVP (1.4) for superlinear case.

Inspired by [3–18], we consider boundary value problem (1.1) in this paper. By the existing fixed point theorems due to Sang et al. [19], we obtain the existence results of multiple nontrivial solutions for BVP (1.1) for asymptotically linear case, including two positive solutions, one sign-changing, and two negative solutions. Characteristic value is an important index of the linear operator. One of the features of this paper is that we give a lemma about the characteristic values of the relevant linear operator about BVP (1.1). The other feature of this paper is that the obtained main theorems are new results for BVP (1.1), which improve and generalize BVP (1.4).

The rest of the paper is organized as follows. We introduce the definition of e-continuous operator and the used fixed point theorems due to Sang et al. [19] in Sect. 2. We give the main lemma about the characteristic values of the relevant linear operator, prove some auxiliary lemmas that we need, and obtain the main result of the existence of multiple solutions for BVP (1.1) in Sect. 3. We provide an example to illustrate our main result in Sect. 4.

2 Preliminaries

In the following, we mainly introduce the e-continuous operator and list the used fixed point theorems due to Sang et al. [19]. For detailed concepts and properties about the cone, we can refer to [21–23].

Let E be a Banach space, P be a cone of E. Let A be an operator. If $Au = u$ with $u \notin P \cup (-P)$, then u is said to be a sign-changing fixed point of A. The linear operator B is said to be positive if $B(P) \subset P$.

Definition 2.1 (see [20]) Let $A : D \to E$ be an operator, $e \in P\backslash\{\theta\}$, and $u_0 \in D$. If, for any given $\epsilon > 0$, there exists $\delta > 0$ such that

$$-\epsilon e \leq Au - Au_0 \leq \epsilon e, \quad \text{for all } u \in D \text{ with } \|u - u_0\| < \delta,$$

then A is said to be e-continuous at u_0. If A is e-continuous at every point $u \in D$, then A is said to be e-continuous on D.

Lemma 2.1 (see [19]) *Let E be a Banach space, P be a normal and total cone of E; $B : E \to E$ be a positive linear completely continuous operator and be also e-continuous on E; $F : E \to E$ be a continuous and bounded increasing operator and $A = BF$. Assume that*

(i) *there exist $x_1 \in P\backslash\{\theta\}$ and $x_2 \in (-P)\backslash\{\theta\}$ such that $Ax_1 \leq x_1, x_2 \leq Ax_2$; and there exists $\alpha > 0$ such that $\alpha e \leq x_1, x_2 \leq -\alpha e$;*

(ii) *$F(\theta) = \theta$, F is Fréchet differentiable at θ; and A'_θ has a characteristic value $\lambda_0 < 1$ with a characteristic function ψ satisfying $\mu_1 e \leq \psi \leq \mu_2 e$, where $\mu_1 > 0, \mu_2 > 0$;*

(iii) *the Fréchet derivative A'_∞ at ∞ exists; A'_∞ is increasing; $r(A'_\infty) > 1$; 1 is not a a characteristic value of A'_∞.*

Then A has at least two positive fixed points, one sign-changing solution, fixed point, and two negative fixed points.

3 Main results

Let $E = C[0, 1]$. Define the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$ on E, then E is a Banach space. Let $P = \{u \in E | u(t) \geq 0, t \in [0, 1]\}$. It is obvious that P is a normal and total cone of E (see [21–23]).

For convenience, we first give the following assumptions to be used in the rest of this paper.

(H_1) The sequence of positive solutions of the equation

$$\cos\sqrt{x} = \sum_{i=1}^{m-2} \alpha_i \cos(\beta_i \sqrt{x})$$

is $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$.

(H_2) $g(t, 0) = 0$ and $g(t, x)$ is increasing in x uniformly on $t \in [0, 1]$.

(H_3) $\lim_{x\to0} \frac{g(t,x)}{x} = \eta$, $\forall t \in [0, 1]$ and $\eta > \lambda_1$, where λ_1 is defined by (H_1).

(H_4) $\lim_{x\to\infty} \frac{g(t,x)}{x} = \gamma$, $\forall t \in [0, 1]$ and $\gamma > \lambda_1$, $\gamma \neq \lambda_k$, where λ_k is defined by (H_1), $k = 2, 3, \ldots$.

(H_5) There exist $\tau > 0$, $w > 0$, and $C > 0$ such that

$$\frac{g(t,\tau)}{\tau} < C, \qquad \frac{g(t,-w)}{-w} < C \quad \text{for any } t \in [0, 1],$$

where $0 < C < (1 - \sum_{i=1}^{m-2} \alpha_i)$.

Lemma 3.1 (see [5]) *For $y(t) \in E$, the following boundary value problem*

$$\begin{cases} u''(t) + y(t) = 0, & t \in [0, 1], \\ u'(0) = 0, & u(1) = \sum_{i=1}^{m-2} \alpha_i u(\beta_i), \end{cases} \tag{3.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)\,ds, \quad 0 \le t \le 1,$$

where

$$G(t,s) = k(t,s) + \sum_{i=1}^{m-2} \widetilde{k}_i(t,s) + (1-s)\left(\sum_{i=1}^{m-2}\alpha_i\right)\left(1-\sum_{i=1}^{m-2}\alpha_i\right)^{-1}, \quad t,s \in [0,1]. \tag{3.2}$$

$$k(t,s) = \begin{cases} 1-s, & 0 \le t \le s \le 1, \\ 1-t, & 0 \le s \le t \le 1, \end{cases} \tag{3.3}$$

$$\widetilde{k}_i(t,s) = \begin{cases} -\alpha_i(1-\sum_{i=1}^{m-2}\alpha_i)^{-1}(\beta_i - s), & 0 \le s \le \beta_i, t \in [0,1], \\ 0, & \beta_i \le s \le 1, t \in [0,1], \end{cases}$$

$$i = 1, 2, \dots, m-2. \tag{3.4}$$

Define the following operators:

$$(Tu)(t) = \int_0^1 G(t,s)g\bigl(s,u(s)\bigr)\,ds, \tag{3.5}$$

$$(Lu)(t) = \int_0^1 G(t,s)u(s)\,ds, \tag{3.6}$$

$$(Gu)(t) = g\bigl(t,u(t)\bigr), \tag{3.7}$$

where $T = LG$, $G(t,s)$ is defined by (3.2).

Obviously, $T : E \to E$ is completely continuous (see [5]).

Lemma 3.2 *Assume that* (H_1) *holds. Then the sequence of positive characteristic values of the linear operator L defined by* (3.6) *is*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$$

and the positive characteristic values λ_n have algebraic multiplicity one.

Proof Let ξ be a positive characteristic value and $u(t)$ be a characteristic function corresponding to the characteristic value ξ. □

From Lemma 3.1, we obtain

$$\begin{aligned} &u''(t) + \xi u(t) = 0, \quad 0 \le t \le 1. \\ &u'(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2}\alpha_i u(\beta_i). \end{aligned} \tag{3.8}$$

Then the form of the general solution for the differential equation (3.8) is

$$u(t) = C_1 \sin(t\sqrt{\xi}) + C_2 \cos(t\sqrt{\xi}), \quad \forall t \in [0,1]. \tag{3.9}$$

Since $u'(0) = 0$, we know that $C_1 = 0$. Then (3.9) can be reduced to

$$u(t) = C_2 \cos(t\sqrt{\xi}), \quad \forall t \in [0,1].$$

From $u(1) = \sum_{i=1}^{m-2} \alpha_i u(\beta_i)$, we have

$$\cos\sqrt{\xi} = \sum_{i=1}^{m-2} \alpha_i \cos(\beta_i\sqrt{\xi}).$$

By (H_1), we know that ξ is one of the values $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, and the corresponding characteristic function is

$$u_n(t) = C\cos(t\sqrt{\lambda_n}), \quad t \in [0,1],$$

where C is a nonzero constant.

By ordinary method, we can know that

$$\dim \operatorname{Ker}(I - \lambda_n L) = 1. \tag{3.10}$$

We need to prove that

$$\operatorname{Ker}(I - \lambda_n L) = \operatorname{Ker}(I - \lambda_n L)^2. \tag{3.11}$$

It is obvious that we only need to prove that

$$\operatorname{Ker}(I - \lambda_n L)^2 \subset \operatorname{Ker}(I - \lambda_n L). \tag{3.12}$$

Take any $u \in \operatorname{Ker}(I - \lambda_n L)^2$. If $(I - \lambda_n L)u \neq \theta$, then $(I - \lambda_n L)u$ is a characteristic function of the linear operator L corresponding to the characteristic value λ_n. So we have

$$(I - \lambda_n L)u = b\cos(t\sqrt{\lambda_n}), \quad t \in [0,1],$$

where b is a nonzero constant.

Hence we easily obtain that

$$\begin{aligned} &u''(t) + \lambda_n u(t) = -\lambda_n b\cos(t\sqrt{\lambda_n}), \quad 0 \le t \le 1. \\ &u'(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\beta_i). \end{aligned} \tag{3.13}$$

Then the form of the general solution for the differential equation (3.13) is

$$u(t) = C_1 \sin(t\sqrt{\lambda_n}) + C_2 \cos(t\sqrt{\lambda_n}) - \frac{b}{4}\cos(2t\sqrt{\lambda_n})\cos(t\sqrt{\lambda_n})$$

$$-\left(\frac{b\sin(2t\sqrt{\lambda_n})}{4}+\frac{bt\sqrt{\lambda_n}}{2}\right)\sin(t\sqrt{\lambda_n}),\quad t\in[0,1]. \tag{3.14}$$

Since $u'(0)=0$, we know that $C_1=0$. By (3.14) and $\cos\sqrt{\lambda_n}=\sum_{i=1}^{m-2}\alpha_i\cos(\beta_i\sqrt{\lambda_n})$, we have

$$\begin{aligned}
u(1) &= C_2\cos\sqrt{\lambda_n}-\frac{b}{4}\cos(2\sqrt{\lambda_n})\cos\sqrt{\lambda_n}-\left(\frac{b\sin(2\sqrt{\lambda_n})}{4}+\frac{b\sqrt{\lambda_n}}{2}\right)\sin\sqrt{\lambda_n}\\
&= C_2\sum_{i=1}^{m-2}\alpha_i\cos(\beta_i\sqrt{\lambda_n})-\frac{b}{4}\bigl(1-2\sin^2\sqrt{\lambda_n}\bigr)\sum_{i=1}^{m-2}\alpha_i\cos(\beta_i\sqrt{\lambda_n})\\
&\quad -\frac{b}{2}\sin^2\sqrt{\lambda_n}\sum_{i=1}^{m-2}\alpha_i\cos(\beta_i\sqrt{\lambda_n})-\frac{b\sqrt{\lambda_n}}{2}\sin\sqrt{\lambda_n}\\
&= C_2\sum_{i=1}^{m-2}\alpha_i\cos(\beta_i\sqrt{\lambda_n})-\frac{b}{4}\sum_{i=1}^{m-2}\alpha_i\cos(\beta_i\sqrt{\lambda_n})-\frac{b\sqrt{\lambda_n}}{2}\sin\sqrt{\lambda_n}
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
\sum_{i=1}^{m-2}\alpha_i u(\beta_i) &= C_2\sum_{i=1}^{m-2}\alpha_i\cos(\beta_i\sqrt{\lambda_n})-\frac{b}{4}\sum_{i=1}^{m-2}\alpha_i\cos(2\beta_i\sqrt{\lambda_n})\cos(\beta_i\sqrt{\lambda_n})\\
&\quad -\frac{b}{4}\sum_{i=1}^{m-2}\alpha_i\sin(2\beta_i\sqrt{\lambda_n})\sin(\beta_i\sqrt{\lambda_n})-\frac{b\sqrt{\lambda_n}}{2}\sum_{i=1}^{m-2}\alpha_i\beta_i\sin(\beta_i\sqrt{\lambda_n})\\
&= C_2\sum_{i=1}^{m-2}\alpha_i\cos(\beta_i\sqrt{\lambda_n})-\frac{b}{4}\sum_{i=1}^{m-2}\alpha_i\cos(\beta_i\sqrt{\lambda_n})\\
&\quad -\frac{b\sqrt{\lambda_n}}{2}\sum_{i=1}^{m-2}\alpha_i\beta_i\sin(\beta_i\sqrt{\lambda_n})
\end{aligned} \tag{3.16}$$

From (3.15), (3.16) and $u(1)=\sum_{i=1}^{m-2}\alpha_i u(\beta_i)$, we have

$$\sin\sqrt{\lambda_n}=\sum_{i=1}^{m-2}\alpha_i\beta_i\sin(\beta_i\sqrt{\lambda_n}). \tag{3.17}$$

Applying the Schwarz inequality and (3.17), we have

$$\begin{aligned}
1-\cos^2\sqrt{\lambda_n}=\sin^2\sqrt{\lambda_n} &= \left(\sum_{i=1}^{m-2}\alpha_i\beta_i\sin(\beta_i\sqrt{\lambda_n})\right)^2\\
&\le \left(\sum_{i=1}^{m-2}\beta_i^2\right)\left(\sum_{i=1}^{m-2}\alpha_i^2\sin^2(\beta_i\sqrt{\lambda_n})\right)\\
&= \left(\sum_{i=1}^{m-2}\beta_i^2\right)\left[\sum_{i=1}^{m-2}\alpha_i^2(1-\cos^2(\beta_i\sqrt{\lambda_n})\right]\\
&= \left(\sum_{i=1}^{m-2}\beta_i^2\right)\left(\sum_{i=1}^{m-2}\alpha_i^2\right)-\left(\sum_{i=1}^{m-2}\beta_i^2\right)\left(\sum_{i=1}^{m-2}\alpha_i^2\cos^2(\beta_i\sqrt{\lambda_n})\right).
\end{aligned}$$

Combining $\cos\sqrt{\lambda_n} = \sum_{i=1}^{m-2}\alpha_i\cos(\beta_i\sqrt{\lambda_n})$, we have

$$\begin{aligned}
1 &\leq \cos^2\sqrt{\lambda_n} + \left(\sum_{i=1}^{m-2}\beta_i^2\right)\left(\sum_{i=1}^{m-2}\alpha_i^2\right) - \left(\sum_{i=1}^{m-2}\beta_i^2\right)\left(\sum_{i=1}^{m-2}\alpha_i^2\cos^2(\beta_i\sqrt{\lambda_n})\right) \\
&= \left(\sum_{i=1}^{m-2}\alpha_i\cos(\beta_i\sqrt{\lambda_n})\right)^2 + \left(\sum_{i=1}^{m-2}\beta_i^2\right)\left(\sum_{i=1}^{m-2}\alpha_i^2\right) - \left(\sum_{i=1}^{m-2}\beta_i^2\right)\left(\sum_{i=1}^{m-2}\alpha_i^2\cos^2(\beta_i\sqrt{\lambda_n})\right) \\
&= \left(\sum_{i=1}^{m-2}\beta_i^2\right)\left(\sum_{i=1}^{m-2}\alpha_i^2\right) + \left[1 - \left(\sum_{i=1}^{m-2}\beta_i^2\right)\right]\left(\sum_{i=1}^{m-2}\alpha_i^2\cos^2(\beta_i\sqrt{\lambda_n})\right) \\
&\quad + \sum_{i\neq j}\alpha_i\alpha_j\cos(\beta_i\sqrt{\lambda_n})\cos(\beta_j\sqrt{\lambda_n}) \\
&\leq \left(\sum_{i=1}^{m-2}\beta_i^2\right)\left(\sum_{i=1}^{m-2}\alpha_i^2\right) + \left[1 - \left(\sum_{i=1}^{m-2}\beta_i^2\right)\right]\sum_{i=1}^{m-2}\alpha_i^2 + \sum_{i\neq j}\alpha_i\alpha_j \\
&= \sum_{i=1}^{m-2}\alpha_i^2 + \sum_{i\neq j}\alpha_i\alpha_j = \left(\sum_{i=1}^{m-2}\alpha_i\right)^2. \qquad (3.18)
\end{aligned}$$

Since $\sum_{i=1}^{m-2}\alpha_i < 1$, we know that (3.18) is a contradiction. So (3.12) holds. By (3.10) and (3.11), we know that the algebraic multiplicity of characteristic value λ_n is 1.

Lemma 3.3 *The linear operator L is $e(t)$-continuous on E.*

Proof Take $u_0 \in E$. For any given $\epsilon > 0$, we choose $\delta = (1 - \sum_{i=1}^{m-2}\alpha_i)\epsilon$, when $\|u - u_0\| \leq \delta$, we have

$$\begin{aligned}
\left|(Lu)(t) - (Lu_0)(t)\right| &\leq \int_0^1 G(t,s)\left|u(t) - u_0(t)\right|\,ds \\
&\leq \|u - u_0\|\left(1 + \frac{\sum_{i=1}^{m-2}\alpha_i}{1 - \sum_{i=1}^{m-2}\alpha_i}\right) < \epsilon,
\end{aligned}$$

where $e(t) = 1$. Hence L is $e(t)$-continuous on $u_0 \in E$. By the arbitrariness of u_0, L is $e(t)$-continuous on E. □

Lemma 3.4 *Suppose that* (H_2) *and* (H_3) *hold. Then T is Fréchet differentiable at θ and $T'_\theta = \eta L$.*

Proof By (H_3), for any given $\epsilon > 0$, there exists $\delta > 0$, when $0 < |x| < \delta$, we obtain

$$\left|\frac{g(t,x)}{x} - \eta\right| < \epsilon, \quad \forall t \in [0,1].$$

Namely,

$$\left|g(t,x) - \eta x\right| < \epsilon|x|, \quad \forall t \in [0,1], 0 < |x| < \delta.$$

So we have

$$\left|g\big(t,u(t)\big)-\eta u(t)\right| < \|Gu-\eta u\| \leq \epsilon\|u\|, \quad \forall t\in[0,1], |u|\leq\delta. \tag{3.19}$$

By (H_2), we have $T\theta=\theta$. By (3.19), we have

$$\begin{aligned}\|Tu-T\theta-\eta Lu\| &= \big\|(LG)u-(\eta L)u\big\| = \big\|L(Gu-\eta u)\big\| \\ &\leq \|L\|\cdot\|Gu-\eta u\| \leq \epsilon\|L\|\cdot\|u\|, \quad \forall\|u\|\leq\delta.\end{aligned}$$

So we have

$$\lim_{\|u\|\to 0}\frac{\|Tu-T\theta-\eta Lu\|}{\|u\|}=0.$$

Namely, $T'_\theta=\eta L$. □

Lemma 3.5 *Suppose that* (H_4) *holds. Then the Fréchet derivative* $T'_\infty=\gamma L$.

Proof By (H_4), for any given $\epsilon>0$, there exists $M>0$, when $|x|\geq M$, we have

$$\left|\frac{g(t,x)}{x}-\gamma\right|<\epsilon, \quad \forall t\in[0,1].$$

Namely,

$$\left|g(t,x)-\gamma x\right|<\epsilon|x|, \quad \forall t\in[0,1], |x|\geq M.$$

Let $\widetilde{M}=\max_{t\in[0,1],|x|\leq M}|g(t,x)-\gamma x|$. Then

$$\left|g(t,x)-\gamma x\right|\leq\widetilde{M}+\epsilon|x|, \quad \forall t\in[0,1], x\in(-\infty,+\infty).$$

So

$$\begin{aligned}\|Tu-\gamma Lu\| &= \big\|(LG)u-(\gamma L)u\big\| = \big\|L(Gu-\gamma u)\big\| \leq \|L\|\cdot\|Gu-\gamma u\| \\ &\leq \big(\widetilde{M}+\epsilon\|u\|\big)\|L\|, \quad \forall u\in E.\end{aligned}$$

Therefore

$$\lim_{\|u\|\to\infty}\frac{\|Tu-\gamma Lu\|}{\|u\|}=0.$$

Namely, $T'_\infty=\gamma L$. □

Theorem 3.1 *Suppose that* (H_1)–(H_5) *hold. Then BVP* (1.1) *has at least five nontrivial solutions*: *two positive solutions*, *one sign-changing solution*, *and two negative solutions*.

Proof (i) By (H_5), we have

$$g(t,\tau)\leq C\tau \quad \text{for } t\in[0,1].$$

So

$$T(\tau) = \int_0^1 G(t,s)g(s,\tau)\,ds \le C\int_0^1 G(t,s)\tau\,ds \le C\tau\left(1 + \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\right) \le \tau. \tag{3.20}$$

Similarly, by (H_5), we have

$$\begin{aligned} T(-w) &= \int_0^1 G(t,s)g(s,-w)\,ds \ge -Cw\int_0^1 G(t,s)\,ds \\ &\ge -Cw\left(1 + \frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\right) \ge -w. \end{aligned} \tag{3.21}$$

Take $\alpha = \min\{C\tau, Cw\}$. Then, by (3.20) and (3.21), we have

$$\alpha e(t) < \tau, \quad -w < -\alpha e(t), \tag{3.22}$$

where $e(t) = 1$. □

From (3.20)–(3.22), condition (i) of Lemma 2.1 is satisfied.

(ii) By (H_2), we have $G(\theta) = \theta$. Similar to the proof of Lemma 3.4, G is Fréchet differentiable at θ. By Lemma 3.4, we know that $\frac{\lambda_n}{\eta}$ is the characteristic value of T'_θ. Since $\eta > \lambda_1$, T'_θ has a characteristic value $\frac{\lambda_1}{\eta} < 1$. Let $\psi(t)$ be a characteristic function corresponding to the characteristic value $\frac{\lambda_1}{\eta} < 1$. Namely,

$$\left(\frac{\lambda_1}{\eta}T'_\theta\psi\right)(t) = \psi(t), \quad \forall t \in [0,1].$$

By (3.2)–(3.4), we have

$$(1-s)\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i} \le G(t,s) \le 1 + \frac{\sum_{i=1}^{m-2}\alpha_i(1-s)}{1-\sum_{i=1}^{m-2}\alpha_i}.$$

So

$$\lambda_1\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^1(1-s)\psi(s)\,ds}{1-\sum_{i=1}^{m-2}\alpha_i}\cdot 1 \le \psi(t) \le \lambda_1\left[1 + \frac{\sum_{i=1}^{m-2}\alpha_i}{2(1-\sum_{i=1}^{m-2}\alpha_i)}\right]\|\psi\|\cdot 1.$$

i.e.,

$$\mu_1 e(t) \le \psi(t) \le \mu_2 e(t),$$

where

$$\mu_1 = \lambda_1\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^1(1-s)\psi(s)\,ds}{1-\sum_{i=1}^{m-2}\alpha_i}, \qquad \mu_2 = \lambda_1\left[1 + \frac{\sum_{i=1}^{m-2}\alpha_i}{2(1-\sum_{i=1}^{m-2}\alpha_i)}\right]\|\psi\|, e(t) = 1.$$

Then condition (ii) of Lemma 2.1 is satisfied.

(iii) From Lemma 3.5, $T'_\infty = \gamma L$. So T'_∞ is increasing and $\frac{\lambda_n}{\gamma}$ is the characteristic value of T'_∞, where λ_n is defined by (H_1). By (H_4), since $r(T'_\infty) = \frac{\gamma}{\lambda_1}$, $\gamma > \lambda_1$, and $\gamma \neq \lambda_n$, we have that $r(T'_\infty) > 1$ and 1 is not a characteristic value of T'_∞. Hence condition (iii) of Lemma 2.1 holds.

From the above proof, Theorem 3.1 holds by Lemma 2.1.

4 Application

The following nonlinear four-point boundary value problem is studied:

$$\begin{cases} -u''(t) = g(t,u(t)), & 0 \le t \le 1, \\ u'(0) = 0, & u(1) = \frac{1}{4}u(\frac{1}{5}) + \frac{1}{2}u(\frac{1}{3}). \end{cases} \tag{4.1}$$

From simple calculations, $\lambda_1 \approx 0.5626$, $\lambda_2 \approx 23.3709$, $\lambda_3 \approx 70.0951$ are solutions of the following equation:

$$\cos\sqrt{x} = \frac{1}{4}\cos\frac{\sqrt{x}}{5} + \frac{1}{2}\cos\frac{\sqrt{x}}{3}.$$

Let

$$g(t,x) = \begin{cases} 2x + \sqrt{x} - \frac{t}{64} - \frac{439}{64}, & t \in [0,1], x \in (4,+\infty), \\ (x-1) + \frac{9}{64} - \frac{t}{64}, & t \in [0,1], x \in (1,4), \\ \frac{9}{64} - \frac{t}{64}, & t \in [0,1], x \in (\frac{1}{8},1] \\ x + (1-t)x^2, & t \in [0,1], x \in [-1,\frac{1}{8}], \\ \frac{1}{72}(x+1) - t, & t \in [0,1], x \in (-27,-1), \\ 2x + \sqrt[3]{x} + \frac{2039}{36} - t, & t \in [0,1], x \in (-\infty,-27]. \end{cases} \tag{4.2}$$

From (4.2), we know that g is continuous and increasing on x; $g(t,0) = 0$.

$$\liminf_{x\to 0} \frac{g(t,x)}{x} = 1 > \lambda_1,$$
$$\limsup_{x\to\infty} \frac{g(t,x)}{x} = 2 > \lambda_1.$$

Let $C = \frac{1}{5}, \tau = 1, w = -27$, then

$$\frac{g(t,\tau)}{\tau} = \frac{9}{64} - \frac{t}{64} < \frac{1}{5};$$
$$\frac{g(t,w)}{w} = \frac{13}{972} + \frac{t}{27} < \frac{1}{5}.$$

So, by Theorem 3.1, BVP (4.1) has at least two positive solutions, one sign-changing solution, and two negative solutions.

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