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A new approach to approximate solutions for a class of nonlinear multi-term fractional differential equations with integral boundary conditions

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Abstract

In this paper, we discuss the existence and uniqueness of solutions for a class of integral boundary value problems of nonlinear multi-term fractional differential equations and propose a new method to obtain their approximate solutions. The existence results are established by the Banach fixed point theorem, and approximate solutions are determined by the Daftardar-Gejji and Jafari iterative method (DJIM) and the Adomian decomposition method (ADM). Finally, we present some examples to illustrate the existence result and the effectiveness of applied approximate techniques.

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1 Introduction

In this paper, we consider the integral boundary value problems of nonlinear multi-term fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}y(t) = f(t, y(t), D_{0+}^{\beta_1}y(t), \dots, D_{0+}^{\beta_n}y(t)), & t \in (0,1), \\ y(0) = 0, \qquad y(1) = \int_0^1 g(s, y(s))\, ds, \end{cases} \tag{1}$$

where $1 < \alpha < 2$, $0 < \beta_1 < \cdots < \beta_n < 1$, $\alpha - \beta_n > 1$, $f : [0,1] \times \mathbf{R}^{n+1} \to \mathbf{R}$, $g : [0,1] \times \mathbf{R} \to \mathbf{R}$ are continuous functions, and $D_{0+}^{\alpha}, D_{0+}^{\beta_1}, \dots, D_{0+}^{\beta_n}$ are the Riemann–Liouville fractional derivatives, respectively.

Fractional differential equations have been attractive to many researchers because they play an important role in describing many phenomena arising in physics, chemistry, biology, aerodynamics, control theory, finance, and social sciences [1–6].

Especially, boundary value problems of fractional differential equations are often regarded as valuable mathematical models in the study of various physical, biological, and chemical processes, such as heat conduction, chemical engineering, thermo-elasticity, computational fluid dynamics, and bacterial self-regularization, and represent very interesting results [7–18].

They include two-point, three-point, multi-point, and nonlocal integral boundary value conditions as special cases. Existence and uniqueness results of solutions for such problems are obtained by using the techniques of nonlinear analysis such as fixed point theorems [2, 3, 13, 15, 17–21], fixed-point index theory [16], monotone iterative method [13, 14], nonlinear alternative of Leray–Schauder type [1, 20].

Bai [19] and Zhang [21], by using some fixed point theorems on cones, investigated the existence of positive solutions for the Riemann–Liouville fractional differential equation

$$D_{0+}^{\alpha}u(t) + f\bigl(t, u(t)\bigr) = 0, \quad 1 < \alpha \leq 2, 0 < t < 1,$$

with boundary conditions

$$u(0) = u(1) = 0,$$

or

$$u(0) + u'(0) = u(1) + u'(0) = 0.$$

Sun and Zhao [16] obtained the existence results of positive solutions for the fractional integral boundary value problem by means of the monotone iteration method

$$\begin{cases} D_{0+}^{\alpha}u(t) + q(t)f(t, u(t)) = 0, & 0 < t < 1, 2 < \alpha \leq 3, \\ u(0) = u'(0) = 0, & u(1) = \int_0^1 g(s)u(s)\,ds, \end{cases}$$

where $f \in C([0,1] \times [0,\infty), [0,\infty))$ and $g \in L^1[0,1]$ is nonnegative.

In [17], Tariboon et al. studied a new class of three-point boundary value problems of fractional differential equations with fractional integral boundary conditions

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t, u(t)), & 1 < \alpha \leq 2, 0 < t < T, \\ u(\eta) = 0, & I^{\nu}u(T) = 0, \end{cases}$$

where $\eta \in (0, T)$ is a given constant, D_{0+}^{α} is the standard Riemann–Liouville fractional derivative and $\nu > 0$.

In [14], Liu considered the existence and uniqueness of solutions for a class of nonlinear fractional differential equations with nonlocal integral boundary conditions

$$\begin{cases} D_{0+}^{\alpha}u(t) + p(t)f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u'(1) = \lambda I_{0+}^{\beta}u(\eta), \end{cases}$$

where $n-1<\alpha\le n$, $0<\eta\le 1$, $\lambda,\beta>0$, $0<\frac{\lambda\Gamma(\alpha)\cdot\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}<1$, and D_{0+}^{α} is the standard Riemann–Liouville fractional derivative of order α. When $n=4$ and $p(t)\equiv 1$, it has been studied in [13].

In [22], Padhi et al. considered the existence of positive solutions for fractional differential equations with nonlinear integral boundary conditions

$$\begin{cases} D_{0+}^{\alpha}x(t)+q(t)f(t,x(t))=0, \quad 0<t<1, \\ x(0)=x'(0)=\cdots=x^{(n-2)}(0)=0, \qquad D_{0+}^{\beta}x(1)=\int_0^1 h(s,x(s))\,dA(s), \end{cases}$$

where $n>2$, $n-1<\alpha\le n$, $\beta\in[1,\alpha-1]$ and D_{0+}^{α}, D_{0+}^{β} are the Riemann–Liouville fractional derivatives.

In [23], Li et al. used the Schauder fixed point theorem and the Banach contraction mapping principle to establish the existence and uniqueness of solutions for the following initial value problem of nonlinear fractional differential equation:

$$\begin{cases} D_{0+}^{\alpha}u(t)=f(t,u(t),D_{0+}^{\beta_1}u(t),\dots,D_{0+}^{\beta_N}u(t)), \quad 0<t\le 1, \\ D_{0+}^{\alpha-k}u(0)=0, \quad k=1,2,\dots,n, \end{cases}$$

where $\alpha>\beta_1>\beta_2>\cdots>\beta_N>0$, $n=[\alpha]+1$ for $\alpha\notin\mathbf{N}$ and $\alpha=n$ for $\alpha\in\mathbf{N}$, $0<\beta_j<1$ for any $j\in\{1,2,\dots,N\}$, $D_{0+}^{\alpha},D_{0+}^{\beta_1},\dots,D_{0+}^{\beta_N}$ are the standard Riemann–Liouville fractional derivatives and $f:[0,1]\times\mathbf{R}^{N+1}\to\mathbf{R}$.

From the previous results, we can see that very little is known about the existence of solutions for integral boundary value problems of nonlinear multi-term fractional differential equations.

On the other hand, although the existence of solutions for nonlinear fractional differential equations has been studied, it is difficult to obtain their analytic solution, so several approximate techniques such as the Adomian decomposition method (ADM) [5, 24–28], the Daftardar-Gejji and Jafari iterative method (DJIM) [25, 28], the variational iteration method (VIM) [29, 30], the homotopy perturbation method (HPM) [30], and the reproducing kernel method (RKM) [31] have been previously proposed to solve nonlinear fractional differential equations.

The ADM and the DJIM are known as highly accurate numerical techniques to solve nonlinear fractional differential equations.

Hu et al. [26] made use of the ADM to present the approximate solution of the following n-term linear fractional differential equation with constant coefficients and showed that the solution by the ADM was the same as the solution by the Green's function:

$$\begin{cases} a_nD^{\beta_n}y(t)+a_{n-1}D^{\beta_{n-1}}y(t)+\cdots+a_1D^{\beta_1}y(t)+a_0D^{\beta_0}y(t)=f(t), \\ y^{(i)}(0)=0, \quad i=0,1,\dots,n, \end{cases}$$

where a_i is a real constant, $D^{\beta_0},D^{\beta_1},\dots,D^{\beta_n}$ are the Riemann–Liouville derivatives, and $n+1>\beta_n\ge n>\beta_{n-1}>\cdots>\beta_1>\beta_0$.

In [24, 25, 27], the authors obtained approximate solutions for some initial value problems of nonlinear fractional differential equations by employing the ADM and its modifications.

Loghmani et al. [28] studied the approximate solutions for the initial value problems of nonlinear fractional differential equations by using the ADM and the DJIM and showed that the ADM and the DJIM were highly accurate numerical techniques to solve them.

In [32], Babolian et al. proposed a method based on the combination of the ADM and the spectral method to solve nonlinear fractional differential equations and applied it to some initial value problems.

In the ADM, the most important part is to compute the Adomian polynomials. It is rather easy to compute Adomian polynomials for initial value problems of fractional differential equations, but it is very difficult to do so for fractional differential equations with boundary conditions, more particularly for the case of nonlinear integral boundary value problems. However, to the best of our knowledge, there is no work concerned with approximate methods for solving nonlinear multi-term fractional differential equations with integral boundary conditions.

Summarizing all the previous results mentioned above motivates us to study problem (1) to establish the existence and uniqueness of the solutions and obtain the approximate solutions by using a new technique. The existence results are based on the Banach fixed point theorem, and approximate solutions that converge to an exact solution rapidly are obtained by the appropriate recursion schemes of the ADM and the DJIM.

The paper is organized as follows:

In Sect. 2, we recall some definitions and lemmas that will be useful to our main results. In Sect. 3, we obtain the corresponding integral equation to problem (1) and prove the existence and uniqueness of solutions for the integral equation by the Banach fixed point theorem. In Sect. 4, we show the procedures of solving our problem, using the ADM and the DJIM. In Sect. 5, we present some examples to illustrate the existence results of solutions and the effectiveness of our methods. In Sect. 6, we summarize our main results.

2 Preliminaries

In this section, we present some definitions and lemmas that will be useful for our main results.

Definition 2.1 ([6]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbf{R}$ is given by

$$\left(I_{0+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2 ([6]) The Riemann–Liouville fractional derivative of order α of a continuous function $f : (0,\infty) \to \mathbf{R}$ is given by

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where $\alpha > 0$, $n = [\alpha] + 1$ and $[\alpha]$ denotes the integral part of α.

Lemma 2.1 ([12]) *The following hold*:

(i) *Let* $y \in L^1(0,1)$ *and* $\nu > \sigma > 0$, *then*

$$I_{0+}^{\nu} I_{0+}^{\sigma} y(t) = I_{0+}^{\nu+\sigma} y(t), \qquad D_{0+}^{\sigma} I_{0+}^{\nu} y(t) = I_{0+}^{\nu-\sigma} y(t), \qquad D_{0+}^{\nu} I_{0+}^{\nu} y(t) = y(t).$$

(ii) *Let* $\alpha > 0$ *and* $\sigma > 0$, *then*

$$D_{0+}^{\alpha} t^{\sigma-1} = \begin{cases} 0, & \sigma - \alpha \in \{0\} \cup \mathbf{Z}^-, \\ \frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)} t^{\sigma-\alpha-1}, & \textit{otherwise}. \end{cases} \tag{2}$$

Lemma 2.2 ([12]) *Let* $\alpha > 0$ *and* $D_{0+}^{\alpha} u \in C(0,T) \cap L(0,T)$, *then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbf{R}$, $i = 1, 2, \ldots, n$, *and* $n = [\alpha] + 1$.

3 Existence and uniqueness results

In this section, we establish the existence and uniqueness of solutions of problem (1) by using the Banach fixed point theorem.

Definition 3.1 A function $y(t)$ is called a *solution* of problem (1) if it satisfies (1) and $D_{0+}^{\alpha} y(t) \in C[0,1]$, $y(t) \in C[0,1]$.

Theorem 3.1 *A function* $y(t)$ *is a solution of* (1) *if and only if* $x(t) := D_{0+}^{\beta_n} y(t)$ *is a solution of the integral equation*

$$\begin{aligned} x(t) = {} & I_{0+}^{\alpha-\beta_n} f\bigl(t, I_{0+}^{\beta_n} x(t), I_{0+}^{\beta_n-\beta_1} x(t), \ldots, x(t)\bigr) + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_n)} \biggl[\int_0^1 g\bigl(s, I_{0+}^{\beta_n} x(s)\bigr)\, ds \\ & - I_{0+}^{\alpha} f\bigl(t, I_{0+}^{\beta_n} x(t), I_{0+}^{\beta_n-\beta_1} x(t), \ldots, x(t)\bigr)\Big|_{t=1} \biggr] t^{\alpha-\beta_n-1}. \end{aligned} \tag{3}$$

Remark A continuous function $x(t)$ is called a *solution* of the integral equation (3) if it satisfies Eq. (3).

Proof Firstly, let $y(t) \in C[0,1]$ be a solution of (1), then $x(t) := D_{0+}^{\beta_n} y(t) \in C[0,1]$. Taking the Riemann–Liouville fractional integral of order β_n on both sides of $x(t) = D_{0+}^{\beta_n} y(t)$ gives

$$I_{0+}^{\beta_n} x(t) = I_{0+}^{\beta_n} D_{0+}^{\beta_n} y(t) = y(t) - \frac{(I_{0+}^{1-\beta_n} y)(0)}{\Gamma(\beta_n)} t^{\beta_n-1}.$$

Since $(I_{0+}^{1-\beta_n} y)(0) = 0$, we obtain $y(t) = I_{0+}^{\beta_n} x(t)$.

In view of Lemma 2.1, we get

$$\begin{aligned} & D_{0+}^{\beta_n} y(t) = D_{0+}^{\beta_n} I_{0+}^{\beta_n} x(t) = x(t), \\ & D_{0+}^{\beta_{n-1}} y(t) = D_{0+}^{\beta_{n-1}} I_{0+}^{\beta_n} x(t) = I_{0+}^{\beta_n-\beta_{n-1}} x(t), \\ & \ldots, \\ & D_{0+}^{\beta_1} y(t) = I_{0+}^{\beta_n-\beta_1} x(t). \end{aligned}$$

By the definition of Riemann–Liouville fractional derivative, we have

$$D_{0+}^{\alpha}y(t) = D_{0+}^{2}I_{0+}^{2-\alpha}y(t) = D_{0+}^{2}I_{0+}^{2-\alpha}I_{0+}^{\beta_n}x(t) = D_{0+}^{2}I_{0+}^{2-\alpha+\beta_n}x(t) = D_{0+}^{\alpha-\beta_n}x(t).$$

Then the equation of (1) can be written as

$$D_{0+}^{\alpha-\beta_n}x(t) = f\big(t, I_{0+}^{\beta_n}x(t), I_{0+}^{\beta_n-\beta_1}x(t), \dots, x(t)\big), \quad t \in [0,1]. \tag{4}$$

Setting $\beta_0 := 0$, $\mu := \alpha - \beta_n$, $\mu_i := \beta_n - \beta_i$ $(i = 0, 1, \dots, n)$, Eq. (4) can be rewritten as

$$D_{0+}^{\mu}x(t) = f\big(t, I_{0+}^{\mu_0}x(t), I_{0+}^{\mu_1}x(t), \dots, x(t)\big), \quad t \in [0,1]. \tag{5}$$

And since $x(t) \in C[0,1]$, $I_{0+}^{\beta_n}x(t) = \frac{1}{\Gamma(\beta_n)}\int_0^t \frac{x(s)}{(t-s)^{1-\beta_n}}\,ds$, we can arbitrarily provide the initial value of $x(t)$ such that $y(0) = I_{0+}^{\beta_n}x(t)|_{t=0} = 0$. Assume that $x(0) = 0$.

Applying the Riemann–Liouville fractional integral I_{0+}^{μ} to both sides of Eq. (5), we get

$$I_{0+}^{\mu}D_{0+}^{\mu}x(t) = I_{0+}^{\mu}f\big(t, I_{0+}^{\mu_0}x(t), I_{0+}^{\mu_1}x(t), \dots, x(t)\big). \tag{6}$$

From $\mu > 1$ and Lemma 2.2, we get

$$I_{0+}^{\mu}D_{0+}^{\mu}x(t) = x(t) + c_1 t^{\mu-1} + c_2 t^{\mu-2},$$

then Eq. (6) is rewritten as

$$x(t) = I_{0+}^{\mu}f\big(t, I_{0+}^{\mu_0}x(t), I_{0+}^{\mu_1}x(t), \dots, x(t)\big) - c_1 t^{\mu-1} - c_2 t^{\mu-2}. \tag{7}$$

Since $x(0) = 0$ and $\mu - 1 > 0$, we obtain that $c_2 = 0$ in Eq. (6).

That is, Eq. (7) can be rewritten as

$$x(t) = I_{0+}^{\mu}f\big(t, I_{0+}^{\mu_0}x(t), I_{0+}^{\mu_1}x(t), \dots, x(t)\big) - c_1 t^{\mu-1}. \tag{8}$$

By the boundary condition $y(1) = I_{0+}^{\beta_n}x(t)|_{t=1} = \int_0^1 g(s, I_{0+}^{\beta_n}x(s))\,ds$ and Lemma 2.1, we get

$$\begin{aligned} y(1) = I_{0+}^{\beta_n}x(t)\big|_{t=1} &= I_{0+}^{\beta_n+\mu}f\big(t, I_{0+}^{\mu_0}x(t), I_{0+}^{\mu_1}x(t), \dots, x(t)\big)\big|_{t=1} - I_{0+}^{\beta_n}c_1 t^{\mu-1}\big|_{t=1} \\ &= I_{0+}^{\beta_n+\mu}f\big(t, I_{0+}^{\mu_0}x(t), I_{0+}^{\mu_1}x(t), \dots, x(t)\big)\big|_{t=1} - c_1\frac{\Gamma(\mu)}{\Gamma(\mu+\beta_n)}t^{\mu+\beta_n-1}\bigg|_{t=1} \\ &= \int_0^1 g\big(s, I_{0+}^{\beta_n}x(s)\big)\,ds, \end{aligned}$$

and since $\mu + \beta_n - 1 = \alpha - \beta_n + \beta_n - 1 = \alpha - 1 > 0$, we have

$$\begin{aligned} \int_0^1 g\big(s, I_{0+}^{\beta_n}x(s)\big)\,ds &= I_{0+}^{\beta_n+\mu}f\big(t, I_{0+}^{\mu_0}x(t), I_{0+}^{\mu_1}x(t), \dots, x(t)\big)\big|_{t=1} - c_1\frac{\Gamma(\mu)}{\Gamma(\mu+\beta_n)} \\ &= I_{0+}^{\beta_n+\mu}f\big(t, I_{0+}^{\mu_0}x(t), I_{0+}^{\mu_1}x(t), \dots, x(t)\big)\big|_{t=1} - c_1\frac{\Gamma(\mu)}{\Gamma(\alpha)} \\ &= I_{0+}^{\alpha}f\big(t, I_{0+}^{\beta_n}x(t), I_{0+}^{\beta_n-\beta_1}x(t), \dots, x(t)\big)\big|_{t=1} - c_1\frac{\Gamma(\alpha-\beta_n)}{\Gamma(\alpha)}. \end{aligned}$$

Therefore, we get

$$c_1 = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_n)}\left[I_{0+}^{\alpha} f\big(t, I_{0+}^{\beta_n}x(t), I_{0+}^{\beta_n-\beta_1}x(t), \ldots, x(t)\big)\big|_{t=1} - \int_0^1 g\big(s, I_{0+}^{\beta_n}x(s)\big)\,ds\right], \tag{9}$$

and substituting the values of c_1 in Eq. (8), we obtain the following equation:

$$\begin{aligned} x(t) = {} & I_{0+}^{\alpha-\beta_n} f\big(t, I_{0+}^{\beta_n}x(t), I_{0+}^{\beta_n-\beta_1}x(t), \ldots, x(t)\big) + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_n)}\biggl[\int_0^1 g\big(s, I_{0+}^{\beta_n}x(s)\big)\,ds \\ & - I_{0+}^{\alpha} f\big(t, I_{0+}^{\beta_n}x(t), I_{0+}^{\beta_n-\beta_1}x(t), \ldots, x(t)\big)\big|_{t=1}\biggr] t^{\alpha-\beta_n-1}. \end{aligned}$$

That is, $x(t) = D_{0+}^{\beta_n}y(t) \in C[0,1]$ is the solution of Eq. (3).

Conversely, let $x(t) = D_{0+}^{\beta_n}y(t) \in C[0,1]$ be the solution of Eq. (3), then by Lemma 2.1 we obtain

$$\begin{aligned} & y(t) = I_{0+}^{\beta_n}x(t), \\ & D_{0+}^{\beta_{n-1}}y(t) = D_{0+}^{\beta_{n-1}} I_{0+}^{\beta_n}x(t) = I_{0+}^{\beta_n-\beta_{n-1}}x(t), \\ & \ldots, \\ & D_{0+}^{\beta_1}y(t) = I_{0+}^{\beta_n-\beta_1}x(t). \end{aligned}$$

Applying the Riemann–Liouville fractional integral $I_{0+}^{\beta_n}$ to both sides of Eq. (3), it can be written as

$$\begin{aligned} I_{0+}^{\beta_n}x(t) = {} & I_{0+}^{\alpha} f\big(t, I_{0+}^{\beta_n}x(t), I_{0+}^{\beta_n-\beta_1}x(t), \ldots, x(t)\big) + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_n)}\biggl[\int_0^1 g\big(s, I_{0+}^{\beta_n}x(s)\big)\,ds \\ & - I_{0+}^{\alpha} f\big(t, I_{0+}^{\beta_n}x(t), I_{0+}^{\beta_n-\beta_1}x(t), \ldots, x(t)\big)\big|_{t=1}\biggr] I_{0+}^{\beta_n} t^{\alpha-\beta_n-1}. \end{aligned} \tag{10}$$

Taking Riemann–Liouville fractional derivative D_{0+}^{α} to both sides of Eq. (10), we have

$$\begin{aligned} D_{0+}^{\alpha}y(t) = {} & D_{0+}^{\alpha} I_{0+}^{\alpha} f\big(t, y(t), D_{0+}^{\beta_1}y(t), \ldots, D_{0+}^{\beta_n}y(t)\big) + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_n)}\biggl[\int_0^1 g\big(s, I_{0+}^{\beta_n}x(s)\big)\,ds \\ & - I_{0+}^{\alpha} f\big(t, y(t), D_{0+}^{\beta_1}y(t), \ldots, D_{0+}^{\beta_n}y(t)\big)\big|_{t=1}\biggr] D_{0+}^{\alpha} I_{0+}^{\beta_n} t^{\alpha-\beta_n-1} \\ = {} & f\big(t, y(t), D_{0+}^{\beta_1}y(t), \ldots, D_{0+}^{\beta_n}y(t)\big) + \biggl[\int_0^1 g\big(s, I_{0+}^{\beta_n}x(s)\big)\,ds \\ & - I_{0+}^{\alpha} f\big(t, y(t), D_{0+}^{\beta_1}y(t), \ldots, D_{0+}^{\beta_n}y(t)\big)\big|_{t=1}\biggr] D_{0+}^{\alpha} t^{\alpha-1} \\ = {} & f\big(t, y(t), D_{0+}^{\beta_1}y(t), \ldots, D_{0+}^{\beta_n}y(t)\big). \end{aligned}$$

On the other hand, by Eq. (3) we have

$$\begin{aligned}x(0) &= I_{0+}^{\alpha-\beta_n} f\big(t, I_{0+}^{\beta_n}x(t), I_{0+}^{\beta_n-\beta_1}x(t), \ldots, x(t)\big)\big|_{t=0} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_n)}\bigg[\int_0^1 g\big(s, I_{0+}^{\beta_n}x(s)\big)\,ds \\ &\quad - I_{0+}^{\alpha} f\big(t, I_{0+}^{\beta_n}x(t), I_{0+}^{\beta_n-\beta_1}x(t), \ldots, x(t)\big)\big|_{t=1}\bigg] t^{\alpha-\beta_n-1}\bigg|_{t=0} \\ &= 0.\end{aligned}$$

Now let us check that the boundary conditions of (1) are satisfied.

Since $y(t) = I_{0+}^{\beta_n}x(t)$, we get $y(0) = I_{0+}^{\beta_n}x(t)|_{t=0} = 0$. Substituting $t = 1$ into Eq. (10) yields

$$\begin{aligned}y(1) &= I_{0+}^{\alpha} f\big(t, I_{0+}^{\beta_n}x(t), I_{0+}^{\beta_n-\beta_1}x(t), \ldots, x(t)\big)\big|_{t=1} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_n)}\bigg[\int_0^1 g\big(s, I_{0+}^{\beta_n}x(s)\big)\,ds \\ &\quad - I_{0+}^{\alpha} f\big(t, I_{0+}^{\beta_n}x(t), I_{0+}^{\beta_n-\beta_1}x(t), \ldots, x(t)\big)\big|_{t=1}\bigg] I_{0+}^{\beta_n} t^{\alpha-\beta_n-1}\bigg|_{t=1}.\end{aligned}$$

By using $I_{0+}^{\beta_n} t^{\alpha-\beta_n-1} = \frac{\Gamma(\alpha-\beta_n)}{\Gamma(\alpha-\beta_n+\beta_n)} t^{\alpha-\beta_n+\beta_n-1} = \frac{\Gamma(\alpha-\beta_n)}{\Gamma(\alpha)} t^{\alpha-1}$, we get

$$\begin{aligned}y(1) &= I_{0+}^{\alpha} f\big(t, I_{0+}^{\beta_n}x(t), I_{0+}^{\beta_n-\beta_1}x(t), \ldots, x(t)\big)\big|_{t=1} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_n)}\bigg[\int_0^1 g\big(s, I_{0+}^{\beta_n}x(s)\big)\,ds \\ &\quad - I_{0+}^{\alpha} f(t, I_{0+}^{\beta_n}x(t), I_{0+}^{\beta_n-\beta_1}x(t), \ldots, x(t)\big|_{t=1}\bigg] \frac{\Gamma(\alpha-\beta_n)}{\Gamma(\alpha)} t^{\alpha-1}\bigg|_{t=1} \\ &= \int_0^1 g\big(s, y(s)\big)\,ds.\end{aligned}$$

Therefore, $y(t)$ is the solution of (1). □

Let us consider the Banach space $X = C[0,1]$ endowed with the norm

$$\|u\| = \max_{0\le t\le 1}\big|u(t)\big|.$$

Define an operator $T : X \to X$ by

$$\begin{aligned}(Tx)(t) &:= I_{0+}^{\alpha-\beta_n} f\big(t, I_{0+}^{\beta_n}x(t), I_{0+}^{\beta_n-\beta_1}x(t), \ldots, x(t)\big) + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_n)}\bigg[\int_0^1 g\big(s, I_{0+}^{\beta_n}x(s)\big)\,ds \\ &\quad - I_{0+}^{\alpha} f\big(t, I_{0+}^{\beta_n}x(t), I_{0+}^{\beta_n-\beta_1}x(t), \ldots, x(t)\big)\big|_{t=1}\bigg] t^{\alpha-\beta_n-1}.\end{aligned} \tag{11}$$

Then Eq. (3) is equivalent to the operator equation

$$x = Tx, \quad x \in X. \tag{12}$$

Obviously, T is continuous on X.

For the existence results of solutions, we need the following assumptions:

(**H1**) There exist constants $l_i > 0$, $i = 0, 1, \ldots, n$, such that

$$\forall t \in [0,1], \forall (y_0, \ldots, y_n), (Y_0, \ldots, Y_n) \in \mathbf{R}^{n+1},$$
$$\bigl|f(t, y_0, y_1, \ldots, y_n) - f(t, Y_0, Y_1, \ldots, Y_n)\bigr| \le \sum_{i=0}^{n} l_i |y_i - Y_i|.$$

(**H2**) There exists a constant $\lambda > 0$ such that

$$\forall x, y \in \mathbf{R}, \quad \bigl|g(t,x) - g(t,y)\bigr| \le \lambda |x - y|.$$

(**H3**) Let $\omega := \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_n)} \frac{\lambda}{\Gamma(\beta_n+1)} + \sum_{i=0}^{n} (\frac{l_i}{\Gamma(\alpha-\beta_i+1)} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_n)} \frac{l_i}{\Gamma(\alpha-\beta_i+1+\beta_n)})$, then $0 < \omega < 1$.

Theorem 3.2 *Assume that hypotheses* (**H1**)–(**H3**) *are satisfied. Then problem* (1) *has a unique solution.*

Proof By Theorem 3.1, the existence of solutions to problem (1) refers to the existence of solutions of Eq. (12). So it is sufficient to prove that Eq. (12) has a unique fixed point.

Let $\beta_0 := 0$, $\mu_i := \beta_n - \beta_i$, $i = 0, 1, \ldots, n$, and $\mu := \alpha - \beta_n$, then by (**H1**), for any $x_1, x_2 \in X$, we have

$$\begin{aligned} &\bigl|f\bigl(t, I_{0+}^{\mu_0} x_1(t), \ldots, x_1(t)\bigr) - f\bigl(t, I_{0+}^{\mu_0} x_2(t), \ldots, x_2(t)\bigr)\bigr| \\ &\quad \le \sum_{i=0}^{n} l_i \cdot \bigl|I_{0+}^{\mu_i} x_1(t) - I_{0+}^{\mu_i} x_2(t)\bigr|. \end{aligned} \tag{13}$$

Applying the Riemann–Liouville fractional integral I_{0+}^{μ} to both sides of inequality (13), we get

$$\begin{aligned} &I_{0+}^{\mu} \bigl|f\bigl(t, I_{0+}^{\mu_0} x_1(t), \ldots, x_1(t)\bigr) - f\bigl(t, I_{0+}^{\mu_0} x_2(t), \ldots, x_2(t)\bigr)\bigr| \\ &\quad \le I_{0+}^{\mu} \sum_{i=0}^{n} l_i \cdot \bigl|I_{0+}^{\mu_i} x_1(t) - I_{0+}^{\mu_i} x_2(t)\bigr| \\ &\quad \le \sum_{i=0}^{n} l_i \cdot I_{0+}^{\mu+\mu_i} \bigl|x_1(t) - x_2(t)\bigr| \\ &\quad \le \|x_1 - x_2\| \sum_{i=0}^{n} \frac{l_i}{\Gamma(\mu+\mu_i+1)}. \end{aligned}$$

On the other hand, by (**H2**) we have

$$\begin{aligned} &\biggl|\frac{\Gamma(\alpha)}{\Gamma(\mu)} \biggl[\int_0^1 g\bigl(s, I_{0+}^{\beta_n} x_1(s)\bigr)\,ds - I_{0+}^{\beta_n+\mu} f\bigl(t, I_{0+}^{\mu_0} x_1(t), I_{0+}^{\mu_1} x_1(t), \ldots, x_1(t)\bigr)\big|_{t=1}\biggr] \cdot t^{\mu-1} \\ &\quad - \frac{\Gamma(\alpha)}{\Gamma(\mu)} \biggl[\int_0^1 g\bigl(s, I_{0+}^{\beta_n} x_2(s)\bigr)\,ds - I_{0+}^{\beta_n+\mu} f\bigl(t, I_{0+}^{\mu_0} x_2(t), I_{0+}^{\mu_1} x_2(t), \ldots, x_2(t)\bigr)\big|_{t=1}\biggr] \cdot t^{\mu-1}\biggr| \end{aligned}$$

$$\leq \frac{\Gamma(\alpha)}{\Gamma(\mu)}\left[\lambda\int_0^1 \left|I_{0+}^{\beta_n}x_1(s)-I_{0+}^{\beta_n}x_2(s)\right|ds + I_{0+}^{\beta_n+\mu}\sum_{i=0}^{n} l_i\cdot I_0^{\mu_i}\left|x_1(t)-x_2(t)\right|\Big|_{t=1}\right]$$
$$\leq \frac{\Gamma(\alpha)}{\Gamma(\mu)}\left[\frac{\lambda}{\Gamma(\beta_n+1)}+\sum_{i=0}^{n}\frac{l_i}{\Gamma(\beta_n+\mu+\mu_i+1)}\right]\|x_1-x_2\|.$$

Therefore, we obtain

$$\begin{aligned}
&\left|T\bigl(x_1(t)\bigr)-T\bigl(x_2(t)\bigr)\right| \\
&\quad\leq\left[\sum_{i=0}^{n}\frac{l_i}{\Gamma(\mu+\mu_i+1)}+\frac{\Gamma(\alpha)}{\Gamma(\mu)}\left(\frac{\lambda}{\Gamma(\beta_n+1)}+\sum_{i=0}^{n}\frac{l_i}{\Gamma(\beta_n+\mu+\mu_i+1)}\right)\right]\|x_1-x_2\| \\
&\quad=\left[\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_n)}\frac{\lambda}{\Gamma(\beta_n+1)}+\sum_{i=0}^{n}\left(\frac{l_i}{\Gamma(\alpha-\beta_i+1)}+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_n)}\frac{l_i}{\Gamma(\alpha-\beta_i+\beta_n+1)}\right)\right] \\
&\qquad\times\|x_1-x_2\|.
\end{aligned}$$

By (**H3**), this yields

$$\left\|T(x_1)-T(x_2)\right\|\leq\omega\|x_1-x_2\|,\quad 0<\omega<1.$$

Therefore, by the Banach fixed point theorem, the operator $T: X\to X$ has a unique fixed point. The proof is completed. □

4 A new approximate method by the ADM and the DJIM

In this section, we discuss how to apply the ADM and the DJIM to our problem. We present appropriate recursion schemes for the approximate solution of Eq. (3) and consider its convergence. Our method is motivated by [24, 25, 28].

Assume that the right-hand side of Eq. (12) is decomposed as follows:

$$(Tx)(t)=L\bigl(x(t)\bigr)+N\bigl(x(t)\bigr)+G(t),$$

where L is a linear operator to be inverted, G is a known function, N represents the nonlinear terms.

So, Eq. (3) can be written as

$$x(t)=L\bigl(x(t)\bigr)+N\bigl(x(t)\bigr)+G(t). \tag{14}$$

Also suppose that the solution of Eq. (14) is expressed by the form of series as follows:

$$x(t)=\sum_{n=0}^{\infty}x_n(t). \tag{15}$$

Then Eq. (14) can be rewritten as

$$\sum_{n=0}^{\infty}x_n(t)=L\left(\sum_{n=0}^{\infty}x_n(t)\right)+N\left(\sum_{n=0}^{\infty}x_n(t)\right)+G(t). \tag{16}$$

Transforming the right-hand side of Eq. (16), we obtain that

$$\begin{aligned}
&L\left(\sum_{n=0}^{\infty} x_n(t)\right) + N\left(\sum_{n=0}^{\infty} x_n(t)\right) + G(t) \\
&\quad = G(t) + L\big(x_0(t)\big) + N\big(x_0(t)\big) + \big[L\big(x_0(t) + x_1(t)\big) + N\big(x_0(t) + x_1(t)\big)\big] \\
&\qquad - \big[L\big(x_0(t)\big) + N\big(x_0(t)\big)\big] + \big[L\big(x_0(t) + x_1(t) + x_2(t)\big) + N\big(x_0(t) + x_1(t) + x_2(t)\big)\big] \\
&\qquad - \big[L\big(x_0(t) + x_1(t)\big) + N\big(x_0(t) + x_1(t)\big)\big] + \cdots + \left[L\left(\sum_{k=0}^{n-1} x_k(t)\right) + N\left(\sum_{k=0}^{n-1} x_k(t)\right)\right] \\
&\qquad - \left[L\left(\sum_{k=0}^{n-2} x_k(t)\right) + N\left(\sum_{k=0}^{n-2} x_k(t)\right)\right] + \cdots .
\end{aligned} \tag{17}$$

From (17) and the linearity of L, we obtain the following iterative schemes:

$$\begin{aligned}
&x_0(t) = G(t), \\
&x_1(t) = L\big(x_0(t)\big) + N\big(x_0(t)\big), \\
&x_2(t) = L\big(x_1(t)\big) + N\big(x_0(t) + x_1(t)\big) - N\big(x_0(t)\big), \\
&\ldots, \\
&x_n(t) = L\big(x_{n-1}(t)\big) + N\left(\sum_{j=0}^{n-1} x_j(t)\right) - N\left(\sum_{j=0}^{n-2} x_j(t)\right), \\
&\ldots .
\end{aligned} \tag{18}$$

Therefore, we can put the n-term approximation solution of Eq. (3) as

$$U_n(t) = \sum_{j=0}^{n} x_j(t). \tag{19}$$

From (19), we have that $x_n(t) = U_n(t) - U_{n-1}(t)$. Then (18) can be rewritten as

$$U_n(t) = U_{n-1}(t) + L\big(U_{n-1}(t) - U_{n-2}(t)\big) + N\big(U_{n-1}(t)\big) - N\big(U_{n-2}(t)\big). \tag{20}$$

If $\|L(x) - L(y)\| \leq k_1\|x - y\|$, $\|N(x) - N(y)\| \leq k_2\|x - y\|$, $0 < k_1, k_2 < 1$, and $k_1 + k_2 < 1$, then in terms of the Banach fixed point theorem, (14) has a unique solution $U^*(t)$. Since for $n \geq 1$,

$$\begin{aligned}
\|U_n - U_{n-1}\| &\leq k_1\|U_{n-1} - U_{n-2}\| + k_2\|U_{n-1} - U_{n-2}\| \\
&= (k_1 + k_2)\|U_{n-1} - U_{n-2}\| \\
&\leq (k_1 + k_2)^2\|U_{n-2} - U_{n-3}\| \\
&\leq \cdots \leq (k_1 + k_2)^{n-1}\|U_1 - U_0\|,
\end{aligned} \tag{21}$$

the sequence $\{U_n\}$ absolutely and uniformly converges to exact solution $U^*(t)$.

In Eq. (16), the ADM decomposes nonlinear term $N(\sum_{n=0}^{\infty} x_n(t))$ into the following series:

$$N\left(\sum_{n=0}^{\infty} x_n\right) = \sum_{n=0}^{\infty} A_n(x_0, \dots, x_n), \tag{22}$$

where $A_n(x_0, \dots, x_n)$ is obtained by the definitional formula

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \left[N\left(\sum_{k=0}^{\infty} x_k \lambda^k\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \dots. \tag{23}$$

Then Eq. (16) can be written as

$$\sum_{n=0}^{\infty} x_n(t) = L\left(\sum_{n=0}^{\infty} x_n(t)\right) + \sum_{n=0}^{\infty} A_n + G(t). \tag{24}$$

Expressing the right-hand side of (24) as

$$\begin{aligned} L&\left(\sum_{n=0}^{\infty} x_n(t)\right) + \sum_{n=0}^{\infty} A_n + G(t) \\ &= G(t) + L\bigl(x_0(t)\bigr) + A_0 + L\bigl(x_1(t)\bigr) \\ &\quad + A_1 + L\bigl(x_2(t)\bigr) + A_2 + \cdots + L\bigl(x_{n-1}(t)\bigr) + A_{n-1} + \cdots, \end{aligned} \tag{25}$$

we get the following recursion schemes:

$$\begin{aligned} &x_0(t) = G(t), \\ &x_1(t) = L\bigl(x_0(t)\bigr) + A_0, \\ &x_2(t) = L\bigl(x_1(t)\bigr) + A_1, \\ &x_3(t) = L\bigl(x_2(t)\bigr) + A_2, \\ &\dots, \\ &x_n(t) = L\bigl(x_{n-1}(t)\bigr) + A_{n-1}, \\ &\dots. \end{aligned} \tag{26}$$

Expressing the N-term approximation solution of Eq. (3) as $U_N(t) = \sum_{n=0}^{N} x_n(t)$, the exact solution of (3) is obtained by

$$x(t) = \lim_{N\to\infty} U_N(t).$$

Therefore, the exact solution of (1) is obtained by $y(t) = I_{0+}^{\beta_n} x(t)$.

5 Examples

Here, we give two examples to illustrate our main results. We will check only the validity of the existence and uniqueness results of the given problem in Example 1, while only the approximate method for solving the problem will be illustrated in Example 2. As can be seen in Sect. 4, it is obvious that hypotheses (**H1**–**H3**) have not been used to obtain the approximate solution to problem (1). Therefore, the functions f, g in Example 2 will be chosen to compare our approximate solutions with the exact one instead of satisfying these hypotheses.

Example 1 Consider the following boundary value problem:

$$\begin{cases} D_{0+}^{1.7}y(t) = \frac{1}{10(1+|y(t)|)}(t^2 + y(t) + \sin(D_{0+}^{0.3}y(t)) + \sin(D_{0+}^{0.5}y(t))), \\ y(0) = 0, \qquad y(1) = \frac{1}{8}\int_0^1 \sin^2 y(s)\,ds. \end{cases} \tag{27}$$

Putting $\alpha = 1.7$, $\beta_1 = 0.3$, $\beta_2 = 0.5$, $l_0 = l_1 = l_2 = 0.1$, $\lambda = 0.25$, $\beta_0 = 0$, we have

$$\begin{aligned} \omega &= \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_2)} \cdot \frac{\lambda}{\Gamma(\beta_2+1)} + \sum_{i=0}^{2}\left(\frac{l_i}{\Gamma(\alpha-\beta_i+1)} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_2)} \cdot \frac{l_i}{\Gamma(\alpha-\beta_i+1+\beta_2)}\right) \\ &\cong 0.674 < 1. \end{aligned}$$

Hence, by Theorem 3.2, problem (27) has a unique solution.

Example 2 In order to demonstrate the effectiveness of our approximate methods, we consider the following nonlinear fractional differential equation with nonlinear integral boundary condition:

$$\begin{cases} D_{0+}^{1.7}y(t) = D_{0+}^{0.5}y(t) + y^2(t) + g(t), \quad t \in (0,1), \\ y(0) = 0, \qquad y(1) = \frac{6}{13}\int_0^1 (y^2(s)+2)\,ds, \end{cases} \tag{28}$$

where $g(t) = \frac{\Gamma(3.5)}{\Gamma(1.8)}t^{0.8} - \frac{\Gamma(3.5)}{2}t^2 - t^5$.

The corresponding integral equation to problem (28) can be written as

$$\begin{aligned} x(t) = {}& I_{0+}^{1.2}\bigl[x(t) + \bigl(I_{0+}^{0.5}x(t)\bigr)^2 + g(t)\bigr] + \frac{\Gamma(1.7)}{\Gamma(1.2)}\biggl[\frac{6}{13}\int_0^1 \bigl(I_{0+}^{0.5}x(t)\bigr)^2\,dt + \frac{12}{13} \\ & - I_{0+}^{1.7}\bigl[x(t) + \bigl(I_0^{0.5}x(t)\bigr)^2 + g(t)\bigr]\big|_{t=1}\biggr]. \end{aligned} \tag{29}$$

The right-hand side of Eq. (29) is decomposed as follows:

$$L\bigl(x(t)\bigr) = I_{0+}^{1.2}x(t) - \frac{\Gamma(1.7)}{\Gamma(1.2)}\bigl[I_{0+}^{1.7}x(t)\bigr]\big|_{t=1}, \tag{30}$$

$$\begin{aligned} N\bigl(x(t)\bigr) = {}& I_{0+}^{1.2}\bigl[\bigl(I_{0+}^{0.5}x(t)\bigr)^2\bigr] + \frac{\Gamma(1.7)}{\Gamma(1.2)}\biggl[\frac{6}{13}\int_0^1 \bigl(I_{0+}^{0.5}x(t)\bigr)^2\,dt \\ & - I_{0+}^{1.7}\bigl[\bigl(I_{0+}^{0.5}x(t)\bigr)^2\bigr]\big|_{t=1}\biggr], \end{aligned} \tag{31}$$

$$G(t) = I_{0+}^{1.2}g(t) + \frac{\Gamma(1.7)}{\Gamma(1.2)}\frac{12}{13} - \frac{\Gamma(1.7)}{\Gamma(1.2)} \cdot \bigl[I_{0+}^{1.7}g(t)\bigr]\big|_{t=1}. \tag{32}$$

Then we can write (29) as

$$x(t) = L(x(t)) + N(x(t)) + G(t). \tag{33}$$

Solution by the DJIM

According to DJIM (18), we have

$$\begin{aligned} &x_0(t) = G(t), \\ &x_1(t) = L(x_0(t)) + N(x_0(t)), \\ &x_2(t) = L(x_1(t)) + N(x_0(t) + x_1(t)) - N(x_0(t)), \\ &\ldots. \end{aligned} \tag{34}$$

The two-term approximate solution of Eq. (29) is obtained by $U_2(t) = x_0(t) + x_1(t) + x_2(t)$, so the two-term approximate solution of (28) is obtained by $y_2(t) = I_{0+}^{0.5} U_2(t)$.

Solution by the ADM

Putting

$$\begin{aligned} &L_1(x(t)) := I_{0+}^{1.2} x(t), \\ &L_2(x(t)) := \frac{\Gamma(1.7)}{\Gamma(1.2)} \frac{6}{13} \int_0^1 x(t)\,dt, \\ &L_3(x(t)) := -\frac{\Gamma(1.7)}{\Gamma(1.2)} I_{0+}^{1.7} \left[\left(I_{0+}^{0.5} x(t) \right)^2 \right] \Big|_{t=1}, \end{aligned} \tag{35}$$

then in (31), $N(x(t))$ can be rewritten as

$$N(x(t)) = L_1\left(\left(I_{0+}^{0.5} x(t) \right)^2 \right) + L_2\left(\left(I_{0+}^{0.5} x(t) \right)^2 \right) + L_3\left(\left(I_{0+}^{0.5} x(t) \right)^2 \right). \tag{36}$$

According to ADM (26), we have

$$\begin{aligned} &x_0(t) = G(t), \\ &x_1(t) = L_0(x_0(t)) + L_1(A_0) + L_2(A_0) + L_3(A_0), \\ &x_2(t) = L_0(x_1(t)) + L_1(A_1) + L_2(A_1) + L_3(A_1), \\ &\ldots, \end{aligned} \tag{37}$$

where A_n is expressed as

$$\begin{aligned} &A_0 = \left(I_{0+}^{0.5} x_0(t) \right)^2, \\ &A_1 = 2\left(I_{0+}^{0.5} x_0(t) \right)\left(I_{0+}^{0.5} x_1(t) \right), \\ &A_2 = \left(I_{0+}^{0.5} x_1(t) \right)^2 + 2\left(I_{0+}^{0.5} x_1(t) \right)\left(I_{0}^{0.5} x_2(t) \right), \\ &A_3 = 2\left[\left(I_{0+}^{0.5} x_1(t) \right)\left(I_{0+}^{0.5} x_2(t) \right) + \left(I_{0+}^{0.5} x_0(t) \right)\left(I_{0+}^{0.5} x_3(t) \right) \right], \\ &\ldots. \end{aligned} \tag{38}$$

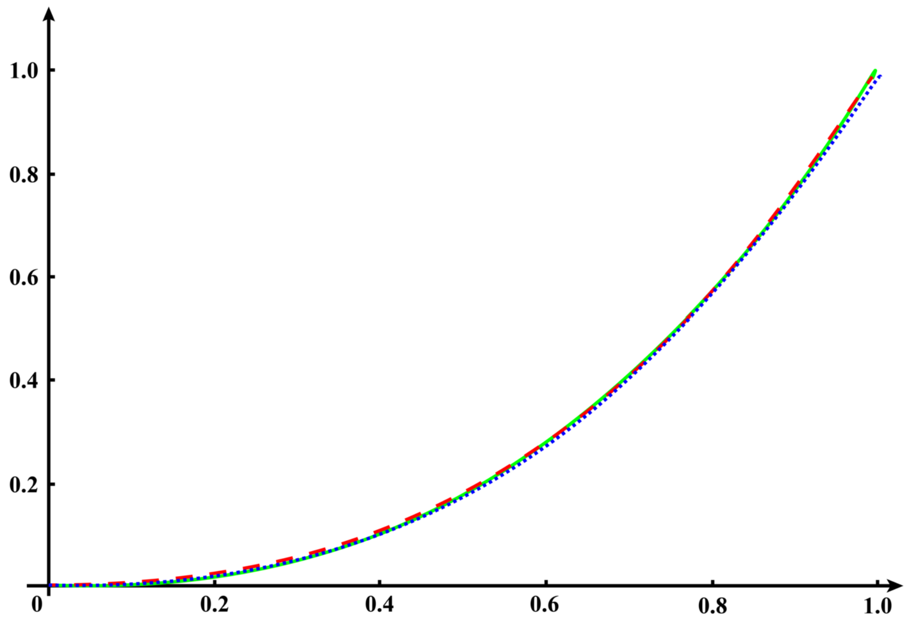

Figure 1 Comparison of solutions. Exact solution (green), DJIM-solution (blue) and ADM-solution (red)

The two-term approximate solution of integral equation is obtained also by $U_2(t) = x_0(t) + x_1(t) + x_2(t)$, so the two-term approximate solution of (29) is obtained by $y_2(t) = I_{0+}^{0.5} U_2(t)$.

The curves of the exact solution $y = t^{2.5}$ and the two-term approximate solutions by the DJIM and the ADM for our problem (29) have been plotted in Fig. 1.

6 Conclusion

In this paper, we considered the existence of solutions for a multi-term fractional differential equation with nonlinear integral boundary conditions and obtained its approximate solution by the appropriate recursion schemes of the ADM and the DJIM. The numerical results show that the ADM and the DJIM yield a very effective and accurate approach to the approximate solution of nonlinear integral boundary problems of fractional differential equations, and therefore, can be widely applied in many boundary value problems of fractional differential equations.

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Authors' contributions
All authors carried out the proof and conceived of the study. All authors read and approved the final manuscript.

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