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A B-spline finite element method for solving a class of nonlinear parabolic equations modeling epitaxial thin-film growth with variable coefficient

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Abstract

In this paper, we propose an efficient B-spline finite element method for a class of fourth order nonlinear differential equations with variable coefficient. For the temporal discretization, we choose the Crank–Nicolson scheme. Boundedness and error estimates are rigorously derived for both semi-discrete and fully discrete schemes. A numerical experiment confirms our theoretical analysis.

Keywords: B-spline; Finite element method; Nonlinear parabolic equation; Variable coefficient; Boundedness

1 Introduction

The epitaxial growth of nanoscale thin films has attracted a lot of attention in recent years [1–6]. The key reason for this concern is that compositions like $\text{YBa}_2\text{Cu}_3\text{O}_{7-\delta}$ (YBCO) are expected to be high temperature superconducting materials that can be used in semiconductor design. King et al. [1] proved the existence, uniqueness, and regularity of solution in an appropriate function space for the initial boundary value problem of the epitaxial thin-film growth. Kohn et al. [3] considered a fourth order parabolic equation, which is a specific example of energy-driven coarsening in two dimension space, and proved that the time-averaged energy per unit area decays no faster than $t^{-\frac{1}{3}}$.

The finite element method (FEM) plays an important role in solving differential equations [7–11]. There are some papers which have already been published to study the FEM for fourth order nonlinear parabolic equations [12–18]. Liu et al. [12] considered a nonlinear model describing epitaxial thin-film growth with constant coefficient and demonstrated that the Hermite FEM has the convergence rate of $O(\Delta t + h^3)$. Choo [13] constructed a finite element scheme for the viscous Cahn–Hilliard equation with a nonconstant gradient energy coefficient and obtained the error estimate using the extended Lax–Richtmyer equivalence theorem. In [18], Qiao et al. presented a mixed FEM for the molecular beam epitaxy mode and showed that the semi-discrete and fully discrete schemes satisfy the nonlinear energy stability property. Moreover, the authors gave the error analysis.

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In 1946, the B-spline method was first introduced by Schoenberg [19]. In 1966, Curry and Schoenberg [20] presented one element B-spline functions. In 1976, B-splines were extended to multiple situations [21]. As a class of piecewise polynomials, B-splines are often used in finite element analysis [22–27]. Erfanian et al. [25] used the linear B-spline FEM and cubic B-spline FEM for solving linear Volterra integro-differential equation in the complex plane. Dhawan et al. [26] applied the linear and quadratic B-spline functions to the advection-diffusion equations.

The main advantages of B-splines are the freedom to choose the order and smoothness, the simple data structure with one parameter in $[0, 1]$, and the exact representation of boundary conditions. Compared with Lagrange and Hermite type elements, B-spline shape functions involve only one type of basis function. Thus the scale of matrix from B-spline FEM is smaller than that from Lagrange and Hermite elements. Moreover, B-spline shape functions are smoother. It is known that quadratic B-splines, which are in $C^1(-\infty, +\infty)$, satisfy the weak form of fourth order differential equations. However, to deal with boundary conditions, the B-spline basis functions need to be modified. In the present work, we choose the cubic B-spline FEM for a fourth order nonlinear parabolic equation with variable coefficient. It is proved that the convergence order of the Crank–Nicolson scheme is higher than that of the backward Euler scheme in [12].

The following sections are organized as follows. In Sect. 2, we introduce the model and some basic preliminaries. In Sect. 3, we show the boundedness and error estimates for the semi-discrete scheme. In Sect. 4, a fully discrete scheme based on the Crank–Nicolson method is studied. In Sect. 5, a numerical experiment is provided to confirm theoretical results.

In this work, we denote L^2, L^k, L^∞, H^k norms in I by $\|\cdot\|, \|\cdot\|_{L^k}, |\cdot|_\infty$, and $\|\cdot\|_k$, respectively.

2 Initial boundary value problem and some preliminaries

In this paper, we consider the following problem:

$$\begin{cases} u_t + (\alpha(x, t)u_{xx})_{xx} - (|u_x|^2u_x - u_x)_x = 0, & (x, t) \in I \times (0, T), \\ u(x, t) = u_x(x, t) = 0, & x \in \partial I, t \in (0, T), \\ u(x, 0) = u_0(x), & x \in I, \end{cases} \tag{1}$$

where $I = [0, 1]$ and $u_t = \frac{\partial u}{\partial t}$.

For the variable coefficient, the following assumptions hold:

$$\alpha(x, t), \quad \frac{\partial \alpha}{\partial t}(x, t) \in C(I \times [0, T]), \tag{2}$$

$$0 < s \leq \alpha(x, t) \leq S < +\infty, \quad \forall x \in I, t \in [0, T], \tag{3}$$

$$\left| \frac{\partial \alpha}{\partial t} \right| \leq M_1, \quad \left| \frac{\partial^2 \alpha}{\partial^2 t} \right| \leq M_2, \quad \forall x \in I, t \in [0, T], \tag{4}$$

where s, S, M_1 , and M_2 are positive constants.

Considering the boundary value conditions, we need the following space:

$$H_0^2(I) = \{w; w \in H^2(I), w(0, t) = w(1, t) = w_x(0, t) = w_x(1, t) = 0\}.$$

The weak formulation associated with problem (1) is: Find $u = u(\cdot, t) \in H_0^2(I)$ ($0 \leq t \leq T$) such that

$$\begin{cases} (u_t, v) + (\alpha(x, t)D^2u, D^2v) + (|Du|^2Du - Du, Dv) = 0, & \forall v \in H_0^2(I), \\ u(x, 0) = u_0(x), & x \in I, \end{cases} \tag{5}$$

where $Du = \frac{\partial u}{\partial x}$.

According to [12], the solution of problem (1) exists.

Theorem 2.1 *Suppose that $u_0 \in H_0^2(I) \cap W^{1,4}(I)$, then there exists a unique global solution $u(x, t)$ for problem (1) such that*

$$u \in C^0([0, T]; L^2(I)) \cap L^\infty([0, T]; W_0^{1,4}(I)) \cap L^2([0, T]; H^4(I)).$$

3 Semi-discrete finite element scheme

Let L be a positive integer. Define a uniform partition $I_h : 0 = x_0 < x_1 < \dots < x_L = 1, h = x_i - x_{i-1} = \frac{1}{L}, I_i = [x_{i-1}, x_i]$. To deal with boundary value conditions, we define six additional knots $x_{-3} = -3h, x_{-2} = -2h, x_{-1} = -h, x_{L+1} = 1 + h, x_{L+2} = 1 + 2h, x_{L+3} = 1 + 3h$.

The cubic B-spline function with integer knots can be defined as follows:

$$N(x) = \begin{cases} \frac{1}{6}x^3, & x \in [0, 1], \\ -\frac{1}{2}x^3 + 2x^2 - 2x + \frac{2}{3}, & x \in [1, 2], \\ \frac{1}{2}x^3 - 4x^2 + 10x - \frac{22}{3}, & x \in [2, 3], \\ -\frac{1}{6}(x - 4)^3, & x \in [3, 4], \\ 0, & \text{else,} \end{cases}$$

then it is easy to get the cubic B-spline in the interval $[x_i, x_{i+4}]$ which is

$$\phi_i(x) = N\left(\frac{x - x_i}{h}\right).$$

The modified basis functions are defined as follows [28]:

$$\left\{ 6\phi_{-3}(x), \phi_{-2}(x) - 4\phi_{-3}(x), \phi_{-1}(x) - \frac{1}{2}\phi_{-2}(x) + \phi_{-3}(x), \phi_0(x), \dots, \phi_{L-4}(x), \phi_{L-3}(x) - \frac{1}{2}\phi_{L-2}(x) + \phi_{L-1}(x), \phi_{L-2}(x) - 4\phi_{L-1}(x), 6\phi_{L-1}(x) \right\}.$$

For the sake of convenience, we denote the modified basis functions with respect to $\{x_i\}$ by $\{\varphi_i(x)\}$, which satisfies the following properties:

$$\begin{aligned} \varphi_{-3}(0) &= 1, & \varphi_i(0) &= 0 \quad (i \neq -3), & \varphi'_i(0) &= 0 \quad (i \neq -3, -2), \\ \varphi_{L-1}(1) &= 1, & \varphi_i(1) &= 0 \quad (i \neq L - 1), & \varphi'_i(1) &= 0 \quad (i \neq L - 1, L - 2). \end{aligned}$$

The modified basis functions can deal with homogeneous as well as non-homogeneous boundary conditions.

Let U_h be the cubic B-spline space. One can see that the cubic B-spline space is in $C^2(-\infty, \infty)$, thus $U_h \subset H_0^2$. The approximation solution $u_h(x, t) \in U_h$ satisfies

$$u_h(x, t) = \sum_{i=-1}^{L-3} \delta_i(t) \varphi_i(x),$$

where $\delta_i(t)$ are time-dependent quantities.

The semi-discrete finite element scheme based on B-splines for problem (5) is: Find $u_h = u_h(\cdot, t) \in U_h$ ($0 < t \leq T$) such that

$$\begin{cases} (u_{h,t}, v_h) + (\alpha(x, t) D^2 u_h, D^2 v_h) + (|Du_h|^2 Du_h - Du_h, Dv_h) = 0, & \forall v_h \in U_h, \\ (u_h(0) - u(0), v_h) = 0, & \forall v_h \in U_h. \end{cases} \tag{6}$$

The bandwidth of stiffness matrix is 7, and the matrix order is $L - 1$, which is only half of the Lagrange and Hermite finite element scheme.

In order to estimate the errors of the B-spline FEM, we introduce the elliptic projection $R_h u$:

$$a(u - R_h u, v_h) \equiv (\alpha(x, t) D^2(u - R_h u), D^2 v_h) = 0, \quad \forall v_h \in U_h, \tag{7}$$

then $R_h u$ is uniquely defined, and

$$a(u, u) \geq C_0 \|u\|_2^2, \quad \forall u \in H_0^2(I), \tag{8}$$

where C_0 is a positive constant depending on $\alpha(x, t)$. Hence, $a(u, v)$ is a symmetrical positive definite bilinear form and (see [12])

$$\|u - R_h u\|_i \leq Ch^{4-i} \|u\|_4, \quad i = 0, 1, 2. \tag{9}$$

We shall discuss the boundedness of the semi-discrete scheme, which is important for error analysis.

Theorem 3.1 *Let $u_h(0) \in H_0^2(I) \cap W^{1,4}(I)$, then there exists a unique solution $u_h(t) \in U_h$ for problem (6) such that*

$$\|u_h(t)\|_2 \leq C \|u_h(0)\|_2, \quad 0 \leq t \leq T, \tag{10}$$

where C is a positive constant depending on $\alpha(x, t)$ and T , independent of mesh size h .

Proof According to ordinary differential equation theory, there exists a unique local solution to problem (5) in the interval $[0, t_n)$. If we have (10), then according to the extension theorem, we can also obtain the existence of unique global solution. So, we only need to prove (10).

Taking $v_h = u_h$ in (6), based on (3), we get

$$\frac{1}{2} \frac{d}{dt} \|u_h\|^2 + s \|D^2 u_h\|^2 + \|Du_h\|_{L^4}^4 \leq \|Du_h\|^2.$$

By the interpolation inequality, we have

$$\frac{1}{2} \frac{d}{dt} \|u_h\|^2 + s \|D^2 u_h\|^2 + \|Du_h\|_{L^4}^4 \leq \frac{s}{2} \|D^2 u_h\|^2 + \frac{1}{2s} \|u_h\|^2.$$

Therefore

$$\frac{d}{dt} \|u_h\|^2 + s \|D^2 u_h\|^2 + 2 \|Du_h\|_{L^4}^4 \leq \frac{1}{s} \|u_h\|^2. \tag{11}$$

According to the method of solving separable equations, we can get the result

$$\frac{d}{dt} (e^{-\frac{t}{s}} \|u_h\|^2) \leq 0. \tag{12}$$

Integrating (12) with respect to t , we have

$$\|u_h(t)\|^2 \leq e^{\frac{t}{s}} \|u_h(0)\|^2 \leq e^{\frac{T}{s}} \|u_h(0)\|^2 \leq C \|u_h(0)\|^2, \quad 0 \leq t \leq T. \tag{13}$$

Integrating (11) with respect to t , we get

$$\|u_h(t)\|^2 - \|u_h(0)\|^2 + s \int_0^t \|D^2 u_h\|^2 dt \leq \frac{1}{s} \int_0^t \|u_h\|^2 dt.$$

By (13), we obtain

$$\int_0^t \|D^2 u_h\|^2 dt \leq C \|u_h(0)\|^2. \tag{14}$$

Choose $v_h = u_{h,t}$ in (6) to get

$$\|u_{h,t}\|^2 + (\alpha(x, t) D^2 u_h, D^2 u_{h,t}) + (|Du_h|^2 Du_h - Du_h, Du_{h,t}) = 0. \tag{15}$$

A direct calculation gives

$$(\alpha(x, t) D^2 u_h, D^2 u_{h,t}) = \frac{1}{2} \frac{d}{dt} (\alpha(x, t) D^2 u_h, D^2 u_h) - \frac{1}{2} \left(\frac{\partial \alpha}{\partial t} D^2 u_h, D^2 u_h \right).$$

Define the energy function

$$E_h(t) = \frac{1}{2} (\alpha(x, t) D^2 u_h, D^2 u_h) + \frac{1}{4} ((1 - |Du_h|^2)^2, 1). \tag{16}$$

It is clear that $E_h(t) \geq 0$. Differentiating $E_h(t)$ with respect to t , we get

$$\begin{aligned} \frac{d}{dt} E_h(t) &= (\alpha(x, t) D^2 u_h, D^2 u_{h,t}) + \frac{1}{2} \left(\frac{\partial \alpha}{\partial t} (x, t) D^2 u_h, D^2 u_h \right) \\ &\quad + (|Du_h|^2 Du_h, Du_{h,t}) - (Du_h, Du_{h,t}). \end{aligned}$$

By (14), we have

$$\frac{d}{dt} E_h(t) = -\|u_{h,t}\|^2 + \frac{1}{2} \left(\frac{\partial \alpha}{\partial t} (x, t) D^2 u_h, D^2 u_h \right) \leq -\|u_{h,t}\|^2 + \frac{M_1}{2} \|D^2 u_h\|^2. \tag{17}$$

Integrating (17) with respect to t , we have

$$E_h(t) - E_h(0) \leq \frac{M_1}{2} \int_0^t \|D^2u_h\|^2 dt \leq C \|u_h(0)\|^2.$$

It is obvious that

$$E_h(t) \leq E_h(0) + C \|u_h(0)\|^2.$$

In view of (3) and (16), we obtain

$$\begin{aligned} & \frac{s}{2} \|D^2u_h\|^2 + \frac{1}{4} \|Du_h\|_{L^4}^4 - \frac{1}{2} \|Du_h\|^2 \\ & \leq \frac{S}{2} \|D^2u_h(0)\|^2 + \frac{1}{4} \|Du_h(0)\|_{L^4}^4 - \frac{1}{2} \|Du_h(0)\|^2 + C \|u_h(0)\|^2. \end{aligned}$$

It is clear to see from Cauchy's inequality that

$$\begin{aligned} & \frac{s}{2} \|D^2u_h\|^2 + \frac{1}{4} \|Du_h\|_{L^4}^4 + \frac{1}{2} \|Du_h(0)\|^2 \\ & \leq \frac{S}{2} \|D^2u_h(0)\|^2 + \frac{1}{4} \|Du_h(0)\|_{L^4}^4 + C \|u_h(0)\|^2 + \frac{1}{2} \|Du_h\|^2 \\ & \leq \frac{S}{2} \|D^2u_h(0)\|^2 + \frac{1}{4} \|Du_h(0)\|_{L^4}^4 + C \|u_h(0)\|^2 + \frac{s}{4} \|D^2u_h\|^2 + \frac{1}{4s} \|u_h\|^2. \end{aligned}$$

Therefore

$$\frac{s}{4} \|D^2u_h\|^2 \leq \frac{S}{2} \|D^2u_h(0)\|^2 + \frac{1}{4} \|Du_h(0)\|_{L^4}^4 + C \|u_h(0)\|^2. \tag{18}$$

It follows that

$$\|D^2u_h\| \leq C \|D^2u_h(0)\|, \quad 0 \leq t \leq T, \tag{19}$$

where C is a positive constant depending on $\alpha(x, t)$ and $u_h(0)$.

Owing to the interpolation inequality, we obtain

$$\|Du_h\|^2 \leq \frac{1}{2} \|D^2u_h\|^2 + \frac{1}{2} \|u_h\|^2.$$

Thus (10) holds. The proof of the theorem is completed. □

Now, we give the error estimates between the solution to problem (5) and the solution in L^2 and H^2 norms.

Theorem 3.2 *Let u be the solution to (5), u_h be the solution to (6), $u(0) \in H^4(I)$, $u, u_t \in L^2(0, T; H^4(I))$, the initial value satisfies*

$$\|u(0) - u_h(0)\| \leq Ch^4 \|u(0)\|_4.$$

As $0 \leq t \leq T$, we have the following error estimate:

$$\|u - u_h\| \leq Ch^4 \left(\|u(0)\|_4^2 + \left(\int_0^t (\|u(\tau)\|_4^2 + \|u_t(\tau)\|_4^2) d\tau \right)^{\frac{1}{2}} \right), \tag{20}$$

where C is a positive constant depending on $\alpha(x, t)$ and T , independent of mesh size h .

Proof Denote $\theta(t) = R_h u - u_h$ and $\rho(t) = u - R_h u$. Then

$$\|u - u_h\| \leq \|\theta(t)\| + \|\rho(t)\|. \tag{21}$$

It follows from (5)–(7) that

$$\begin{aligned} &(\theta_t, v_h) + (\alpha(x, t)D^2\theta, D^2v_h) \\ &= -(\rho_t, v_h) - (|Du|^2Du - |Du_h|^2Du_h, Dv_h) + (Du - Du_h, Dv_h). \end{aligned} \tag{22}$$

Taking $v_h = \theta$ in (22), we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\theta\|^2 + s \|D^2\theta\|^2 \\ &\leq |-(\rho_t, \theta)| + |-(|Du|^2Du - |Du_h|^2Du_h, D\theta)| + |(D\theta + D\rho, D\theta)|. \end{aligned} \tag{23}$$

By estimating the nonlinear term, we have

$$\begin{aligned} &|-(|Du|^2Du - |Du_h|^2Du_h, D\theta)| \\ &= |-((|Du|^2 + DuDu_h + |Du_h|^2)D\theta, D\theta + D\rho)| \\ &= |((|Du|^2 + DuDu_h + |Du_h|^2)D^2\theta, \theta + \rho) \\ &\quad + ((2DuD^2u + D^2uDu_h + DuD^2u_h + 2Du_hD^2u_h)D\theta, \theta + \rho)| \\ &\leq \| |Du|^2 + DuDu_h + |Du_h|^2 \|_\infty \cdot \|D^2\theta\| \cdot \|\theta + \rho\| \\ &\quad + \|2DuD^2u + D^2uDu_h + DuD^2u_h + 2Du_hD^2u_h\| \cdot \|D\theta\| \cdot \|\theta + \rho\| \\ &\leq (\|Du\|_\infty^2 + \|Du\|_\infty \cdot \|Du_h\|_\infty + \|Du_h\|_\infty^2) \cdot \|D^2\theta\| \cdot \|\theta + \rho\| \\ &\quad + (2\|Du\|_\infty \cdot \|D^2u\| + \|D^2u\| \cdot \|Du_h\|_\infty + \|Du\|_\infty \cdot \|D^2u_h\| \\ &\quad + 2\|Du_h\|_\infty \cdot \|D^2u_h\|) \cdot \|D\theta\| \cdot \|\theta + \rho\|. \end{aligned}$$

Based on Sobolev’s embedding theorem, $H^2(I) \hookrightarrow W^{1,\infty}(I)$, i.e.,

$$\|Du\|_\infty \leq C\|u\|_2, \quad \|Du_h\|_\infty \leq C\|u_h\|_2,$$

we obtain

$$\begin{aligned} &-(|Du|^2Du - |Du_h|^2Du_h, D\theta) \leq C(\|D\theta\| + \|D^2\theta\|)\|\theta + \rho\| \\ &\leq \frac{s}{8} \|D^2\theta\|^2 + C(\|\theta\|^2 + \|D\theta\|^2 + \|\rho\|^2) \leq \frac{s}{4} \|D^2\theta\|^2 + C(\|\theta\|^2 + \|\rho\|^2). \end{aligned} \tag{24}$$

In addition, it is easy to get

$$(D\theta + D\rho, D\theta) = -(\theta + \rho, D^2\theta) \leq \frac{s}{4} \|D^2\theta\|^2 + C(\|\theta\|^2 + \|\rho\|^2).$$

Using (3), Hölder’s inequality, and Young’s inequality, we can deduce

$$\frac{1}{2} \frac{d}{dt} \|\theta\|^2 + s \|D^2\theta\|^2 \leq \frac{1}{2} \|\rho_t\|^2 + \frac{1}{2} \|\theta\|^2 + \frac{s}{2} \|D^2\theta\|^2 + C(\|\theta\|^2 + \|\rho\|^2). \tag{25}$$

Hence

$$\frac{d}{dt} \|\theta\|^2 + s \|D^2\theta\|^2 \leq C(\|\theta\|^2 + \|\rho\|^2 + \|\rho_t\|^2). \tag{26}$$

By Gronwall’s inequality, we have

$$\|\theta\|^2 \leq C \left(\|\theta(0)\|^2 + \int_0^t (\|\rho\|^2 + \|\rho_t\|^2) d\tau \right). \tag{27}$$

Moreover, using the triangle inequality, we know

$$\|\theta(0)\| = \|u(0) - u_h(0) + R_h u(0) - u(0)\| \leq \|u(0) - u_h(0)\| + \|\rho(0)\|. \tag{28}$$

Hence, when $0 \leq t \leq T$, it follows from (21) and (27)–(28) that formula (20) is derived. This completes the proof. \square

Theorem 3.3 *Let u be the solution to (5), u_h be the solution to (6), $u(0) \in H^4(I)$, $u, u_t \in L^2(0, T; H^4(I))$, and the initial value satisfies*

$$|u(0) - u_h(0)|_2 \leq Ch^2 \|u(0)\|_4. \tag{29}$$

Then we have the following error estimate:

$$|u(t) - u_h(t)|_2 \leq Ch^2 \left(\|u(0)\|_4 + \left(\int_0^t (\|u(\tau)\|_4^2 + h^4 \|u_t(\tau)\|_4^2) d\tau \right)^{\frac{1}{2}} \right), \tag{30}$$

where C is a positive constant depending on $\alpha(x, t)$, independent of mesh size h .

Proof Letting $v_h = \theta_t$ in (22), we get

$$\begin{aligned} & \|\theta_t\|^2 + \frac{d}{dt} (\alpha(x, t) D^2\theta, D^2\theta) - \frac{1}{2} \left(\frac{\partial}{\partial t} \alpha(x, t) D^2\theta, D^2\theta \right) \\ &= -(\rho_t, \theta_t) - (|Du|^2 Du - |Du_h|^2 Du_h, D\theta_t) + (D\theta + D\rho, D\theta_t) \\ &= -(\rho_t, \theta_t) + (D(|Du|^2 Du - |Du_h|^2 Du_h), \theta_t) - (D^2\theta + D^2\rho, \theta_t) \\ &\leq \frac{1}{2} \|\theta_t\|^2 + C(\|\rho_t\|^2 + \|D(|Du|^2 Du - |Du_h|^2 Du_h)\|^2 + \|D^2\theta\|^2 + \|D^2\rho\|^2). \end{aligned}$$

Using the triangle inequality and Sobolev’s embedding theorem, we get

$$\|D(|Du|^2 Du - |Du_h|^2 Du_h)\|$$

$$\begin{aligned}
 &\leq \|(D^2u - D^2u_h)(|Du|^2 + DuDu_h + |Du_h|^2)\| \\
 &\quad + \|(Du - Du_h)(2DuD^2u + D^2uDu_h + DuD^2u_h + 2Du_hD^2u_h)\| \\
 &\leq \|D^2u - D^2u_h\| (|Du|_\infty^2 + |Du|_\infty|Du_h|_\infty + |Du_h|_\infty^2) \\
 &\quad + \|Du - Du_h\| (2|Du|_\infty\|D^2u\| + \|D^2u\| |Du_h|_\infty \\
 &\quad + |Du|_\infty\|D^2u_h\| + 2|Du_h|_\infty\|D^2u_h\|) \\
 &\leq C(\|D\theta\| + \|D^2\theta\| + \|D\rho\| + \|D^2\rho\|) \\
 &\leq C(\|D\theta\| + \|D^2\theta\| + \|\rho\|_2). \tag{31}
 \end{aligned}$$

Based on (4) and ε -inequality, we have

$$\begin{aligned}
 &\|\theta_t\|^2 + \frac{d}{dt}(\alpha(x,t)D^2\theta, D^2\theta) \\
 &\leq \frac{M_1}{2}\|D^2\theta\|^2 + \frac{1}{2}\|\theta_t\|^2 + C(\|\rho_t\|^2 + \|\rho\|_2^2 + \|D\theta\|^2 + \|D^2\theta\|^2) \\
 &\leq \frac{1}{2}\|\theta_t\|^2 + C(\|\rho_t\|^2 + \|\rho\|_2^2 + \|\theta\|^2 + \|D^2\theta\|^2). \tag{32}
 \end{aligned}$$

Integrating (32) with respect to t , we find

$$\begin{aligned}
 &(\alpha(x,t)D^2\theta, D^2\theta) - (\alpha(x,0)D^2\theta(0), D^2\theta(0)) \\
 &\leq C \int_0^t (\|\rho_\tau\|^2 + \|\rho\|_2^2 + \|\theta\|^2 + \|D^2\theta\|^2) d\tau.
 \end{aligned}$$

By (3), we have

$$s\|D^2\theta\|^2 \leq S\|D^2\theta(0)\|^2 + C \int_0^t (\|\rho_\tau\|^2 + \|\rho\|_2^2 + \|\theta\|^2 + \|D^2\theta\|^2) d\tau.$$

Then

$$\|D^2\theta\|^2 \leq C\left(\|D^2\theta(0)\|^2 + \int_0^t (\|\rho_\tau\|^2 + \|\rho\|_2^2 + \|\theta\|^2) d\tau\right). \tag{33}$$

Note that

$$\|D^2\theta(0)\| \leq \|D^2u(0) - D^2u_h(0)\| + \|D^2R_hu(0) - D^2u(0)\|. \tag{34}$$

Combining (33) and (34), we have

$$\|D^2\theta\| \leq Ch^2\left(\|u(0)\|_4 + \left(\int_0^t (\|u(\tau)\|_4^2 + h^4\|u_t(\tau)\|_4^2) d\tau\right)^{\frac{1}{2}}\right).$$

Finally, using (29), we obtain (30). The proof is completed. □

4 Fully discrete finite element scheme

To construct the Crank–Nicolson scheme, we define the following function:

$$H(Du_h^n) = \frac{1}{4}(1 - |Du_h^n|^2)^2, \tag{35}$$

where $H(Du_h^n)$ is a double well potential function. Obviously, $H'(Du_h) = |Du_h|^2 Du_h - Du_h$.

The fully discrete finite element scheme for problem (1) is: Find $u_h^n \in U_h$ ($n = 1, 2, \dots, N$) such that

$$\begin{cases} (\partial_t u_h^n, v_h) + (\alpha^{n-\frac{1}{2}} D^2 u_h^{n-\frac{1}{2}}, D^2 v_h) + (\frac{H(Du_h^n) - H(Du_h^{n-1})}{Du_h^n - Du_h^{n-1}}, Dv_h) = 0, \\ \forall v_h \in U_h, \\ (u(0) - u_h^0, v_h) = 0, \quad \forall v_h \in U_h, \end{cases} \tag{36}$$

where N is a given positive integer, $\Delta t = T/N$ denotes the time step size, $t_n = n\Delta t$ and

$$\begin{aligned} \partial_t u_h^n &= (u_h^n - u_h^{n-1}) / \Delta t, \\ \alpha^{n-\frac{1}{2}} &= \alpha(x, t^{n-\frac{1}{2}}), \\ u_h^{n-\frac{1}{2}} &= (u_h^n + u_h^{n-1}) / 2, \\ t^{n-\frac{1}{2}} &= (t^n + t^{n-1}) / 2. \end{aligned}$$

Firstly, we analyze the boundedness of the fully discrete scheme (36). It is a key step for deducing the error estimate.

Theorem 4.1 *Let $u_h^0 \in H_0^2(I) \cap W^{1,4}(I)$, then there exists a unique solution u_h^n for problem (36) such that*

$$\|u_h^n\|_2 \leq C \|u_h^0\|_2, \quad 0 \leq t \leq T, \tag{37}$$

where C is a positive constant depending on $\alpha(x, t)$ and T , independent of h and Δt .

Proof A direct calculation gives

$$\begin{aligned} & \frac{H(Du_h^n) - H(Du_h^{n-1})}{Du_h^n - Du_h^{n-1}} \\ &= \frac{1}{4} (Du_h^n + Du_h^{n-1}) (|Du_h^n|^2 + |Du_h^{n-1}|^2) - \frac{1}{2} (Du_h^n + Du_h^{n-1}). \end{aligned} \tag{38}$$

Setting $v_h = u_h^n + u_h^{n-1}$ in (36), we get

$$\begin{aligned} & \frac{1}{\Delta t} (\|u_h^n\|^2 - \|u_h^{n-1}\|^2) + \frac{s}{2} \|D^2 u_h^n + D^2 u_h^{n-1}\|^2 \\ &+ \frac{1}{4} ((Du_h^n + Du_h^{n-1})^2, |Du_h^n|^2 + |Du_h^{n-1}|^2) \leq \frac{1}{2} \|Du_h^n + Du_h^{n-1}\|^2. \end{aligned} \tag{39}$$

Using Cauchy’s inequality, we obtain

$$\begin{aligned} & \frac{1}{\Delta t} (\|u_h^n\|^2 - \|u_h^{n-1}\|^2) + \frac{s}{2} \|D^2 u_h^n + D^2 u_h^{n-1}\|^2 \\ & \leq \frac{s}{4} \|D^2 u_h^n + D^2 u_h^{n-1}\|^2 + \frac{1}{4s} \|u_h^n + u_h^{n-1}\|^2. \end{aligned}$$

Further, we derive

$$\frac{1}{\Delta t} (\|u_h^n\|^2 - \|u_h^{n-1}\|^2) \leq \frac{1}{4s} \|u_h^n + u_h^{n-1}\|^2 \leq \frac{1}{2s} (\|u_h^n\|^2 + \|u_h^{n-1}\|^2). \tag{40}$$

Letting $\gamma = \frac{1}{2s}$, we have

$$\|u_h^n\|^2 \leq \frac{1 + \gamma \Delta t}{1 - \gamma \Delta t} \|u_h^{n-1}\|^2 \leq \dots \leq \left(\frac{1 + \gamma \Delta t}{1 - \gamma \Delta t} \right)^n \|u_h^0\|^2. \tag{41}$$

It is easy to show

$$\left(\frac{1 + \gamma \Delta t}{1 - \gamma \Delta t} \right)^n = \left(1 + \frac{2\gamma \Delta t}{1 - \gamma \Delta t} \right)^{\frac{1 - \gamma \Delta t}{2\gamma \Delta t} \cdot \frac{2\gamma n \Delta t}{1 - \gamma \Delta t}}.$$

If Δt is small enough, we conclude

$$\|u_h^n\|^2 \leq C \|u_h^0\|^2. \tag{42}$$

Choosing $v_h = \partial_t u_h^n$ in (36), we have

$$\begin{aligned} & \|\partial_t u_h^n\|^2 + \frac{1}{2\Delta t} (\alpha(x, t^{n-\frac{1}{2}}) (|D^2 u_h^n|^2 - |D^2 u_h^{n-1}|^2), 1) \\ & + \frac{1}{\Delta t} (H(Du_h^n) - H(Du_h^{n-1}), 1) = 0. \end{aligned} \tag{43}$$

Then we get

$$\begin{aligned} & \left(\frac{1}{2} \alpha(x, t^{n-\frac{1}{2}}) |D^2 u_h^n|^2 + H(Du_h^n), 1 \right) \\ & \leq \left(\frac{1}{2} \alpha(x, t^{n-\frac{1}{2}}) |D^2 u_h^{n-1}|^2 + H(Du_h^{n-1}), 1 \right). \end{aligned} \tag{44}$$

Define the function

$$G(u_h^n, t^{n-\frac{1}{2}}) = \left(\frac{1}{2} \alpha(x, t^{n-\frac{1}{2}}) |D^2 u_h^n|^2 + H(Du_h^n), 1 \right), \tag{45}$$

then $G(u_h^n, t^{n-\frac{1}{2}}) \geq 0$. By (44) and (45), we have

$$G(u_h^n, t^{n-\frac{1}{2}}) \leq G(u_h^{n-1}, t^{n-\frac{3}{2}}) + \frac{1}{2} ((\alpha(x, t^{n-\frac{1}{2}}) - \alpha(x, t^{n-\frac{3}{2}})) |D^2 u_h^{n-1}|^2, 1).$$

With the differential mean value theorem and the boundedness of variable coefficient, we obtain

$$\begin{aligned} G(u_h^n, t^{n-\frac{1}{2}}) &\leq G(u_h^{n-1}, t^{n-\frac{3}{2}}) + \frac{\Delta t}{2} \left| \frac{\partial \alpha}{\partial t}(x, \xi) \right| \|D^2 u_h^{n-1}\|^2 \\ &\leq G(u_h^{n-1}, t^{n-\frac{3}{2}}) + \frac{M_1 \Delta t}{2} \|D^2 u_h^{n-1}\|^2, \end{aligned}$$

where $t^{n-\frac{3}{2}} < \xi < t^{n-\frac{1}{2}}$. Then

$$G(u_h^n, t^{n-\frac{1}{2}}) - G(u_h^{n-1}, t^{n-\frac{3}{2}}) \leq \frac{M_1 \Delta t}{2} \|D^2 u_h^{n-1}\|^2.$$

Taking the sum over n , we get

$$G(u_h^n, t^{n-\frac{1}{2}}) - G(u_h^1, t^{\frac{1}{2}}) \leq \frac{M_1 \Delta t}{2} \sum_{j=2}^{n-1} \|D^2 u_h^j\|^2. \tag{46}$$

It is obvious that

$$G(u_h^n, t^{n-\frac{1}{2}}) \geq \frac{s}{2} \|D^2 u_h^n\|^2 + (H(Du_h^n), 1) \geq \frac{s}{2} \|D^2 u_h^n\|^2.$$

Therefore we know

$$G(u_h^n, t^{n-\frac{1}{2}}) - G(u_h^1, t^{\frac{1}{2}}) \leq \frac{M_1 \Delta t}{s} \sum_{j=2}^{n-1} G(u_h^j, t^{j-\frac{1}{2}}).$$

Based on (44) and $u_h^0 \in H_0^2(I) \cap W^{1,4}(I)$, we have

$$\begin{aligned} G(u_h^1, t^{\frac{1}{2}}) &= \left(\frac{1}{2} \alpha(x, t^{\frac{1}{2}}) |D^2 u_h^1|^2 + H(Du_h^1), 1 \right) \\ &\leq \left(\frac{1}{2} \alpha(x, t^{\frac{1}{2}}) |D^2 u_h^0|^2 + H(Du_h^0), 1 \right) \leq C(u_h^0), \end{aligned}$$

where $C(u_h^0)$ is a constant depending on u_h^0 . Then

$$G(u_h^n, t^{n-\frac{1}{2}}) \leq C(u_h^0) + \frac{M_1 \Delta t}{s} \sum_{j=2}^{n-1} G(u_h^j, t^{j-\frac{1}{2}}). \tag{47}$$

Using discrete Gronwall’s inequality, we derive

$$G(u_h^n, t^{n-\frac{1}{2}}) \leq C, \quad C = C(u_h^0, s, M_1, T). \tag{48}$$

Based on (48), it is easy to see

$$\|D^2 u_h^n\| \leq C \|D^2 u_h^0\|. \tag{49}$$

We also know

$$\|Du_h^n\|^2 \leq \frac{1}{2}(\|u_h^n\|^2 + \|D^2u_h^n\|^2).$$

By (42) and (49), we obtain (37). The proof is completed. □

Next, we give the error estimate in L^2 norm.

Theorem 4.2 *Let u^n be the solution to problem (5), u_h^n be the solution to the fully discrete scheme (36), $u(0) \in H^4(I)$, $u_t \in L^2(0, T; H^4(I)) \cap L^2(0, T; W^{1,4}(I))$, $u_{ttt} \in L^2(0, T; L^2(I))$ and $u_h^0 \in U_h$ satisfying*

$$\|u(0) - u_h^0\| \leq Ch^4 \|u(0)\|_4. \tag{50}$$

Then we have the following error estimate:

$$\|u^n - u_h^n\| \leq C((\Delta t)^2 + h^3), \tag{51}$$

where C is a positive constant depending on $\alpha(x, t)$ and T , independent of mesh size h .

Proof Denote $u_t^n = u_t(x, t^n)$ and $u^n = u(x, t^n)$. Setting $t = t^{n-1}$ and $t = t^n$ in (5), respectively, we obtain

$$\begin{aligned} & \left(\frac{u_t^n + u_t^{n-1}}{2}, v_h \right) + \left(\frac{\alpha(x, t^n)D^2u^n + \alpha(x, t^{n-1})D^2u^{n-1}}{2}, D^2v_h \right) \\ & + \left(\frac{|Du^n|^2Du^n + |Du^{n-1}|^2Du^{n-1} - Du^n - Du^{n-1}}{2}, Dv_h \right) = 0. \end{aligned} \tag{52}$$

Denote

$$\begin{aligned} & \Phi(D^2u^n, D^2u^{n-1}, D^2u_h^{n-\frac{1}{2}}) \\ & = \frac{\alpha(x, t^n)D^2u^n + \alpha(x, t^{n-1})D^2u^{n-1}}{2} - \alpha(x, t^{n-\frac{1}{2}})D^2u_h^{n-\frac{1}{2}} \end{aligned} \tag{53}$$

and

$$\begin{aligned} & F(Du^n, Du^{n-1}, Du_h^n, Du_h^{n-1}) \\ & = \frac{|Du^n|^2Du^n + |Du^{n-1}|^2Du^{n-1} - Du^n - Du^{n-1}}{2} - \frac{H(Du_h^n) - H(Du_h^{n-1})}{Du_h^n - Du_h^{n-1}}. \end{aligned} \tag{54}$$

It follows from (52)–(54) and (36) that

$$\begin{aligned} & \left(\frac{u_t^n + u_t^{n-1}}{2} - \partial_t u_h^n, v_h \right) + (\Phi(D^2u^n, D^2u^{n-1}, D^2u_h^{n-\frac{1}{2}}), D^2v_h) \\ & + (F(Du^n, Du^{n-1}, Du_h^n, Du_h^{n-1}), Dv_h) = 0. \end{aligned} \tag{55}$$

Let $\rho^n = u^n - R_h u^n$ and $\theta^n = R_h u^n - u_h^n$, then $u^n - u_h^n = \rho^n + \theta^n$. It is clear to get

$$\begin{aligned} \frac{u_t^n + u_t^{n-1}}{2} - \partial_t u_h^n &= \frac{u_t^n + u_t^{n-1}}{2} - \partial_t u^n + \partial_t u^n - \partial_t u_h^n \\ &= \frac{u_t^n + u_t^{n-1}}{2} - \partial_t u^n + \partial_t (u^n - R_h u^n + R_h u^n - u_h^n) = \partial_t \theta^n - r^n, \end{aligned} \tag{56}$$

where

$$r^n = \partial_t R_h u^n - \partial_t u^n + \partial_t u^n - \frac{u_t(t_j) + u_t(t_{j-1})}{2}.$$

An easy calculation gives

$$\begin{aligned} &\Phi(D^2 u^n, D^2 u^{n-1}, D^2 u_h^{n-\frac{1}{2}}) \\ &= \frac{1}{2}((\alpha(x, t^n) - \alpha(x, t^{n-\frac{1}{2}}))D^2 u^n + (\alpha(x, t^{n-1}) - \alpha(x, t^{n-\frac{1}{2}}))D^2 u^{n-1} \\ &\quad + \alpha(x, t^{n-\frac{1}{2}})(D^2 u^n + D^2 u^{n-1} - D^2 u_h^n - D^2 u_h^{n-1})) \\ &= \frac{1}{2}((\alpha(x, t^n) - \alpha(x, t^{n-\frac{1}{2}}))D^2 u^n + (\alpha(x, t^{n-1}) - \alpha(x, t^{n-\frac{1}{2}}))D^2 u^{n-1} \\ &\quad + \alpha(x, t^{n-\frac{1}{2}})(D^2 \theta^n + D^2 \theta^{n-1} + D^2 \rho^n + D^2 \rho^{n-1})). \end{aligned}$$

Using Taylor’s theorem, we have

$$\alpha(x, t^n) = \alpha(x, t^{n-\frac{1}{2}}) + \frac{\Delta t}{2} \frac{\partial \alpha}{\partial t}(x, t^{n-\frac{1}{2}}) + \frac{(\Delta t)^2}{6} \frac{\partial^2 \alpha}{\partial t^2}\left(x, t^{n-\frac{1}{2}} + \xi_1 \frac{\Delta t}{2}\right), \quad 0 < \xi_1 < 1$$

and

$$\alpha(x, t^{n-1}) = \alpha(x, t^{n-\frac{1}{2}}) - \frac{\Delta t}{2} \frac{\partial \alpha}{\partial t}(x, t^{n-\frac{1}{2}}) + \frac{(\Delta t)^2}{6} \frac{\partial^2 \alpha}{\partial t^2}\left(x, t^{n-\frac{1}{2}} + \xi_2 \frac{\Delta t}{2}\right), \quad -1 < \xi_2 < 0.$$

With (4), we get

$$\begin{aligned} &\Phi(D^2 u^n, D^2 u^{n-1}, D^2 u_h^{n-\frac{1}{2}}) \\ &= \frac{\Delta t}{2} \frac{\partial \alpha}{\partial t}(x, t^{n-\frac{1}{2}})(D^2 u^n - D^2 u^{n-1}) + O((\Delta t)^2) \\ &\quad + \frac{1}{2} \alpha(x, t^{n-\frac{1}{2}})(D^2 \theta^n + D^2 \theta^{n-1} + D^2 \rho^n + D^2 \rho^{n-1}). \end{aligned} \tag{57}$$

From (7), we have

$$\begin{aligned} &(\partial_t \theta^n, v_h) + \frac{1}{2}(\alpha^{n-\frac{1}{2}}(D^2 \theta^n + D^2 \theta^{n-1}), D^2 v_h) \\ &\quad + \frac{\Delta t}{2} \left(\frac{\partial \alpha}{\partial t}(x, t^{n-\frac{1}{2}})(D^2 u^n - D^2 u^{n-1}), D^2 v_h \right) + (O((\Delta t)^2), D^2 v_h) \\ &= (r^n, v_h) - (F(Du^n, Du^{n-1}, Du_h^n, Du_h^{n-1}), Dv_h). \end{aligned} \tag{58}$$

Setting $v_h = \theta^n + \theta^{n-1}$ in (58), we get

$$\begin{aligned} & \frac{1}{\Delta t} (\|\theta^n\|^2 - \|\theta^{n-1}\|^2) + \frac{s}{2} \|D^2\theta^n + D^2\theta^{n-1}\|^2 \\ & \leq \|r^n\|^2 + \frac{1}{4} \|\theta^n + \theta^{n-1}\|^2 + \|F(Du^n, Du^{n-1}, Du_h^n, Du_h^{n-1})\|^2 \\ & \quad + \frac{1}{4} \|D\theta^n + D\theta^{n-1}\|^2 + \frac{M_1\Delta t}{2} \|D^2u^n - D^2u^{n-1}\| \|D^2\theta^n + D^2\theta^{n-1}\| \\ & \leq \|r^n\|^2 + \|F(Du^n, Du^{n-1}, Du_h^n, Du_h^{n-1})\|^2 + \frac{1}{4} \left(1 + \frac{1}{2s}\right) \|\theta^n + \theta^{n-1}\|^2 \\ & \quad + \frac{s}{4} \|D^2\theta^n + D^2\theta^{n-1}\|^2 + \frac{(M_1\Delta t)^2}{2s} \|D^2u^n - D^2u^{n-1}\|^2. \end{aligned}$$

Based on the Newton–Leibniz formula and Hölder’s inequality, we have

$$|D^2u^n - D^2u^{n-1}| = \left| \int_{t_{n-1}}^{t_n} D^2u_t(t) dt \right| \leq \Delta t^{\frac{1}{2}} \left(\int_{t_{n-1}}^{t_n} |D^2u_t(t)|^2 dt \right)^{\frac{1}{2}}.$$

Thus

$$\begin{aligned} & \frac{1}{\Delta t} (\|\theta^n\|^2 - \|\theta^{n-1}\|^2) + \frac{s}{2} \|D^2\theta^n + D^2\theta^{n-1}\|^2 \\ & \leq \|r^n\|^2 + \|F(Du^n, Du^{n-1}, Du_h^n, Du_h^{n-1})\|^2 + \frac{1}{4} \left(1 + \frac{1}{2s}\right) \|\theta^n + \theta^{n-1}\|^2 \\ & \quad + \frac{s}{4} \|D^2\theta^n + D^2\theta^{n-1}\|^2 + \frac{M_1^2(\Delta t)^3}{2s} \int_{t_{n-1}}^{t_n} \|D^2u_t(t)\|^2 dt. \end{aligned} \tag{59}$$

A direct calculation gives

$$\begin{aligned} & \|F(Du^n, Du^{n-1}, Du_h^n, Du_h^{n-1})\| \\ & = \left\| \frac{1}{2} ((Du^n)^3 + (Du^{n-1})^3) - \frac{1}{4} (Du^n + Du^{n-1})(|Du^n|^2 + |Du^{n-1}|^2) \right. \\ & \quad + \frac{1}{4} (Du^n + Du^{n-1})(|Du^n|^2 + |Du^{n-1}|^2) \\ & \quad - \frac{1}{4} (Du_h^n + Du_h^{n-1})(|Du_h^n|^2 + |Du_h^{n-1}|^2) \\ & \quad \left. - \frac{1}{2} (Du^n + Du^{n-1}) + \frac{1}{2} (Du_h^n + Du_h^{n-1}) \right\|. \end{aligned}$$

From (37) and Sobolev’s embedding theorem, $H_0^2(I) \hookrightarrow H^{1,\infty}(I)$, we know

$$|Du^n|_\infty \leq C \|u^n\|_2 \leq C, \quad |Du_h^n|_\infty \leq C \|u_h^n\|_2 \leq C. \tag{60}$$

Using Hölder’s inequality, we have

$$|Du^n - Du^{n-1}| = \left| \int_{t_{n-1}}^{t_n} Du_t(t) dt \right| \leq C(\Delta t)^{\frac{1}{2}} \left(\int_{t_{n-1}}^{t_n} |Du_t(t)|^2 dt \right)^{\frac{1}{2}}. \tag{61}$$

From (60) and (61), we have

$$\begin{aligned}
 & \left\| \frac{1}{2} \left((Du^n)^3 + (Du^{n-1})^3 \right) - \frac{1}{4} (Du^n + Du^{n-1}) (|Du^n|^2 + |Du^{n-1}|^2) \right\| \\
 &= \frac{1}{4} \left\| |Du^n|^2 Du^n - |Du^n|^2 Du^{n-1} - Du^n |Du^{n-1}|^2 + |Du^{n-1}|^2 Du^{n-1} \right\| \\
 &= \frac{1}{4} \left\| (Du^n + Du^{n-1}) (Du^n - Du^{n-1})^2 \right\| \\
 &\leq \frac{1}{4} (|Du^n|_\infty + |Du^{n-1}|_\infty) \left\| (Du^n - Du^{n-1})^2 \right\| \\
 &\leq C \Delta t \int_{t_{n-1}}^{t_n} \|Du_t(t)\|^2 dt. \tag{62}
 \end{aligned}$$

Due to (60), we get

$$\begin{aligned}
 & \left\| (Du^n + Du^{n-1}) (|Du^n|^2 + |Du^{n-1}|^2) \right. \\
 & \quad \left. - (Du_h^n + Du_h^{n-1}) (|Du_h^n|^2 + |Du_h^{n-1}|^2) \right\| \\
 &= \left\| (Du^n + Du^{n-1}) (|Du^n|^2 + |Du^{n-1}|^2) \right. \\
 & \quad - (Du_h^n + Du_h^{n-1}) (|Du^n|^2 + |Du^{n-1}|^2) \\
 & \quad + (Du_h^n + Du_h^{n-1}) (|Du^n|^2 + |Du^{n-1}|^2) \\
 & \quad \left. - (Du_h^n + Du_h^{n-1}) (|Du_h^n|^2 + |Du_h^{n-1}|^2) \right\| \\
 &\leq (|Du^n|_\infty^2 + |Du^{n-1}|_\infty^2) \left\| (Du^n + Du^{n-1}) - (Du_h^n + Du_h^{n-1}) \right\| \\
 & \quad + (|Du_h^n|_\infty + |Du_h^{n-1}|_\infty) \left\| (Du^n + Du_h^n) (Du^n - Du_h^n) \right. \\
 & \quad \left. + (Du^{n-1} + Du_h^{n-1}) (Du^{n-1} - Du_h^{n-1}) \right\| \\
 &\leq (|Du^n|_\infty^2 + |Du^{n-1}|_\infty^2) (\|D\theta^n + D\theta^{n-1}\| + \|D\rho^n + D\rho^{n-1}\|) \\
 & \quad + (|Du_h^n|_\infty + |Du_h^{n-1}|_\infty) (|Du^n|_\infty + |Du_h^n|_\infty + |Du^{n-1}|_\infty + |Du_h^{n-1}|_\infty) \\
 & \quad \times (\|D\theta^n + D\theta^{n-1}\| + \|D\rho^n + D\rho^{n-1}\|) \\
 &\leq C (\|D\theta^n + D\theta^{n-1}\| + \|D\rho^n + D\rho^{n-1}\|). \tag{63}
 \end{aligned}$$

By the triangle inequality, we obtain

$$\begin{aligned}
 & \left\| (Du^n + Du^{n-1}) - (Du_h^n + Du_h^{n-1}) \right\| \\
 &= \|D\theta^n + D\rho^n + D\theta^{n-1} + D\rho^{n-1}\| \\
 &\leq \|D\theta^n + D\theta^{n-1}\| + \|D\rho^n + D\rho^{n-1}\|. \tag{64}
 \end{aligned}$$

In view of (62)–(64) and (9), we have

$$\begin{aligned}
 & \|F(Du^n, Du^{n-1}, Du_h^n, Du_h^{n-1})\| \\
 &\leq C \left(\|D\theta^n + D\theta^{n-1}\| + \|D\rho^n + D\rho^{n-1}\| + \Delta t \int_{t_{n-1}}^{t_n} \|Du_t(t)\|^2 dt \right)
 \end{aligned}$$

$$\leq C \left(\|D\theta^n + D\theta^{n-1}\| + h^3 + \Delta t \int_{t_{n-1}}^{t_n} \|Du_t(t)\|^2 dt \right).$$

Based on the ε -inequality and Hölder’s inequality, we obtain

$$\begin{aligned} & \|F(Du^n, Du^{n-1}, Du_h^n, Du_h^{n-1})\|^2 \\ & \leq C \left(\|\theta^n + \theta^{n-1}\|^2 + h^6 + (\Delta t)^3 \int_{t_{n-1}}^{t_n} \|Du_t(t)\|^4 dt \right) + \frac{s}{8} \|D^2\theta^n + D^2\theta^{n-1}\|^2. \end{aligned} \tag{65}$$

Let $r^n = r_1^n + r_2^n$, where

$$\begin{aligned} r_1^j &= \partial_t R_h u(t_j) - \partial_t u(t_j) = \frac{1}{\Delta t} \int_{t_{j-1}}^{t_j} (R_h - I) u_t dt, \\ r_2^j &= \partial_t u(t_j) - \frac{u_t(t_j) + u_t(t_{j-1})}{2}. \end{aligned}$$

It is clear to see that

$$\|r_1^j\| \leq \frac{1}{\Delta t} C h^4 \int_{t_{j-1}}^{t_j} \|u_t\|_4 dt \leq C (\Delta t)^{-\frac{1}{2}} h^4 \left(\int_{t_{j-1}}^{t_j} \|u_t\|_4^2 dt \right)^{\frac{1}{2}}.$$

Using Taylor’s formula, we derive

$$\|r_2^j\| \leq C \Delta t \int_{t_{j-1}}^{t_j} \|u_{ttt}\| dt \leq C (\Delta t)^{\frac{3}{2}} \left(\int_{t_{j-1}}^{t_j} \|u_{ttt}\|^2 dt \right)^{\frac{1}{2}}.$$

We easily get

$$\sum_{j=1}^n \|r^j\|^2 \leq C (\Delta t)^{-1} ((\Delta t)^4 + h^8) \int_0^{t_n} (\|u_t\|_4^2 + \|u_{ttt}\|^2) dt. \tag{66}$$

Adding (59), (65), and (66), we have

$$\begin{aligned} & (\|\theta^n\|^2 - \|\theta^{n-1}\|^2) + \frac{s\Delta t}{8} \|D^2\theta^n + D^2\theta^{n-1}\|^2 \\ & \leq C \left(\Delta t (\|\theta^n + \theta^{n-1}\|^2 + h^6) \right. \\ & \quad \left. + ((\Delta t)^4 + h^8) \int_{t_{n-1}}^{t_n} (\|u_t\|_4^2 + \|Du_t\|^4 + \|D^2u_t\|^2 + \|u_{ttt}\|^2) dt \right). \end{aligned}$$

We know

$$\|\theta^n + \theta^{n-1}\|^2 \leq 2(\|\theta^n\|^2 + \|\theta^{n-1}\|^2).$$

Then

$$(\|\theta^n\|^2 - \|\theta^{n-1}\|^2) + \frac{s\Delta t}{8} \|D^2\theta^n + D^2\theta^{n-1}\|^2$$

$$\begin{aligned} &\leq C\left(\Delta t(\|\theta^n\|^2 + \|\theta^{n-1}\|^2 + h^6)\right. \\ &\quad \left.+ ((\Delta t)^4 + h^8) \int_{t_{n-1}}^{t_n} (\|u_t\|_4^2 + \|Du_t\|^4 + \|D^2u_t\|^2 + \|u_{ttt}\|^2) dt\right). \end{aligned} \tag{67}$$

Taking the sum over n , by $n\Delta t = t_n \leq T$, we have

$$\begin{aligned} &\|\theta^n\|^2 - \|\theta^0\|^2 + \frac{s\Delta t}{8} \sum_{i=1}^n \|D^2\theta^i + D^2\theta^{i-1}\|^2 \\ &\leq C\left(\Delta t \sum_{i=1}^n (\|\theta^i\|^2 + \|\theta^{i-1}\|^2) + Th^6\right. \\ &\quad \left.+ ((\Delta t)^4 + h^8) \int_0^{t_n} (\|u_t\|_4^2 + \|Du_t\|^4 + \|D^2u_t\|^2 + \|u_{ttt}\|^2) dt\right). \end{aligned}$$

Hence

$$(1 - C\Delta t)\|\theta^n\|^2 \leq (1 + C\Delta t)\|\theta^0\|^2 + C\left(\Delta t \sum_{i=1}^{n-1} \|\theta^i\|^2 + Th^6 + (\Delta t)^4 + h^8\right).$$

If Δt is small enough, we have

$$\|\theta^n\|^2 \leq \frac{1 + C\Delta t}{1 - C\Delta t}\|\theta^0\|^2 + \frac{C}{1 - C\Delta t}\left(\Delta t \sum_{i=1}^{n-1} \|\theta^i\|^2 + Th^6 + (\Delta t)^4 + h^8\right).$$

By discrete Gronwall’s inequality, it gives

$$\|\theta^n\| \leq C((\Delta t)^2 + h^3).$$

Using (9) and (50), we get

$$\|\theta^0\| \leq \|u(0) - u_h(0)\| + \|u(0) - R_h u(0)\| \leq Ch^4 \|u(0)\|_4.$$

Finally, we obtain (51). The proof is completed. □

In the following theorem, we introduce the error estimate in H^2 norm.

Theorem 4.3 *Let u^n be the solution to (5), u_h^n be the solution to the fully discrete problem (36), $u(0) \in H^4(I)$, $u_t \in L^2(0, T; H^4(I)) \cap L^2(0, T; W^{2,4}(I))$, $u_{ttt} \in L^2(0, T; L^2(I))$, and $u_h^0 \in U_h$ satisfying*

$$\|u(0) - u_h^0\|_2 \leq Ch^2 \|u(0)\|_4. \tag{68}$$

Then we have the following error estimate:

$$\|u^n - u_h^n\|_2 \leq C(\Delta t + h^2). \tag{69}$$

Proof Letting $v_h = \partial_t \theta^n$ in (58), we get

$$\begin{aligned} & \|\partial_t \theta^n\|^2 + \frac{1}{2\Delta t} (\alpha^{n-\frac{1}{2}}(D^2\theta^n + D^2\theta^{n-1}), D^2\theta^n - D^2\theta^{n-1}) \\ & \quad + \frac{1}{2} \left(\frac{\partial \alpha}{\partial t}(x, t^{n-\frac{1}{2}})(D^2u^n - D^2u^{n-1}), D^2\theta^n - D^2\theta^{n-1} \right) \\ & \leq \|r^n\|^2 + \|DF(Du^n, Du^{n-1}, Du_h^n, Du_h^{n-1})\|^2 + \frac{1}{2} \|\partial_t \theta^n\|^2. \end{aligned}$$

By Cauchy’s inequality, we have

$$\begin{aligned} & (\alpha^{n-\frac{1}{2}}D^2\theta^n, D^2\theta^n) - (\alpha^{n-\frac{1}{2}}D^2\theta^{n-1}, D^2\theta^{n-1}) \\ & \leq M_1\Delta t \|D^2u^n - D^2u^{n-1}\| \|D^2\theta^n - D^2\theta^{n-1}\| \\ & \quad + 2\Delta t (\|r^n\|^2 + \|DF(Du^n, Du^{n-1}, Du_h^n, Du_h^{n-1})\|^2) \\ & \leq \frac{M_1^2\Delta t}{2} \|D^2u^n - D^2u^{n-1}\|^2 + \Delta t (\|D^2\theta^n\|^2 + \|D^2\theta^{n-1}\|^2) \\ & \quad + 2\Delta t (\|r^n\|^2 + \|DF(Du^n, Du^{n-1}, Du_h^n, Du_h^{n-1})\|^2). \end{aligned} \tag{70}$$

Using the Newton–Leibniz formula and Hölder’s inequality, we obtain

$$|D^2u^n - D^2u^{n-1}| \leq \left| \Delta t \int_{t^{n-1}}^{t^n} |D^2u_t| dt \right| \leq \Delta t \int_{t^{n-1}}^{t^n} |D^2u_t|^2 dt.$$

Based on (70), we have

$$\begin{aligned} & (\alpha^{n-\frac{1}{2}}D^2\theta^n, D^2\theta^n) - (\alpha^{n-\frac{1}{2}}D^2\theta^{n-1}, D^2\theta^{n-1}) \\ & \leq \frac{M_1^2(\Delta t)^2}{2} \int_{t^{n-1}}^{t^n} \|D^2u_t\|^2 dt + \Delta t (\|D^2\theta^n\|^2 + \|D^2\theta^{n-1}\|^2) \\ & \quad + 2\Delta t (\|r^n\|^2 + \|DF(Du^n, Du^{n-1}, Du_h^n, Du_h^{n-1})\|^2). \end{aligned} \tag{71}$$

There exists $\xi \in (t^{n-\frac{3}{2}}, t^{n-\frac{1}{2}})$ such that

$$\begin{aligned} & (\alpha^{n-\frac{1}{2}}(D^2\theta^n + D^2\theta^{n-1}), D^2\theta^n - D^2\theta^{n-1}) \\ & = (\alpha^{n-\frac{1}{2}}D^2\theta^n, D^2\theta^n) - (\alpha^{n-\frac{3}{2}}D^2\theta^{n-1}, D^2\theta^{n-1}) - ((\alpha^{n-\frac{1}{2}} - \alpha^{n-\frac{3}{2}})D^2\theta^{n-1}, D^2\theta^{n-1}) \\ & = (\alpha^{n-\frac{1}{2}}D^2\theta^n, D^2\theta^n) - (\alpha^{n-\frac{3}{2}}D^2\theta^{n-1}, D^2\theta^{n-1}) - \Delta t \left(\frac{\partial \alpha}{\partial t}(x, \xi) D^2\theta^{n-1}, D^2\theta^{n-1} \right). \end{aligned}$$

Then we have

$$\begin{aligned} & (\alpha^{n-\frac{1}{2}}D^2\theta^n, D^2\theta^n) - (\alpha^{n-\frac{3}{2}}D^2\theta^{n-1}, D^2\theta^{n-1}) - \Delta t \left(\frac{\partial \alpha}{\partial t}(x, \xi) D^2\theta^{n-1}, D^2\theta^{n-1} \right) \\ & \leq \frac{M_1^2(\Delta t)^2}{2} \int_{t^{n-1}}^{t^n} \|D^2u_t\|^2 dt + \Delta t (\|D^2\theta^n\|^2 + \|D^2\theta^{n-1}\|^2) \\ & \quad + 2\Delta t (\|r^n\|^2 + \|DF(Du^n, Du^{n-1}, Du_h^n, Du_h^{n-1})\|^2). \end{aligned} \tag{72}$$

Taking the sum over n and using (4), we can obtain

$$\begin{aligned}
 & (\alpha^{n-\frac{1}{2}}D^2\theta^n, D^2\theta^n) - (\alpha^{\frac{1}{2}}D^2\theta^1, D^2\theta^1) \\
 & \leq C\Delta t \sum_{j=2}^n (\|D^2\theta^{j-1}\|^2 + \|D^2\theta^j\|^2 + \|r^j\|^2 + \|DF(Du^j, Du^{j-1}, Du_h^j, Du_h^{j-1})\|^2) \\
 & \quad + C(\Delta t)^2 \int_0^{t^n} \|D^2u_t\|^2 dt.
 \end{aligned} \tag{73}$$

It follows from (3) that

$$(\alpha^{n-\frac{1}{2}}D^2\theta^n, D^2\theta^n) \geq s\|D^2\theta^n\|^2, \quad -(\alpha^{\frac{1}{2}}D^2\theta^1, D^2\theta^1) \geq -S\|D^2\theta^1\|^2.$$

Then one has

$$\begin{aligned}
 & s\|D^2\theta^n\|^2 - S\|D^2\theta^1\|^2 \\
 & \leq C\Delta t \sum_{j=2}^n (\|D^2\theta^j\|^2 + \|r^j\|^2 + \|DF(Du^j, Du^{j-1}, Du_h^j, Du_h^{j-1})\|^2) \\
 & \quad + C(\Delta t)^2 \int_0^{t^n} \|D^2u_t\|^2 dt.
 \end{aligned} \tag{74}$$

Introducing some symbols I_1, I_2, I_3 , then a direct calculation gives

$$\begin{aligned}
 & \|DF(Du^n, Du^{n-1}, Du_h^n, Du_h^{n-1})\| \\
 & = \left\| D\left(\frac{1}{2}((Du^n)^3 + (Du^{n-1})^3) - \frac{1}{2}(Du^n + Du^{n-1}) - \frac{H(Du_h^n) - H(Du_h^{n-1})}{Du_h^n - Du_h^{n-1}}\right) \right\| \\
 & = \left\| D\left(\frac{1}{2}((Du^n)^3 + (Du^{n-1})^3) - \frac{1}{4}(Du^n + Du^{n-1})(|Du^n|^2 + |Du^{n-1}|^2)\right) \right. \\
 & \quad \left. + \frac{1}{4}D((Du^n + Du^{n-1})(|Du^n|^2 + |Du^{n-1}|^2) - (Du_h^n + Du_h^{n-1})(|Du_h^n|^2 + |Du_h^{n-1}|^2)) \right. \\
 & \quad \left. - \frac{1}{2}D((Du^n + Du^{n-1}) - (Du_h^n + Du_h^{n-1})) \right\| \\
 & = \|I_1 + I_2 + I_3\|.
 \end{aligned}$$

It is obvious that

$$\|DF(Du^n, Du^{n-1}, Du_h^n, Du_h^{n-1})\| \leq \|I_1\| + \|I_2\| + \|I_3\|.$$

First, applying the triangle inequality to $\|I_1\|$, we get

$$\begin{aligned}
 \|I_1\| & = \frac{1}{4} \left\| D((Du^n)^3 - |Du^n|^2 Du^{n-1} - Du^n |Du^{n-1}|^2 + (Du^{n-1})^3) \right\|^2 \\
 & = \frac{1}{4} \left\| D((Du^n + Du^{n-1})(Du^n - Du^{n-1})^2) \right\|^2 \\
 & = \frac{1}{4} \left\| (D^2u^n + D^2u^{n-1})(Du^n - Du^{n-1})^2 \right\|^2
 \end{aligned}$$

$$\begin{aligned}
 &+ 2(Du^n + Du^{n-1})(Du^n - Du^{n-1})(D^2u^n - D^2u^{n-1})\|^2 \\
 \leq &\frac{1}{4}(\|D^2u^n\|^2 + \|D^2u^{n-1}\|^2)\|(Du^n - Du^{n-1})^2\|^2 \\
 &+ \frac{1}{2}(|Du^n|_\infty^2 + |Du^{n-1}|_\infty^2)\|Du^n - Du^{n-1}\|^2\|(D^2u^n - D^2u^{n-1})\|^2.
 \end{aligned}$$

Based on Sobolev’s embedding theorem, we have

$$\begin{aligned}
 \|I_1\| &\leq C(\|(Du^n - Du^{n-1})^2\|^2 + \|Du^n - Du^{n-1}\|^2\|(D^2u^n - D^2u^{n-1})\|^2) \\
 &\leq C(\Delta t)^2\left(\left(\int_{t_{n-1}}^{t_n} \|Du_t(t)\|^2 dt\right)^2 + \int_{t_{n-1}}^{t_n} \|Du_t(t)\|^2 dt \int_{t_{n-1}}^{t_n} \|D^2u_t(t)\|^2 dt\right) \\
 &\leq C(\Delta t)^2\left(\left(\int_{t_{n-1}}^{t_n} \|Du_t(t)\|^2 dt\right)^2 + \left(\int_{t_{n-1}}^{t_n} \|D^2u_t(t)\|^2 dt\right)^2\right).
 \end{aligned}$$

Further, Hölder’s inequality yields

$$\|I_1\| \leq C(\Delta t)^3\left(\int_{t_{n-1}}^{t_n} \|Du_t(t)\|^4 dt + \int_{t_{n-1}}^{t_n} \|D^2u_t(t)\|^4 dt\right). \tag{75}$$

Second, we analyze $\|I_2\|$. A direct calculation gives

$$\begin{aligned}
 I_2 &= D((Du^n + Du^{n-1})(|Du^n|^2 + |Du^{n-1}|^2) - (Du_h^n + Du_h^{n-1})(|Du^n|^2 + |Du^{n-1}|^2)) \\
 &\quad + (Du_h^n + Du_h^{n-1})(|Du^n|^2 + |Du^{n-1}|^2) - (Du_h^n + Du_h^{n-1})(|Du_h^n|^2 + |Du_h^{n-1}|^2)) \\
 &= D((Du^n + Du^{n-1}) - (Du_h^n + Du_h^{n-1}))(|Du^n|^2 + |Du^{n-1}|^2) \\
 &\quad + ((Du^n + Du^{n-1}) - (Du_h^n + Du_h^{n-1}))D(|Du^n|^2 + |Du^{n-1}|^2) \\
 &\quad + D(Du_h^n + Du_h^{n-1})((|Du^n|^2 + |Du^{n-1}|^2) - (|Du_h^n|^2 + |Du_h^{n-1}|^2)) \\
 &\quad + (Du_h^n + Du_h^{n-1})D((|Du^n|^2 + |Du^{n-1}|^2) - (|Du_h^n|^2 + |Du_h^{n-1}|^2)).
 \end{aligned}$$

With the help of Sobolev’s embedding theorem, we can obtain

$$\begin{aligned}
 \|I_2\| &\leq (|Du^n|_\infty^2 + |Du^{n-1}|_\infty^2)\|(D^2u^n + D^2u^{n-1}) - (D^2u_h^n + D^2u_h^{n-1})\| \\
 &\quad + 2(|Du^n|_\infty\|D^2u^n\| + |Du^{n-1}|_\infty\|D^2u^{n-1}\|) \\
 &\quad \times \|(Du^n + Du^{n-1}) - (Du_h^n + Du_h^{n-1})\| \\
 &\quad + (\|D^2u_h^n\| + \|D^2u_h^{n-1}\|) \\
 &\quad \times \|(Du^n + Du_h^n)(Du^n - Du_h^n) + (Du^{n-1} + Du_h^{n-1})(Du^{n-1} - Du_h^{n-1})\| \\
 &\quad + (|Du_h^n|_\infty + |Du_h^{n-1}|_\infty) \\
 &\quad \times \|D((Du^n + Du_h^n)(Du^n - Du_h^n) + (Du^{n-1} + Du_h^{n-1})(Du^{n-1} - Du_h^{n-1}))\| \\
 &\leq C(\|D\theta^n\| + \|D\theta^{n-1}\| + \|D\rho^n\| + \|D\rho^{n-1}\| \\
 &\quad + \|D^2\theta^n\| + \|D^2\theta^{n-1}\| + \|D^2\rho^n\| + \|D^2\rho^{n-1}\|).
 \end{aligned}$$

Then

$$\|I_2\| \leq C(\|\theta^n\| + \|\theta^{n-1}\| + \|D^2\theta^n\| + \|D^2\theta^{n-1}\| + h^2). \tag{76}$$

For $\|I_3\|$, by the triangle inequality and (9), one can have

$$\begin{aligned} \|I_3\| &= \|D^2\theta^n + D^2\rho^n + D^2\theta^{n-1} + D^2\rho^{n-1}\| \\ &\leq \|D^2\theta^n\| + \|D^2\theta^{n-1}\| + \|D^2\rho^n\| + \|D^2\rho^{n-1}\| \\ &\leq \|D^2\theta^n\| + \|D^2\theta^{n-1}\| + Ch^2. \end{aligned} \tag{77}$$

By (75)–(77), we get

$$\begin{aligned} &\|DF(Du^n, Du^{n-1}, Du_h^n, Du_h^{n-1})\|^2 \\ &\leq C(\|\theta^n\|^2 + \|\theta^{n-1}\|^2 + \|D^2\theta^n\|^2 + \|D^2\theta^{n-1}\|^2 + h^4) \\ &\quad + C(\Delta t)^3 \left(\int_{t_{n-1}}^{t_n} \|Du_t(t)\|^4 dt + \int_{t_{n-1}}^{t_n} \|D^2u_t(t)\|^4 dt \right). \end{aligned} \tag{78}$$

Substituting (66) and (78) into (74), we obtain

$$\begin{aligned} &s\|D^2\theta^n\|^2 - S\|D^2\theta^1\|^2 \\ &\leq C\Delta t \sum_{j=1}^n (\|D^2\theta^j\|^2 + \|\theta^j\|^2) \\ &\quad + C((\Delta t)^2 + h^4) \int_0^{t_n} (\|u_t\|_4^2 + \|u_{ttt}\|^2 + \|Du_t(t)\|^4 + \|D^2u_t(t)\|^4) dt. \end{aligned} \tag{79}$$

Letting $n = 1$ in (71), based on (66) and (78), we have

$$\|D^2\theta^1\| \leq C\|D^2\theta^0\| + O(\Delta t). \tag{80}$$

By (79) and (80), we get

$$\|D^2\theta^n\|^2 \leq C(\|D^2\theta^0\|^2 + (\Delta t)^2 + h^4 + \Delta t \sum_{j=1}^{n-1} (\|D^2\theta^j\|^2 + \|\theta^j\|^2)).$$

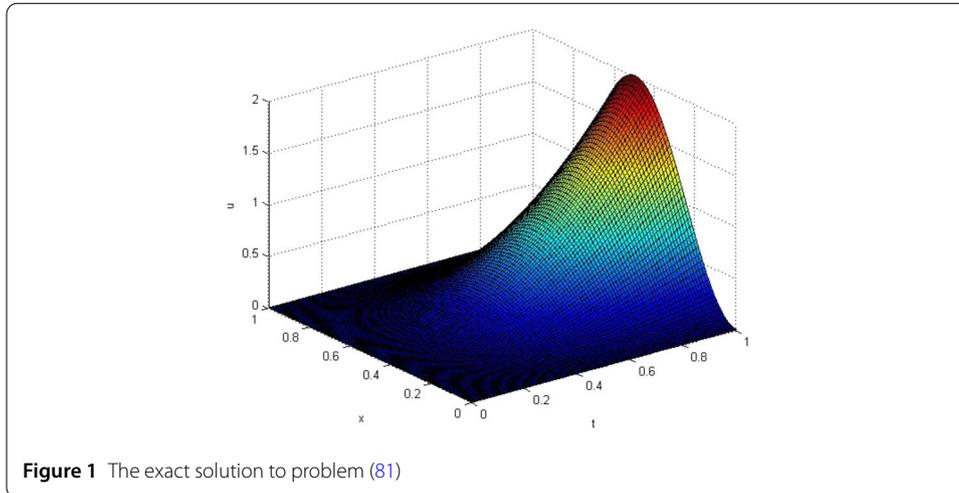
Using (51), we have

$$\|D^2\theta^n\|^2 \leq C \left(\|D^2\theta^0\|^2 + (\Delta t)^2 + h^4 + \Delta t \sum_{j=1}^{n-1} \|D^2\theta^j\|^2 \right).$$

If Δt is sufficiently small, discrete Gronwall’s inequality yields

$$\|D^2\theta^n\| \leq C(\Delta t + h^2).$$

This completes the proof. □



5 Numerical approximation

In this section, a numerical example is provided to illustrate the proposed B-spline FEM for solving the nonlinear parabolic equation. The efficiency of the cubic B-spline finite element scheme is tested. We consider

$$\begin{cases} u_t + (\alpha(x, t)u_{xx})_{xx} - (|u_x|^2 u_x - u_x)_x = f(x, t), & (x, t) \in (0, 1) \times (0, 1], \\ u(x, t) = u_x(x, t) = 0, & x = 0, 1, t \in (0, 1], \\ u(x, 0) = u_0(x), & x \in [0, 1], \end{cases} \quad (81)$$

where $\alpha(x, t) = 1 + xt$ is selected to satisfy the primary assumptions. We take the analytical solution $u(x, t) = t^2(1 - \cos 2\pi x)$. Then the concrete functional form of $f(x, t)$ is

$$\begin{aligned} f(x, t) = & 2t + (4\pi^2 t^2 - 2(1 + 8\pi^4 x)t - 16\pi^4) \cos 2\pi x \\ & - 16\pi^3 \sin 2\pi x - 48\pi^4 \sin^2 \pi x \cos 2\pi x. \end{aligned}$$

Figure 1 illustrates the behavior of the exact solution to problem (81), and Fig. 2 demonstrates the profile of the solution to the fully discrete scheme.

In this example, the numerical solution is in good accordance with the exact solution, indicating that the numerical scheme is valid and efficient.

Then choosing $t = 1$, the corresponding errors and convergence rates of the cubic B-spline FEM are shown in Tables 1–3.

In Table 1, to analyze the spatial convergence order, we take the time step $\Delta t = \frac{1}{8000}$. The values in Table 1 indicate that with the decreasing of the space size, the error is monotone decreasing. We also find that the numerical solution to the scheme is fourth order convergent in L^2 norm and is second order convergent in H^2 norm.

In Table 2, we consider the error estimates and convergence orders in time direction when the space step is fixed to $h = \frac{1}{1000}$. It is easy to see that the orders of error estimate both are second order in L^2 and H^2 norms.

In Table 3, we analyze the convergence rate when the space and time step change at the same time. We choose $(\Delta t, h) = (\frac{1}{100}, \frac{1}{10}), (\frac{1}{400}, \frac{1}{20}), (\frac{1}{1600}, \frac{1}{40}), (\frac{1}{6400}, \frac{1}{80})$, respectively. One can see that the relation between the space step and the time step is $\Delta t/h^2 = 1$. This shows that the B-spline finite element scheme is very stable.

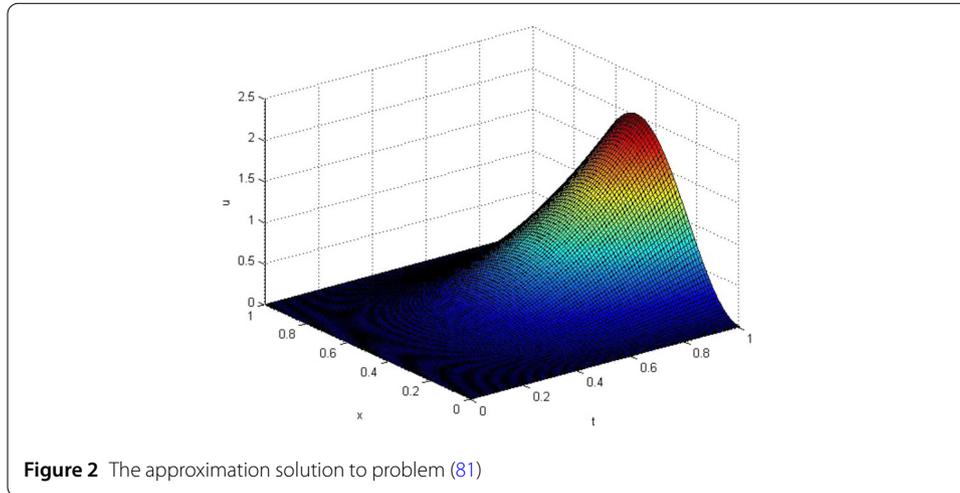


Table 1 The errors for different space step h at $t = 1$ and convergence orders

$(\Delta t, h)$	$\ u - u_h\ $	rate	$\ u - u_h\ _1$	rate	$\ u - u_h\ _2$	rate
(1/8000, 1/10)	$2.0642e^{-4}$		$7.0170e^{-3}$		$4.3022e^{-1}$	
(1/8000, 1/20)	$1.4750e^{-5}$	4.1349	$8.1325e^{-4}$	3.1091	$1.0389e^{-1}$	2.0500
(1/8000, 1/40)	$7.1398e^{-7}$	4.0406	$9.9680e^{-5}$	3.0283	$2.5745e^{-2}$	2.0127
(1/8000, 1/80)	$4.1614e^{-8}$	4.1614	$1.2399e^{-5}$	3.0071	$6.4221e^{-3}$	2.0032

Table 2 The errors for different time step Δt at $t = 1$ and convergence orders

$(\Delta t, h)$	$\ u - u_h\ $	rate	$\ u - u_h\ _1$	rate	$\ u - u_h\ _2$	rate
(1/20, 1/1000)	$1.3225e^{-3}$		$4.7367e^{-3}$		$2.9876e^{-2}$	
(1/40, 1/1000)	$3.4476e^{-4}$	1.9396	$1.2381e^{-3}$	1.9357	$7.7977e^{-3}$	1.9379
(1/80, 1/1000)	$8.6004e^{-5}$	2.0031	$3.1124e^{-4}$	1.9920	$1.9581e^{-3}$	1.9936
(1/160, 1/1000)	$2.1555e^{-5}$	1.9964	$7.8398e^{-5}$	1.9892	$4.9482e^{-4}$	1.9845

Table 3 The errors for different time step Δt and space step h at $t = 1$ and convergence orders

$(\Delta t, h)$	$\ u - u_h\ $	rate	$\ u - u_h\ _1$	rate	$\ u - u_h\ _2$	rate
(1/100, 1/10)	$1.8855e^{-4}$		$7.0332e^{-3}$		$4.3020e^{-1}$	
(1/400, 1/20)	$1.0720e^{-5}$	4.1366	$8.1374e^{-4}$	3.1116	$1.0389e^{-1}$	2.0500
(1/1600, 1/40)	$6.5496e^{-7}$	4.0327	$9.9694e^{-5}$	3.0290	$2.5745e^{-2}$	2.0127
(1/6400, 1/80)	$4.0685e^{-8}$	4.0088	$1.2399e^{-5}$	3.0073	$6.4221e^{-3}$	2.0032

The numerical experiment indicates that the cubic B-spline FEM is an efficient approximation tool for solving the fourth order nonlinear parabolic equation.

6 Conclusion

In this paper, we propose the B-spline FEM for a class of fourth order nonlinear parabolic equations. On the one hand, B-splines have better smoothness than the Lagrange and Hermite type elements. On the other hand, B-spline finite element only has one type of basis functions, so the scale of matrix from B-spline FEM is lower.

The coefficient $\alpha(x, t)$ is variable, which broadens the application fields and also increases the difficulty of analysis. By defining the biharmonic projection operator and the energy function, we prove the boundedness of the semi-discrete and fully discrete schemes based on B-splines. Further, the error estimates in L^2 norm and H^2 norm are deduced by

using the boundedness, Sobolev's embedding theorem, and so on. The results of numerical example confirm our theoretical analysis.

In general, the B-spline FEM is an efficient method for solving higher order nonlinear parabolic equations. By using central difference, the convergence rate in time direction, which can be improved, is of second order.

Acknowledgements

The authors would like to express their deep thanks for the referee's valuable suggestions about the revision and improvement of the manuscript.

Funding

This paper is supported by the Science and Technology Fund Project of Guizhou Health Commission (gzwjkj2019-1-050).

Availability of data and materials

Please contact author for data requests.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

DQ wrote the first draft. DQ and WH made the figure of numerical solution and errors, BL and JT corrected and improved the final version. All authors read and approved the final draft.

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Received: 2 January 2020 Accepted: 8 April 2020 Published online: 22 April 2020

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