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Existence of positive solution for BVP of nonlinear fractional differential equation with integral boundary conditions

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Abstract

This paper is concerned with the following boundary value problem of nonlinear fractional differential equation with integral boundary conditions:

$$\begin{cases} ({}^C D_{0+}^q u)(t) + f(t,u(t)) = 0, \quad t \in [0,1], \\ u''(0) = 0, \\ \alpha u(0) - \beta u'(0) = \int_0^1 h_1(s)u(s)\,ds, \\ \gamma u(1) + \delta({}^C D_{0+}^\sigma u)(1) = \int_0^1 h_2(s)u(s)\,ds, \end{cases}$$

where $2 < q \le 3$, $0 < \sigma \le 1$, $\alpha, \gamma, \delta \ge 0$, and $\beta > 0$ satisfying $0 < \rho := (\alpha + \beta)\gamma + \frac{\alpha\delta}{\Gamma(2-\sigma)} < \beta[\gamma + \frac{\delta\Gamma(q)}{\Gamma(q-\sigma)}]$. ${}^C D_{0+}^q$ denotes the standard Caputo fractional derivative. First, Green's function of the corresponding linear boundary value problem is constructed. Next, some useful properties of the Green's function are obtained. Finally, existence results of at least one positive solution for the above problem are established by imposing some suitable conditions on f and h_i $(i = 1,2)$. The method employed is Guo–Krasnoselskii's fixed point theorem. An example is also included to illustrate the main results of this paper.

MSC: 34A08; 34B18

Keywords: Fractional differential equation; Integral boundary condition; Boundary value problem; Positive solution; Existence

1 Introduction

Fractional calculus has widespread applications in many fields of science and engineering, for example, physics, viscoelasticity, continuum mechanics, bioengineering, rheology, electrical networks, control theory of dynamical systems, optics and signal processing, and so on [1, 2].

Since the discussion of many problems can be summed up in the study of boundary value problems (BVPs for short) to nonlinear fractional differential equations, recently, the existence of solutions or positive solutions of BVPs for nonlinear fractional differential equations has received considerable attention from many authors, see [3–26] and the references therein.

In particular, in 2009, by using nonlinear alternative of Leray–Schauder type and Guo–Krasnoselskii's fixed point theorem, Bai and Qiu [5] obtained the existence of a positive solution to the singular BVP

$$\begin{cases} ({}^C D_{0+}^q u)(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(1) = u''(0) = 0, \end{cases} \tag{1}$$

where $2 < q \leq 3$ is a real number, $f : (0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, and $\lim_{t\to 0^+} f(t,\cdot) = +\infty$.

In 2012, Cabada and Wang [7] studied the existence of a positive solution for the BVP with integral boundary conditions

$$\begin{cases} ({}^C D_{0+}^q u)(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u''(0) = 0, \qquad u(1) = \lambda \int_0^1 u(s)\,ds, \end{cases} \tag{2}$$

where $2 < q < 3$, $0 < \lambda < 2$, and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous. Their analysis relied on Guo–Krasnoselskii's fixed point theorem.

In 2014, Cabada and Hamdi [25] investigated the BVP with integral boundary conditions

$$\begin{cases} (D_{0+}^q u)(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = 0, \qquad u(1) = \lambda \int_0^1 u(s)\,ds, \end{cases} \tag{3}$$

where $2 < q \leq 3$, D_{0+}^q denotes the Riemann–Liouville fractional derivative, $0 < \lambda < q$, and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is a continuous function. The authors proved the existence of a positive solution to BVP (3) by employing Guo–Krasnoselskii's fixed point theorem.

As it has been stated in [7], BVPs with integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, and so forth. Motivated by the above-mentioned works, in this paper, we consider the existence of a positive solution for the following BVP of nonlinear fractional differential equation with integral boundary conditions:

$$\begin{cases} ({}^C D_{0+}^q u)(t) + f(t, u(t)) = 0, \quad t \in [0,1], \\ u''(0) = 0, \\ \alpha u(0) - \beta u'(0) = \int_0^1 h_1(s) u(s)\,ds, \\ \gamma u(1) + \delta({}^C D_{0+}^\sigma u)(1) = \int_0^1 h_2(s) u(s)\,ds. \end{cases} \tag{4}$$

Throughout this paper, we always assume that $2 < q \leq 3$, $0 < \sigma \leq 1$, $\alpha, \gamma, \delta \geq 0$, and $\beta > 0$ satisfying $0 < \rho := (\alpha + \beta)\gamma + \frac{\alpha\delta}{\Gamma(2-\sigma)} < \beta[\gamma + \frac{\delta\Gamma(q)}{\Gamma(q-\sigma)}]$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ and h_i ($i = 1,2$): $[0,1] \to [0,+\infty)$ are continuous.

The main tool used is the following well-known Guo–Krasnoselskii's fixed point theorem [27, 28].

Theorem 1.1 *Let E be a Banach space and K be a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E such that $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ be a completely continuous operator such that either*

(1) $\|Tu\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_2$, *or*

(2) $\|Tu\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2 Preliminaries

Let $[a,b]$ $(-\infty < a < b < +\infty)$ be a finite interval on the real axis $\mathbb{R}$, $\mathbb{N} = \{1,2,3,\ldots\}$, $\mu > 0$ and $[\mu]$ be the integer part of μ.

First, we present definitions of some spaces.

Let $\mathrm{AC}[a,b]$ be the space of functions u which are absolutely continuous on $[a,b]$. For $n \in \mathbb{N}$, we denote by $\mathrm{AC}^n[a,b]$ the space of functions u which have continuous derivatives up to order $n-1$ on $[a,b]$ such that $u^{(n-1)} \in \mathrm{AC}[a,b]$. In particular, $\mathrm{AC}^1[a,b] = \mathrm{AC}[a,b]$.

For $m \in \mathbb{N}_0 = \{0,1,2,\ldots\}$, we denote by $C^m[a,b]$ the space of functions u which are m times continuously differentiable on $[a,b]$. In particular, for $m = 0$, $C^0[a,b] = C[a,b]$ is the space of continuous functions u on $[a,b]$.

Next, we give the definitions of the Riemann–Liouville fractional integrals and fractional derivatives and the Caputo fractional derivatives on $[a,b]$, which may be found in [1].

Definition 2.1 The Riemann–Liouville fractional integrals $I^{\mu}_{a+}u$ and $I^{\mu}_{b-}u$ of order μ are defined by

$$\big(I^{\mu}_{a+}u\big)(t) := \frac{1}{\Gamma(\mu)} \int_a^t \frac{u(s)\,ds}{(t-s)^{1-\mu}} \quad (t > a)$$

and

$$\big(I^{\mu}_{b-}u\big)(t) := \frac{1}{\Gamma(\mu)} \int_t^b \frac{u(s)\,ds}{(s-t)^{1-\mu}} \quad (t < b),$$

respectively, where

$$\Gamma(\mu) = \int_0^{+\infty} s^{\mu-1} e^{-s}\,ds.$$

Definition 2.2 The Riemann–Liouville fractional derivatives $D^{\mu}_{a+}u$ and $D^{\mu}_{b-}u$ of order μ are defined by

$$\big(D^{\mu}_{a+}u\big)(t) := \left(\frac{d}{dt}\right)^n \big(I^{n-\mu}_{a+}u\big)(t) = \frac{1}{\Gamma(n-\mu)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{u(s)\,ds}{(t-s)^{\mu-n+1}} \quad (t > a)$$

and

$$\big(D^{\mu}_{b-}u\big)(t) := \left(-\frac{d}{dt}\right)^n \big(I^{n-\mu}_{b-}u\big)(t) = \frac{1}{\Gamma(n-\mu)} \left(-\frac{d}{dt}\right)^n \int_t^b \frac{u(s)\,ds}{(s-t)^{\mu-n+1}} \quad (t < b),$$

respectively, where $n = [\mu] + 1$.

Definition 2.3 Let $D^{\mu}_{a+}[u(s)](t) \equiv (D^{\mu}_{a+}u)(t)$ and $D^{\mu}_{b-}[u(s)](t) \equiv (D^{\mu}_{b-}u)(t)$ be the Riemann–Liouville fractional derivatives of order μ, respectively. The Caputo fractional derivatives ${}^{C}D^{\mu}_{a+}u$ and ${}^{C}D^{\mu}_{b-}u$ of order μ on $[a,b]$ are defined by

$$\bigl({}^{C}D^{\mu}_{a+}u\bigr)(t) := \left(D^{\mu}_{a+}\left[u(s) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(s-a)^k\right]\right)(t)$$

and

$$\bigl({}^{C}D^{\mu}_{b-}u\bigr)(t) := \left(D^{\mu}_{b-}\left[u(s) - \sum_{k=0}^{n-1}\frac{u^{(k)}(b)}{k!}(b-s)^k\right]\right)(t),$$

respectively, where

$$n = \begin{cases} [\mu] + 1, & \mu \notin \mathbb{N}, \\ \mu, & \mu \in \mathbb{N}. \end{cases} \tag{5}$$

Lemma 2.1 (see [2]) *Let $\nu > \mu$. Then the equation $({}^{C}D^{\mu}_{0+}I^{\nu}_{0+}u)(t) = (I^{\nu-\mu}_{0+}u)(t)$, $t \in [0,1]$ is satisfied for $u \in C[0,1]$.*

Lemma 2.2 (see [1]) *Let n be given by (5). Then the following relations hold*:
(1) *for $k \in \{0,1,2,\dots,n-1\}$, ${}^{C}D^{\mu}_{0+}t^k = 0$;*
(2) *if $\nu > n$, ${}^{C}D^{\mu}_{0+}t^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu-\mu)}t^{\nu-\mu-1}$.*

Lemma 2.3 (see [1]) *Let n be given by (5). If $u \in \mathrm{AC}^n[0,1]$ or $u \in C^n[0,1]$, then*

$$\bigl(I^{\mu}_{0+}\,{}^{C}D^{\mu}_{0+}u\bigr)(t) = u(t) - \sum_{k=0}^{n-1}\frac{u^k(0)}{k!}t^k.$$

For convenience, we denote

$$P_i = \frac{1}{\rho}\int_0^1 (\alpha s + \beta)h_i(s)\,ds$$

and

$$Q_i = \frac{1}{\rho\Gamma(2-\sigma)}\int_0^1 \bigl[\gamma\Gamma(2-\sigma)(1-s) + \delta\bigr]h_i(s)\,ds, \quad i = 1,2.$$

Lemma 2.4 *Let $(1-Q_1)(1-P_2) \neq P_1Q_2$. Then, for any $y \in C[0,1]$, the BVP*

$$\begin{cases} ({}^{C}D^{q}_{0+}u)(t) + y(t) = 0, & t \in [0,1], \\ u''(0) = 0, \\ \alpha u(0) - \beta u'(0) = \int_0^1 h_1(s)u(s)\,ds, \\ \gamma u(1) + \delta({}^{C}D^{\sigma}_{0+}u)(1) = \int_0^1 h_2(s)u(s)\,ds \end{cases} \tag{6}$$

has a unique solution

$$u(t)=\int_0^1 H(t,s)y(s)\,ds,\quad t\in[0,1],$$

here

$$H(t,s)=G(t,s)+\sum_{i=1}^{2}\phi_i(t)\int_0^1 G(\tau,s)h_i(\tau)\,d\tau,\quad (t,s)\in[0,1]\times[0,1],$$

where

$$G(t,s)=\frac{\alpha t+\beta}{\rho}\left[\frac{\gamma(1-s)^{q-1}}{\Gamma(q)}+\frac{\delta(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}\right]-\begin{cases}\frac{(t-s)^{q-1}}{\Gamma(q)}, & 0\le s\le t\le 1,\\ 0, & 0\le t\le s\le 1,\end{cases}$$

$$\phi_1(t)=\frac{\Gamma(2-\sigma)Q_2(\alpha t+\beta)+(1-P_2)[\gamma\Gamma(2-\sigma)(1-t)+\delta]}{\rho\Gamma(2-\sigma)[(1-Q_1)(1-P_2)-P_1Q_2]},\quad t\in[0,1],$$

and

$$\phi_2(t)=\frac{\Gamma(2-\sigma)(1-Q_1)(\alpha t+\beta)+P_1[\gamma\Gamma(2-\sigma)(1-t)+\delta]}{\rho\Gamma(2-\sigma)[(1-Q_1)(1-P_2)-P_1Q_2]},\quad t\in[0,1].$$

Proof In view of the equation in (6), Lemma 2.3, and $u''(0)=0$, we have

$$u(t)=-\bigl(I_{0+}^{q}y\bigr)(t)+u(0)+u'(0)t,\quad t\in[0,1].\tag{7}$$

By (7), Lemma 2.1, and Lemma 2.2, we obtain

$$\bigl({}^{C}D_{0+}^{\sigma}u\bigr)(t)=-\bigl(I_{0+}^{q-\sigma}y\bigr)(t)+\frac{u'(0)}{\Gamma(2-\sigma)}t^{1-\sigma},\quad t\in[0,1].\tag{8}$$

It follows from (7), (8), and the boundary conditions in (6) that

$$\begin{aligned}u(0)=\frac{1}{\rho}\biggl[&\beta\gamma\bigl(I_{0+}^{q}y\bigr)(1)+\beta\delta\bigl(I_{0+}^{q-\sigma}y\bigr)(1)+\frac{\gamma\Gamma(2-\sigma)+\delta}{\Gamma(2-\sigma)}\int_0^1 h_1(s)u(s)\,ds\\&+\beta\int_0^1 h_2(s)u(s)\,ds\biggr]\end{aligned}$$

and

$$u'(0)=\frac{1}{\rho}\left[\alpha\gamma\bigl(I_{0+}^{q}y\bigr)(1)+\alpha\delta\bigl(I_{0+}^{q-\sigma}y\bigr)(1)-\gamma\int_0^1 h_1(s)u(s)\,ds+\alpha\int_0^1 h_2(s)u(s)\,ds\right],$$

which together with (7) shows that

$$\begin{aligned}u(t)=&\int_0^t\left\{-\frac{(t-s)^{q-1}}{\Gamma(q)}+\frac{\alpha t+\beta}{\rho}\left[\frac{\gamma(1-s)^{q-1}}{\Gamma(q)}+\frac{\delta(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}\right]\right\}y(s)\,ds\\&+\int_t^1\left\{\frac{\alpha t+\beta}{\rho}\left[\frac{\gamma(1-s)^{q-1}}{\Gamma(q)}+\frac{\delta(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}\right]\right\}y(s)\,ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{\gamma\Gamma(2-\sigma)(1-t)+\delta}{\rho\Gamma(2-\sigma)}\int_0^1 h_1(s)u(s)\,ds+\frac{\alpha t+\beta}{\rho}\int_0^1 h_2(s)u(s)\,ds \\
&=\int_0^1 G(t,s)y(s)\,ds+\frac{\gamma\Gamma(2-\sigma)(1-t)+\delta}{\rho\Gamma(2-\sigma)}\int_0^1 h_1(s)u(s)\,ds \\
&+\frac{\alpha t+\beta}{\rho}\int_0^1 h_2(s)u(s)\,ds,\quad t\in[0,1].
\end{aligned}
\tag{9}
$$

From (9), we get

$$
(1-Q_1)\int_0^1 h_1(s)u(s)\,ds-P_1\int_0^1 h_2(s)u(s)\,ds=\int_0^1 h_1(s)\int_0^1 G(s,\tau)y(\tau)\,d\tau\,ds
$$

and

$$
-Q_2\int_0^1 h_1(s)u(s)\,ds+(1-P_2)\int_0^1 h_2(s)u(s)\,ds=\int_0^1 h_2(s)\int_0^1 G(s,\tau)y(\tau)\,d\tau\,ds,
$$

and so,

$$
\begin{aligned}
&\int_0^1 h_1(s)u(s)\,ds \\
&\quad=\frac{(1-P_2)\int_0^1 h_1(s)\int_0^1 G(s,\tau)y(\tau)\,d\tau\,ds+P_1\int_0^1 h_2(s)\int_0^1 G(s,\tau)y(\tau)\,d\tau\,ds}{(1-Q_1)(1-P_2)-P_1Q_2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 h_2(s)u(s)\,ds \\
&\quad=\frac{Q_2\int_0^1 h_1(s)\int_0^1 G(s,\tau)y(\tau)\,d\tau\,ds+(1-Q_1)\int_0^1 h_2(s)\int_0^1 G(s,\tau)y(\tau)\,d\tau\,ds}{(1-Q_1)(1-P_2)-P_1Q_2},
\end{aligned}
$$

which together with (9) implies that

$$
\begin{aligned}
u(t)&=\int_0^1 G(t,s)y(s)\,ds+\sum_{i=1}^{2}\phi_i(t)\int_0^1 h_i(s)\int_0^1 G(s,\tau)y(\tau)\,d\tau\,ds \\
&=\int_0^1 G(t,s)y(s)\,ds+\sum_{i=1}^{2}\phi_i(t)\int_0^1 h_i(\tau)\int_0^1 G(\tau,s)y(s)\,ds\,d\tau \\
&=\int_0^1 G(t,s)y(s)\,ds+\sum_{i=1}^{2}\phi_i(t)\int_0^1 y(s)\int_0^1 G(\tau,s)h_i(\tau)\,d\tau\,ds \\
&=\int_0^1\left[G(t,s)+\sum_{i=1}^{2}\phi_i(t)\int_0^1 G(\tau,s)h_i(\tau)\,d\tau\right]y(s)\,ds \\
&=\int_0^1 H(t,s)y(s)\,ds,\quad t\in[0,1].
\end{aligned}
$$

□

In what follows, we let

$$
g(s)=\frac{\alpha+\beta}{\rho}\left[\frac{\gamma(1-s)^{q-1}}{\Gamma(q)}+\frac{\delta(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}\right],\quad s\in[0,1]
$$

and

$$\eta(s)=\frac{\beta\delta\Gamma(q)-\Gamma(q-\sigma)(\rho-\beta\gamma)(1-s)^{\sigma}}{(\alpha+\beta)[\gamma\Gamma(q-\sigma)+\delta\Gamma(q)]},\quad s\in[0,1].$$

Lemma 2.5 *$G(t,s)$ satisfies the following properties*:

(1) $G(t,s)\le g(s)$, $(t,s)\in[0,1]\times[0,1]$;

(2) $G(t,s)\ge\eta(s)g(s)$, $(t,s)\in[0,1]\times[0,1]$.

Proof Since (1) is obvious, we only need to prove that (2) holds.

First, it is clear that $G(t,1)\ge\eta(1)g(1)$ for $t\in[0,1]$.

Now, we verify that $G(t,s)\ge\eta(s)g(s)$ for $(t,s)\in[0,1]\times[0,1)$. In fact, if $s\le t$, then

$$\begin{aligned}\frac{G(t,s)}{g(s)}&=\frac{(\alpha t+\beta)[\gamma\Gamma(q-\sigma)(1-s)^{q-1}+\delta\Gamma(q)(1-s)^{q-\sigma-1}]-\rho\Gamma(q-\sigma)(t-s)^{q-1}}{(\alpha+\beta)[\gamma\Gamma(q-\sigma)(1-s)^{q-1}+\delta\Gamma(q)(1-s)^{q-\sigma-1}]}\\&\ge\frac{\beta\gamma\Gamma(q-\sigma)(1-s)^{\sigma}+\beta\delta\Gamma(q)-\rho\Gamma(q-\sigma)(1-s)^{\sigma}}{(\alpha+\beta)[\gamma\Gamma(q-\sigma)(1-s)^{\sigma}+\delta\Gamma(q)]}\\&\ge\frac{\beta\delta\Gamma(q)-\Gamma(q-\sigma)(\rho-\beta\gamma)(1-s)^{\sigma}}{(\alpha+\beta)[\gamma\Gamma(q-\sigma)+\delta\Gamma(q)]}\\&=\eta(s)\end{aligned}$$

and if $t\le s$, then

$$\begin{aligned}\frac{G(t,s)}{g(s)}&=\frac{(\alpha t+\beta)[\gamma\Gamma(q-\sigma)(1-s)^{q-1}+\delta\Gamma(q)(1-s)^{q-\sigma-1}]}{(\alpha+\beta)[\gamma\Gamma(q-\sigma)(1-s)^{q-1}+\delta\Gamma(q)(1-s)^{q-\sigma-1}]}\\&\ge\frac{\beta\gamma\Gamma(q-\sigma)(1-s)^{\sigma}+\beta\delta\Gamma(q)}{(\alpha+\beta)[\gamma\Gamma(q-\sigma)(1-s)^{\sigma}+\delta\Gamma(q)]}\\&\ge\frac{\beta\gamma\Gamma(q-\sigma)(1-s)^{\sigma}+\beta\delta\Gamma(q)-\rho\Gamma(q-\sigma)(1-s)^{\sigma}}{(\alpha+\beta)[\gamma\Gamma(q-\sigma)(1-s)^{\sigma}+\delta\Gamma(q)]}\\&\ge\frac{\beta\delta\Gamma(q)-\Gamma(q-\sigma)(\rho-\beta\gamma)(1-s)^{\sigma}}{(\alpha+\beta)[\gamma\Gamma(q-\sigma)+\delta\Gamma(q)]}\\&=\eta(s).\end{aligned}$$

□

By the definition of η and the condition $0<\rho<\beta[\gamma+\frac{\delta\Gamma(q)}{\Gamma(q-\sigma)}]$, we may obtain the following remark.

Remark 2.1 η is increasing on $[0,1]$ and $0<\eta(s)<1$ for $s\in[0,1]$.

In the remainder of this paper, we always assume that the following conditions are satisfied:

$$Q_1<1,\qquad P_2<1\quad\text{and}\quad(1-Q_1)(1-P_2)>P_1Q_2.$$

Lemma 2.6 *$H(t,s)$ has the following property*:

$$m\eta(s)g(s)\le H(t,s)\le Mg(s),\quad(t,s)\in[0,1]\times[0,1],$$

where

$$m = 1 + \sum_{i=1}^{2} \min_{t\in[0,1]} \phi_i(t) \int_0^1 h_i(\tau)\,d\tau$$

and

$$M = 1 + \sum_{i=1}^{2} \max_{t\in[0,1]} \phi_i(t) \int_0^1 h_i(\tau)\,d\tau.$$

Proof On the one hand, in view of (1) of Lemma 2.5, we have

$$\begin{aligned} H(t,s) &= G(t,s) + \sum_{i=1}^{2} \phi_i(t) \int_0^1 G(\tau,s)h_i(\tau)\,d\tau \\ &\le \left(1 + \sum_{i=1}^{2} \phi_i(t) \int_0^1 h_i(\tau)\,d\tau\right) g(s) \\ &\le Mg(s), \quad (t,s) \in [0,1] \times [0,1]. \end{aligned}$$

On the other hand, by (2) of Lemma 2.5, we get

$$\begin{aligned} H(t,s) &= G(t,s) + \sum_{i=1}^{2} \phi_i(t) \int_0^1 G(\tau,s)h_i(\tau)\,d\tau \\ &\ge \left(1 + \sum_{i=1}^{2} \phi_i(t) \int_0^1 h_i(\tau)\,d\tau\right) \eta(s)g(s) \\ &\ge m\eta(s)g(s), \quad (t,s) \in [0,1] \times [0,1]. \end{aligned}$$

□

Let $E = C[0,1]$ be equipped with norm $\|u\| = \max_{t\in[0,1]} |u(t)|$ and

$$K = \bigl\{u \in E : u(t) \ge \theta\|u\|, t \in [0,1]\bigr\},$$

where $0 < \theta = \frac{m\eta(0)}{M} < 1$. Then it is easy to check that E is a Banach space and K is a cone in E.

Now, we define an operator T on K by

$$(Tu)(t) = \int_0^1 H(t,s)f\bigl(s,u(s)\bigr)\,ds, \quad u \in K, t \in [0,1].$$

Obviously, if u is a fixed point of T, then u is a nonnegative solution of BVP (4).

Lemma 2.7 *$T : K \to K$ is completely continuous.*

Proof Let $u \in K$. Then, in view of Lemma 2.6, we have

$$\|Tu\| \le M \int_0^1 g(s)f\bigl(s,u(s)\bigr)\,ds,$$

which together with Lemma 2.6 and Remark 2.1 implies that

$$
\begin{aligned}
(Tu)(t) &\geq m \int_0^1 \eta(s)g(s)f\bigl(s,u(s)\bigr)\,ds \\
&\geq m\eta(0) \int_0^1 g(s)f\bigl(s,u(s)\bigr)\,ds \\
&\geq \theta \|Tu\|, \quad t \in [0,1].
\end{aligned}
$$

This indicates that $Tu \in K$. Furthermore, it is easy to prove that T is completely continuous by an application of Arzela–Ascoli theorem [29]. □

3 Main results

Define

$$
\begin{aligned}
f^0 &= \limsup_{u\to 0^+} \max_{t\in[0,1]} \frac{f(t,u)}{u}, \qquad f^\infty = \limsup_{u\to +\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u}, \\
f_0 &= \liminf_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{u}, \qquad f_\infty = \liminf_{u\to +\infty} \min_{t\in[0,1]} \frac{f(t,u)}{u}.
\end{aligned}
$$

Theorem 3.1 *Suppose that one of the following conditions is satisfied*:

(i) $f_0 = +\infty$ *and* $f^\infty = 0$, *or*

(ii) $f^0 = 0$ *and* $f_\infty = +\infty$.

Then BVP (4) *has at least one positive solution*.

Proof First, we consider case (i): $f_0 = +\infty$ and $f^\infty = 0$.

In view of $f_0 = +\infty$, there exists $r_1 > 0$ such that

$$
f(t,u) \geq G_1 u, \quad (t,u) \in [0,1] \times [0,r_1], \tag{10}
$$

where $G_1 \geq \frac{1}{m\theta \int_0^1 \eta(s)g(s)\,ds}$.

Let $\Omega_1 = \{u \in E : \|u\| < r_1\}$. Then, for any $u \in K \cap \partial\Omega_1$, by Lemma 2.6 and (10), we get

$$
\begin{aligned}
(Tu)(t) &\geq m \int_0^1 \eta(s)g(s)f\bigl(s,u(s)\bigr)\,ds \\
&\geq mG_1 \int_0^1 \eta(s)g(s)u(s)\,ds \\
&\geq mG_1\theta \|u\| \int_0^1 \eta(s)g(s)\,ds \\
&\geq \|u\|, \quad t \in [0,1],
\end{aligned}
$$

which shows that

$$
\|Tu\| \geq \|u\| \quad \text{for } u \in K \cap \partial\Omega_1. \tag{11}
$$

On the other hand, since $f^\infty = 0$, there exists $U_1 > 0$ such that

$$
f(t,u) \leq \varepsilon_1 u, \quad (t,u) \in [0,1] \times (U_1,+\infty),
$$

where $\varepsilon_1 > 0$ satisfies $\varepsilon_1 \leq \frac{1}{2M\int_0^1 g(s)\,ds}$.

Let $M^* = \max_{(t,u)\in[0,1]\times[0,U_1]} f(t,u)$. Then we have

$$f(t,u) \le M^* + \varepsilon_1 u, \quad (t,u) \in [0,1] \times [0,+\infty). \tag{12}$$

If we choose $r_2 = \max\{2r_1, 2MM^*\int_0^1 g(s)\,ds\}$ and let $\Omega_2 = \{u \in E : \|u\| < r_2\}$, then for any $u \in K \cap \partial\Omega_2$, from Lemma 2.6 and (12), we obtain

$$\begin{aligned}(Tu)(t) &\le M\int_0^1 g(s)f\bigl(s,u(s)\bigr)\,ds \\ &\le MM^*\int_0^1 g(s)\,ds + M\varepsilon_1\|u\|\int_0^1 g(s)\,ds \\ &\le \frac{\|u\|}{2} + \frac{\|u\|}{2} \\ &= \|u\|, \quad t \in [0,1],\end{aligned}$$

which indicates that

$$\|Tu\| \le \|u\| \quad \text{for } u \in K \cap \partial\Omega_2. \tag{13}$$

Therefore, it follows from Theorem 1.1, Lemma 2.7, (11), and (13) that T has a fixed point $u \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which is a desired positive solution of BVP (4).

Next, we consider case (ii): $f^0 = 0$ and $f_\infty = +\infty$.

In view of $f^0 = 0$, there exists $r_3 > 0$ such that

$$f(t,u) \le \varepsilon_2 u, \quad (t,u) \in [0,1] \times [0,r_3], \tag{14}$$

where $\varepsilon_2 > 0$ satisfies $\varepsilon_2 \le \frac{1}{M\int_0^1 g(s)\,ds}$.

Let $\Omega_3 = \{u \in E : \|u\| < r_3\}$. Then, for any $u \in K \cap \partial\Omega_3$, by Lemma 2.6 and (14), we get

$$\begin{aligned}(Tu)(t) &\le M\int_0^1 g(s)f\bigl(s,u(s)\bigr)\,ds \\ &\le M\varepsilon_2\int_0^1 g(s)u(s)\,ds \\ &\le M\varepsilon_2\|u\|\int_0^1 g(s)\,ds \\ &\le \|u\|, \quad t \in [0,1],\end{aligned}$$

which shows that

$$\|Tu\| \le \|u\| \quad \text{for } u \in K \cap \partial\Omega_3. \tag{15}$$

On the other hand, since $f_\infty = +\infty$, there exists $U_2 > 0$ such that

$$f(t,u) \ge G_2 u, \quad (t,u) \in [0,1] \times [U_2,+\infty), \tag{16}$$

where $G_2 \ge \frac{1}{m\theta\int_0^1 \eta(s)g(s)\,ds}$.

If we choose $r_4 = \max\{\frac{U_2}{\theta}, 2r_3\}$ and let $\Omega_4 = \{u \in E : \|u\| < r_4\}$, then for any $u \in K \cap \partial\Omega_4$, we know

$$u(t) \geq \theta\|u\| = \theta r_4 \geq U_2, \quad t \in [0,1],$$

which together with Lemma 2.6 and (16) implies that

$$\begin{aligned}(Tu)(t) &\geq m\int_0^1 \eta(s)g(s)f\bigl(s,u(s)\bigr)\,ds \\ &\geq mG_2\int_0^1 \eta(s)g(s)u(s)\,ds \\ &\geq mG_2\theta\|u\|\int_0^1 \eta(s)g(s)\,ds \\ &\geq \|u\|, \quad t \in [0,1].\end{aligned}$$

This indicates that

$$\|Tu\| \geq \|u\| \quad \text{for } u \in K \cap \partial\Omega_4. \tag{17}$$

Therefore, it follows from Theorem 1.1, Lemma 2.7, (15), and (17) that T has a fixed point $u \in K \cap (\overline{\Omega}_4 \setminus \Omega_3)$, which is a desired positive solution of BVP (4). □

Example 3.1 Consider the following BVP:

$$\begin{cases}({}^C D_{0+}^{\frac{5}{2}}u)(t) + [\sin(\frac{\pi t}{2}) + 1]u^2(t) = 0, \quad t \in [0,1], \\ u''(0) = 0, \\ u(0) - 4u'(0) = \int_0^1 su(s)\,ds, \\ u(1) + ({}^C D_{0+}^{\frac{1}{2}}u)(1) = \int_0^1 (1-s)u(s)\,ds.\end{cases} \tag{18}$$

In view of $q = \frac{5}{2}$, $\sigma = \frac{1}{2}$, $\alpha = \gamma = \delta = 1$, $\beta = 4$, $h_1(s) = s$, and $h_2(s) = 1 - s$, $s \in [0,1]$, a simple calculation shows that

$$0 < \rho = 5 + \frac{2}{\pi}\sqrt{\pi} < \beta\left[\gamma + \frac{\delta\Gamma(q)}{\Gamma(q-\sigma)}\right] = 4 + 3\sqrt{\pi}$$

and

$$P_1 = \frac{7\sqrt{\pi}}{3(5\sqrt{\pi}+2)}, \qquad P_2 = \frac{13\sqrt{\pi}}{6(5\sqrt{\pi}+2)},$$
$$Q_1 = \frac{\sqrt{\pi}+6}{6(5\sqrt{\pi}+2)}, \qquad Q_2 = \frac{\sqrt{\pi}+3}{3(5\sqrt{\pi}+2)}.$$

Obviously, $Q_1 < 1$, $P_2 < 1$ and

$$(1-Q_1)(1-P_2) = \frac{(29\sqrt{\pi}+6)(17\sqrt{\pi}+12)}{36(5\sqrt{\pi}+2)^2} > P_1Q_2 = \frac{7\sqrt{\pi}(\sqrt{\pi}+3)}{9(5\sqrt{\pi}+2)^2}.$$

Moreover, since $f(t,u) = [\sin(\frac{\pi t}{2}) + 1]u^2$, $(t,u) \in [0,1] \times [0,+\infty)$, it is easy to know that $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and

$$f^0 = 0, \qquad f_\infty = +\infty.$$

Therefore, it follows from Theorem 3.1 that BVP (18) has at least one positive solution.

4 Conclusion

In this paper, by applying Guo–Krasnoselskii's fixed point theorem, we obtain the existence of at least one positive solution for a class of nonlinear boundary value problems involving fractional differential equation and integral boundary conditions. An illustrative example is also given to show the effectiveness of theoretical results.

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