

RESEARCH

Open Access



Positive solutions of a second-order nonlinear Robin problem involving the first-order derivative

Zhilin Yang^{1*}

*Correspondence:
zhilinyang@gmail.com;
zhilinyang@sina.com;
zhilinyang@qtech.edu.cn

¹Department of Mathematics,
Qingdao University of Technology,
No 11 Fushun Road, Qingdao,
Shandong Province, P.R. China

Abstract

This paper is concerned with the second-order nonlinear Robin problem involving the first-order derivative:

$$\begin{cases} u'' + f(t, u, u') = 0, \\ u(0) = u'(1) - \alpha u(1) = 0, \end{cases}$$

where $f \in C([0, 1] \times \mathbb{R}_+^2, \mathbb{R}_+)$ and $\alpha \in]0, 1[$. Based on a priori estimates, we use fixed point index theory to establish some results on existence, multiplicity and uniqueness of positive solutions thereof, with the unique positive solution being the limit of an iterative sequence. The results presented here generalize and extend the corresponding ones for nonlinearities independent of the first-order derivative.

MSC: 34B16; 34B18; 47H07; 47H11

Keywords: Robin problem; Positive solution; A priori estimate; Fixed point index; Iterative sequence; Super- and sub-solution

1 Introduction

In this paper, we study the existence, multiplicity and uniqueness of positive solutions for the second-order nonlinear Robin problem involving the first-order derivative:

$$\begin{cases} -u'' = f(t, u, u'), \\ u(0) = 0, \quad u'(1) = \alpha u(1), \end{cases} \quad (1)$$

where $f \in C([0, 1] \times \mathbb{R}_+^2, \mathbb{R}_+)$ ($\mathbb{R}_+ := [0, \infty[$) and $\alpha \in]0, 1[$.

In 1994, Erbe et al. [10] studied the existence of positive solutions for the second-order boundary value problem below:

$$\begin{cases} u'' + f(t, u) = 0, \\ \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{cases} \quad (2)$$

© The Author(s) 2021. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

where $f \in C([0, 1] \times \mathbb{R}_+, \mathbb{R}_+)$, $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, $\delta \geq 0$, $\gamma\beta + \alpha\gamma + \alpha\delta > 0$. The main conditions used in [10] are as follows:

$$(A1) \lim_{u \rightarrow 0^+} \min_{t \in [0, 1]} \frac{f(t, u)}{u} = +\infty, \lim_{u \rightarrow +\infty} \min_{t \in [0, 1]} \frac{f(t, u)}{u} = +\infty;$$

$$(A2) \lim_{u \rightarrow 0^+} \max_{t \in [0, 1]} \frac{f(t, u)}{u} = 0, \lim_{u \rightarrow +\infty} \max_{t \in [0, 1]} \frac{f(t, u)}{u} = 0.$$

Let $\lambda_1 > 0$ be the first characteristic value of the following characteristic value problem:

$$\begin{cases} -u'' = \lambda u, \\ \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0. \end{cases}$$

In 1996 and 1998, Liu et al. [20, 23] generalized and extended the above work, replacing (A1) and (A2) with the following sharp conditions:

$$(B1) \lim_{u \rightarrow 0^+} \min_{t \in [0, 1]} \frac{f(t, u)}{u} > \lambda_1, \lim_{u \rightarrow +\infty} \min_{t \in [0, 1]} \frac{f(t, u)}{u} > \lambda_1,$$

$$(B2) \lim_{u \rightarrow 0^+} \max_{t \in [0, 1]} \frac{f(t, u)}{u} < \lambda_1, \lim_{u \rightarrow +\infty} \max_{t \in [0, 1]} \frac{f(t, u)}{u} < \lambda_1.$$

Now the main results in [20, 23] have been generalized to many other boundary value problems, including ones with delays, ones in measure chains, ones with three-point and multipoint conditions, ones with integral conditions and ones with generalized Lidstone conditions. See [18, 22, 28, 29, 33–35] and the references therein (see also [25–27]).

Notice that our boundary value conditions in Problem (1) correspond to $\alpha = 1$, $\beta = 0$, $\gamma = 1$, $\delta = -\alpha < 0$ in (2), so the boundary value conditions in (1) are *faux* special cases of (2). What is more important, the nonlinearity in (1) involves the first-order derivative, whereas the nonlinearity in (2) does not. This means that (1) is much more difficult to deal with than (2). When tackling the boundary value problems involving the first order derivative, one usually employs the Leggett–Williams fixed point theorem [19] and all kinds of its generalizations [4–6], or the coincidence degree theory [11]. The methods mentioned above cannot be applied to generalize the sharp results in [20, 23]. Consequently, in this paper, we shall use neither the Leggett–Williams fixed point theorems and its variants, nor the coincidence degree theory to deal with our problem (1), rather utilize the fixed point index theory on cones of Banach spaces to do that. The most important and difficult ingredients in our proofs are the establishing of the a priori estimates of positive solutions for some associated problems, in particular these for the first-order derivatives. In order to facilitate our proofs, we first establish an integral identity and an integral inequality that are of vital importance in the proofs of our main results. For the cases of superlinear nonlinearity at ∞ , the Bernstein–Nagumo type condition [7, 9, 25] is introduced to enable us to obtain the a priori estimate of the first-order derivative for associated boundary value problems. Our results generalize and extend the ones in [20, 23], are strikingly different from the ones in [1–3, 8, 12, 14, 15, 17, 21] and also complement the main ones in [16, 24].

This paper is also a continuation of [32] (see also [30, 31]), where we studied (1) with $\alpha = 0$ by the integro-differential equation argument. In contrast to [32], with the introduction of a cone in $C^1[0, 1]$ and the function $g(x, y) := (1 - \alpha)x + 2y$, we shall tackle Problem (1) using fixed point index theory more directly, thereby rendering the statements and proofs of the main results clearer and more concise.

The remainder of this paper is organized as follows. Some basic lemmas, including two versatile results, i.e. an integral identity and an integral inequality, are stated and proved in Sect. 2. The main results, Theorems 3.1–3.3, are presented and proved in Sect. 3. Also,

three extra results parallel with Theorems 3.1–3.3 are given in this section. The uniqueness of positive solutions and its iteration convergence are offered in Sect. 4.

2 Preliminaries and basic lemmas

For the sake of convenience, we denote $\beta := 1 - \alpha$ from now on.

Notice that (1) is equivalent to the integral equation below:

$$u(t) = \int_0^1 k(t, s) f(s, u(s), u'(s)) \, ds, \quad (3)$$

where $k(t, s)$ is the Green function defined by

$$k(t, s) := \begin{cases} \frac{t(1-\alpha(1-s))}{\beta}, & 0 \leq t \leq s \leq 1, \\ \frac{s(1-\alpha(1-t))}{\beta}, & 0 \leq s \leq t \leq 1. \end{cases}$$

Let $E := C^1[0, 1]$ and $P := \{u \in E : u(t) \geq 0, u'(t) \geq 0, t \in [0, 1]\}$. Define the norm $\|u\|$ by

$$\|u\| := \max_{t \in [0, 1]} \{|u(t)|, |u'(t)|\}.$$

Then $(E, \|\cdot\|)$ is a Banach space and P is a normal cone in E . Define $A : P \rightarrow P$ by

$$(Au)(t) := \int_0^1 k(t, s) f(s, u(s), u'(s)) \, ds.$$

Under the condition $f \in C([0, 1] \times \mathbb{R}_+^2, \mathbb{R}_+)$, the operator $A : P \rightarrow P$ is completely continuous. In our setting, the existence of positive solutions for (1) is equivalent to that of positive fixed points of the completely continuous nonlinear operator $A : P \rightarrow P$.

Lemma 2.1 (see [13]) *Let E be a real Banach space and P a cone in E . Suppose that $\Omega \subset E$ is a bounded open set and that $T : \overline{\Omega} \cap P \rightarrow P$ is a completely continuous operator. If there exists $w_0 \in P \setminus \{0\}$ such that*

$$w - Tw \neq \lambda w_0, \quad \forall \lambda \geq 0, w \in \partial\Omega \cap P,$$

then $i(T, \Omega \cap P, P) = 0$, where i indicates the fixed point index.

Lemma 2.2 (see [13]) *Let E be a real Banach space and P a cone in E . Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $T : \overline{\Omega} \cap P \rightarrow P$ is a completely continuous operator. If*

$$w - \lambda Tw \neq 0, \quad \forall \lambda \in [0, 1], w \in \partial\Omega \cap P,$$

then $i(T, \Omega \cap P, P) = 1$.

What follows are two versatile lemmas in the proofs of our main results.

Lemma 2.3 *If $w \in C^2[0, 1]$ with $w(0) = w'(1) - \alpha w(1) = 0$, then*

$$\int_0^1 (-w''(t))te^{\beta t} dt = \beta \int_0^1 (2w'(t) + \beta w(t))te^{\beta t} dt. \quad (4)$$

Proof By integration by parts, we have

$$\begin{aligned} \int_0^1 (-w''(t))te^{\beta t} dt &= -w'(1)e^{\beta} + \int_0^1 w'(t)(1 + \beta t)e^{\beta t} dt \\ &= -w'(1)e^{\beta} + \int_0^1 2\beta w'(t)te^{\beta t} dt \\ &\quad + \int_0^1 w'(t)(1 - \beta t)e^{\beta t} dt \end{aligned}$$

and

$$\int_0^1 w'(t)(1 - \beta t)e^{\beta t} dt = \alpha e^{\beta} w(1) + \int_0^1 \beta^2 w(t)te^{\beta t} dt.$$

Combining the preceding two equalities and noting $w(0) = w'(1) - \alpha w(1) = 0$, we obtain the desired result. \square

Lemma 2.4 *If $w \in P$ with $w(0) = 0$, then*

$$w(1) \leq \frac{\beta}{1 - \alpha e^{\beta}} \int_0^1 (\beta w(t) + 2w'(t))te^{\beta t} dt.$$

Proof Integrating by parts, we have

$$\int_0^1 \beta w(t)te^{\beta t} dt = -\frac{\alpha e^{\beta} w(1)}{\beta} + \frac{1}{\beta} \int_0^1 w'(t)(1 - \beta t)e^{\beta t} dt,$$

and thus

$$\begin{aligned} \int_0^1 (\beta w(t) + 2w'(t))te^{\beta t} dt &= -\frac{\alpha e^{\beta} w(1)}{\beta} + \frac{1}{\beta} \int_0^1 w'(t)(1 + \beta t)e^{\beta t} dt \\ &\geq -\frac{\alpha e^{\beta} w(1)}{\beta} + \frac{1}{\beta} \int_0^1 w'(t) dt \\ &= \frac{1 - \alpha e^{\beta}}{\beta} w(1). \end{aligned}$$

Then the desired inequality follows immediately. \square

3 Existence and multiplicity of positive solutions of (1)

For the sake of convenience, we define the function g by

$$g(x, y) := \beta x + 2y.$$

Recall that we have defined $\beta := 1 - \alpha$ in the preceding section. We make the following hypotheses in this section.

(H1) $f \in C([0, 1] \times \mathbb{R}_+^2, \mathbb{R}_+)$.

(H2) There are two constants $a > \beta$ and $c_1 > 0$ such that

$$f(t, x, y) \geq ag(x, y) - c_1, \quad \forall x \geq 0, y \geq 0, t \in [0, 1].$$

(H3) For every $M > 0$ there is a function $\Phi_M \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$f(t, x, y) \leq \Phi_M(y), \quad \forall t \in [0, 1], x \in [0, M], y \in \mathbb{R}_+$$

and $\int_0^\infty \frac{\xi d\xi}{\Phi_M(\xi) + \delta} = \infty$ for every $\delta > 0$.

(H4) There are two constants $r > 0$, $b \in]0, \beta[$ such that

$$f(t, x, y) \leq bg(x, y), \quad t \in [0, 1], x \in [0, r], y \in [0, r].$$

(H5) There are two constants $r > 0$, $d > \beta$ such that

$$f(t, x, y) \geq dg(x, y), \quad t \in [0, 1], x \in [0, r], y \in [0, r].$$

(H6) There are two constants $c \in]0, \beta[$, $c_2 > 0$ such that

$$f(t, x, y) \leq cg(x, y) + c_2, \quad t \in [0, 1], x \in \mathbb{R}_+, y \in \mathbb{R}_+.$$

(H7) $f(t, x, y)$ is increasing in x, y and there is a constant $\omega > 0$ such that $\int_0^1 (1 - \alpha(1 - s))f(s, \omega, \omega) ds < \beta\omega$.

Remark 3.1 By saying that $f(t, x, y)$ is increasing in x, y , we mean that, if $x_2 \geq x_1 \geq 0$ and $y_2 \geq y_1 \geq 0$, then the inequality $f(t, x_2, y_2) \geq f(t, x_1, y_1)$ holds for all $t \in [0, 1]$.

Theorem 3.1 *If (H1)–(H4) hold, then (1) has at least one positive solution.*

Proof Let

$$\mathcal{M}_1 := \{u \in P : u = Au + \lambda\varphi, \text{ for some } \lambda \geq 0\},$$

where $\varphi(t) := te^{-\beta t}$. Clearly, if $u \in \mathcal{M}_1$, then u is increasing on $[0, 1]$, and $u(t) \geq (Au)(t)$, $t \in [0, 1]$. We shall prove that \mathcal{M}_1 is bounded. We first establish the a priori bound of $\|u\|_0 := \max\{u(t) : 0 \leq t \leq 1\} = u(1)$ for \mathcal{M}_1 . If $u \in \mathcal{M}_1$, then $Au \in C^2[0, 1]$ (whence $u \in C^2[0, 1]$) and there is $\lambda \geq 0$ such that $u = Au + \lambda\varphi$, and equivalently, $-u''(t) = f(t, u(t), u'(t)) + \lambda(2 - \beta t)e^{-\beta t}$. Therefore, u is concave on $[0, 1]$ and $-u''(t) \geq f(t, u(t), u'(t))$. By (H2), we have

$$-u''(t) \geq ag(u(t), u'(t)) - c_1 = a(\beta u(t) + 2u'(t)) - c_1.$$

Multiply the preceding inequalities by $\psi(t) := te^{\beta t}$ and integrate over $[0, 1]$ and invoke Lemma 2.3 to obtain

$$\beta \int_0^1 (\beta u(t) + 2u'(t))\psi(t) dt \geq a \int_0^1 (\beta u(t) + 2u'(t))\psi(t) dt - \frac{c_1(1 - \alpha e^\beta)}{\beta^2}$$

and whence

$$\int_0^1 (\beta u(t) + 2u'(t))\psi(t) dt \leq \frac{c_1(1 - \alpha e^\beta)}{(a - \beta)\beta^2}, \quad \forall u \in \mathcal{M}_1.$$

Invoking Lemma 2.4 yields

$$\|u\|_0 = u(1) \leq \frac{c_1}{(a - \beta)\beta} := M, \quad \forall u \in \mathcal{M}_1,$$

thereby establishing the a priori bound of $\|u\|_0$ for \mathcal{M}_1 . Now we turn to establish the a priori bound of $\|u'\|_0$ for \mathcal{M}_1 . Indeed, if $u \in \mathcal{M}_1$, then $u \in C^2[0, 1]$ (as explained previously) and there is a constant $\lambda \geq 0$ such that $u = Au + \lambda\psi$, which can be equivalently rewritten as

$$-u''(t) = f(t, u(t), u'(t)) + \lambda\beta(2 - \beta t)e^{-\beta t}.$$

Let

$$\mu := \sup\{\lambda \geq 0 : u = Au + \lambda\psi \text{ for some } u \in P\}.$$

Then $\mu < \infty$ and, for every $u \in \mathcal{M}_1$, we have

$$-u''(t) \leq f(t, u(t), u'(t)) + \mu\beta(2 - \beta t)e^{-\beta t} \leq f(t, u, u') + 2\mu\beta.$$

By (H3), there is a function $\Phi_M \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$-u''(t) \leq \Phi_M(u'(t)) + 2\mu\beta.$$

This implies that

$$\begin{aligned} \int_0^1 \frac{u''(t)u'(t) dt}{\Phi_M(u'(t)) + 2\mu\beta} &= \int_{u'(1)}^{u'(0)} \frac{\xi d\xi}{\Phi_M(\xi) + 2\mu\beta} = \int_{\alpha u(1)}^{u'(0)} \frac{\xi d\xi}{\Phi_M(\xi) + 2\mu\beta} \\ &\leq \int_0^1 u'(t) dt = u(1). \end{aligned}$$

Noticing $u(1) \leq M, \forall u \in \mathcal{M}_1$, we obtain

$$\int_{\alpha M}^{u'(0)} \frac{\xi d\xi}{\Phi_M(\xi) + 2\mu\beta} \leq \int_{\alpha u(1)}^{u'(0)} \frac{\xi d\xi}{\Phi_M(\xi) + 2\mu\beta} \leq u(1) \leq M, \quad \forall u \in \mathcal{M}_1.$$

Now (H3) means that there is a constant $M_1 > 0$ such that

$$\|u'\|_0 = u'(0) \leq M_1, \quad \forall u \in \mathcal{M}_1,$$

thereby establishing the a priori bound of u' for \mathcal{M}_1 . This, together with $\|u\|_0 \leq M$, implies that the \mathcal{M}_1 is bounded. Taking $R > \sup\{\|u\| : u \in \mathcal{M}_1\}$, we have

$$u \neq Au + \lambda\psi, \quad \forall u \in \partial B_R \cap P, \lambda \geq 0,$$

where $B_R := \{u \in E : \|u\| < R\}$. Invoking Lemma 2.1 gives

$$i(A, B_R \cap P, P) = 0. \quad (5)$$

On the other hand, let

$$\mathcal{M}_2 := \{u \in \overline{B}_r \cap P : u = \lambda Au \text{ for some } \lambda \in [0, 1]\},$$

where $r > 0$ is specified by (H4). We are in a position to prove that $\mathcal{M}_2 = \{0\}$. Indeed, if $u \in \mathcal{M}_2$, then $Au \in C^2[0, 1]$ (whence $u \in C^2[0, 1]$) and there is $\lambda \in [0, 1]$ such that $u = \lambda Au$, which is equivalent to $-u''(t) = \lambda f(t, u(t), u'(t))$. By (H4), we have

$$-u''(t) \leq f(t, u(t), u'(t)) \leq b(\beta u(t) + 2u'(t)), \quad \forall u \in \overline{B}_r \cap P.$$

Multiply the preceding inequalities by $\psi(t) := te^{\beta t}$ and integrate over $[0, 1]$ and use Lemma 2.3 to obtain

$$\beta \int_0^1 (2u'(t) + \beta u(t))te^{\beta t} dt \leq b \int_0^1 (2u'(t) + \beta u(t))te^{\beta t} dt,$$

whence $u = 0$, and, in turn, $\mathcal{M}_2 = \{0\}$, as desired. As a result of this, we have

$$u \neq \lambda Au, \quad \forall u \in \partial B_r \cap P, \lambda \in [0, 1].$$

Then invoking Lemma 2.2 yields

$$i(A, B_r \cap P, P) = 1. \quad (6)$$

Obviously, we may assume $R > r$ in (5) and (6). Therefore, combining (5) and (6), we arrive at

$$i(A, (B_R \setminus \overline{B}_r) \cap P, P) = i(A, B_R \cap P, P) - i(A, B_r \cap P, P) = 0 - 1 = -1.$$

Consequently, the operator A has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$. Equivalently, (1) has at least one positive solution u . This completes the proof. \square

Theorem 3.2 *If (H1), (H5) and (H6) hold, then (1) has at least one positive solution.*

Proof Let

$$\mathcal{M}_3 := \{u \in P : u = \lambda Au \text{ for some } \lambda \in [0, 1]\}.$$

We are going to prove that \mathcal{M}_3 is bounded. Indeed, if $u \in \mathcal{M}_3$, then $Au \in C^2[0, 1]$ (whence $u \in C^2[0, 1]$) and there is $\lambda \in [0, 1]$ such that $u = \lambda Au$, which is equivalent to $-u''(t) = \lambda f(t, u(t), u'(t))$. By (H6), we have

$$-u''(t) \leq f(t, u(t), u'(t)) \leq c(\beta u(t) + 2u'(t)) + c_2.$$

Multiply the preceding inequalities by $\psi(t) := te^{\beta t}$ and integrate over $[0, 1]$ and use Lemma 2.3 to obtain

$$\beta \int_0^1 (2u'(t) + \beta u(t))te^{\beta t} dt \leq c \int_0^1 (2u'(t) + \beta u(t))te^{\beta t} dt + \frac{c_2(1 - \alpha e^{\beta})}{\beta^2}$$

whence

$$\int_0^1 (2u'(t) + \beta u(t))te^{\beta t} dt \leq \frac{c_2(1 - \alpha e^{\beta})}{(\beta - c)\beta^2}, \quad \forall u \in \mathcal{M}_3.$$

Now Lemma 2.4 implies

$$\|u\|_0 = u(1) \leq \frac{c_2}{(\beta - c)\beta} := M_2, \quad \forall u \in \mathcal{M}_3.$$

By (H6) again, we have

$$-u''(t) \leq 2cu'(t) + M_2 + c_2, \quad \forall u \in \mathcal{M}_3.$$

Some basic calculations, along with the boundary value condition $u'(1) = \alpha u(1)$, imply

$$u'(0) \leq \alpha e^{2c}u(1) + \frac{M_2(e^{2c} - 1)}{2c} \leq \alpha e^{2c}M_2 + \frac{M_2(e^{2c} - 1)}{2c} := M_3, \quad \forall u \in \mathcal{M}_3.$$

This establishes the a priori bound of $\|u'\|_0$ for \mathcal{M}_3 , which, together with the a priori bound of $\|u\|_0$, implies the boundedness of \mathcal{M}_3 . Taking $R > \sup\{\|u\| : u \in \mathcal{M}_3\}$, we have

$$u \neq \lambda Au, \quad \forall u \in \partial B_R \cap P, \lambda \in [0, 1].$$

Invoking Lemma 2.2 yields

$$i(A, B_R \cap P, P) = 1. \quad (7)$$

On the other hand, let

$$\mathcal{M}_4 := \{u \in \overline{B}_r \cap P : u = Au + \lambda \varphi \text{ for some } \lambda \geq 0\},$$

where $r > 0$ is specified in (H5). We shall prove that $\mathcal{M}_4 \subset \{0\}$. Indeed, if $u \in \mathcal{M}_4$, then $Au \in C^2[0, 1]$ (whence $u \in C^2[0, 1]$) and there is $\lambda \geq 0$ such that $u = Au + \lambda \varphi$, which is equivalent to $-u''(t) = f(t, u(t), u'(t)) + \lambda \beta(2 + (1 - \lambda)t)e^{\beta t}$. By (H5), we have

$$-u''(t) \geq f(t, u(t), u'(t)) \geq d(\beta u(t) + 2u'(t)).$$

Multiply the preceding inequalities by $\psi(t) := te^{\beta t}$ and integrate over $[0, 1]$ and use Lemma 2.3 to obtain

$$\beta \int_0^1 (2u'(t) + \beta u(t))te^{\beta t} dt \geq d \int_0^1 (2u'(t) + \beta u(t))te^{\beta t} dt,$$

whence

$$\int_0^1 (2u'(t) + \beta u(t)) t e^{\beta t} dt = 0.$$

Therefore $u = 0$, and, in turn, $\mathcal{M}_4 \subset \{0\}$, as desired, and this implies

$$u \neq Au + \lambda \varphi, \quad \forall u \in \partial B_r \cap P, \lambda \geq 0.$$

Now invoking Lemma 2.1 gives

$$i(A, B_r \cap P, P) = 0. \quad (8)$$

Combining (7) and (8), we arrive at

$$i(A, (B_R \setminus \overline{B_r}) \cap P, P) = i(A, B_R \cap P, P) - i(A, B_r \cap P, P) = 1 - 0 = 1. \quad (9)$$

Consequently, the operator A has at least one fixed point on $(B_R \setminus \overline{B_r}) \cap P$. Equivalently, (1) has at least one positive solution u . This completes the proof. \square

Theorem 3.3 *If (H1)–(H3), (H5) and (H7) hold, then (1) has at least two positive solutions.*

Proof (H1)–(H3) and (H5) imply that (5) and (8) hold (see the proofs of Theorems 3.1 and 3.2). On the other hand, by (H7), we have

$$\begin{aligned} \|Au\|_0 &= (Au)(1) = \int_0^1 k(1, s) f(s, u(s), u'(s)) ds \\ &\leq \frac{1}{\beta} \int_0^1 s f(s, \omega, \omega) ds \\ &\leq \frac{1}{\beta} \int_0^1 (1 - \alpha(1 - s)) f(s, \omega, \omega) ds \\ &< \omega, \quad \forall u \in \partial B_\omega \cap P \end{aligned}$$

and

$$\begin{aligned} \|(Au)'\|_0 &= (Au)'(0) = \frac{1}{\beta} \int_0^1 (1 - \alpha(1 - s)) k(1, s) f(s, u(s), u'(s)) ds \\ &\leq \frac{1}{\beta} \int_0^1 (1 - \alpha(1 - s)) f(s, \omega, \omega) ds \\ &< \omega, \quad \forall u \in \partial B_\omega \cap P, \end{aligned}$$

whence

$$\|Au\| < \|u\| = \omega, \quad \forall u \in \partial B_\omega \cap P.$$

This implies

$$u \neq \lambda Au, \quad \forall u \in \partial B_\omega \cap P, \lambda \in [0, 1].$$

Invoking Lemma 2.2 yields

$$i(A, B_\omega \cap P, P) = 1. \quad (10)$$

This, along with (5) and (8), leads to

$$i(A, (B_R \setminus \overline{B}_\omega) \cap P, P) = 0 - 1 = -1$$

and

$$i(A, (B_\omega \setminus \overline{B}_r) \cap P, P) = 1 - 0 = 1.$$

Therefore, A has at least two positive fixed points, one on $(B_R \setminus \overline{B}_\omega) \cap P$ and the other on $(B_\omega \setminus \overline{B}_r) \cap P$. This implies (1) has at least two positive solutions, which completes the proof. \square

An example of multiple positive solutions Let

$$f(t, x, y) := \lambda(x^{p_1} + y^{q_1} + x^{p_2} + y^{q_2}),$$

where $p_1 > 1$, $2 \geq q_1 > 1$, $1 > p_2$, $q_2 > 0$, $\frac{1-\alpha}{4-2\alpha} > \lambda > 0$. Then (H1)–(H3), (H5) and (H7) hold. By Theorem 3.3, (1), with f being defined as above, has two positive solutions.

Now we present some extra results parallell with Theorems 3.1–3.3.

Let λ_1 be the minimal positive solution of the transcendental equation

$$\sqrt{\lambda} = \alpha \tan \sqrt{\lambda}.$$

Then λ_1 is the first characteristic value of

$$\begin{cases} -u'' = \lambda u, \\ u(0) = u'(1) - \alpha u(1) = 0. \end{cases}$$

And $\varphi_1(t) := \sin \sqrt{\lambda_1} t$ is a characteristic function thereof.

It is worthwhile to point out that (H2) and (H4)–(H6) all are described in terms of the function $g(x, y) := \beta x + 2y$. We replace (H2) and (H5) with the conditions below described in terms of λ_1 :

(H2)' There are two constants $a > \lambda_1$ and $c_1 > 0$ such that

$$f(t, x, y) \geq ax - c_1, \quad \forall x \geq 0, y \geq 0, t \in [0, 1].$$

(H5)' There are two constants $r > 0$, $d > \lambda_1$ such that

$$f(t, x, y) \geq dx, \quad t \in [0, 1], x \in [0, r], y \in [0, r].$$

What follows is a result obtained by elementary calculus.

Lemma 3.1 *If $u \in P$ is concave on $[0, 1]$ with $u(0) = 0$, then*

$$u(1) \leq \frac{\alpha^2}{\beta} \sin \sqrt{\lambda_1} \sec^2 \sqrt{\lambda_1} \int_0^1 u(t) \sin \sqrt{\lambda_1} t \, dt.$$

The following three results are parallel counterparts of Theorems 3.1–3.3, acquired by replacing (H2) and (H5) with (H2)' and (H5)', respectively.

Theorem 3.1' *If (H1), (H2)', (H3) and (H4) hold, then (1) has at least one positive solution.*

Proof Recall that (H2) serves to establish the a priori bound of $\|u\|_0$ (see the proof of Theorem 3.1 for more details). The obtaining of the a priori bound of $\|u\|_0$ can proceed as that of the proof of Theorem 3.1 with replacing $\psi(t) := te^{\beta t}$ with $\varphi_1(t) := \sin \sqrt{\lambda_1} t$ and employing Lemma 3.1. Then we may keep the remainder of the proof of Theorem 3.1 unchanged. This completes the proof. \square

Theorem 3.2' *If (H1), (H5)' and (H6) hold, then (1) has at least one positive solution.*

Theorem 3.3' *If (H1), (H2)', (H3), (H5) and (H7) hold, then (1) has at least two positive solutions.*

4 Uniqueness of positive solutions and iteration convergence

Let

$$w_0(t) := \frac{(2 - \alpha)t - \beta t^2}{2\beta} = \int_0^1 k(t, s) \, ds.$$

Then

$$w'_0(t) = \frac{2 - \alpha - 2\beta t}{2\beta} \geq \frac{\alpha}{2\beta}, \quad t \in [0, 1].$$

Below is a result that is easy to prove.

Lemma 4.1 *If $h \in C([0, 1], \mathbb{R}_+)$, $h(t) \not\equiv 0$, then there are two positive constants $M_h \geq m_h$ such that*

$$m_h w_0(t) \leq \int_0^1 k(t, s) h(s) \, ds \leq M_h w_0(t), \quad m_h w'_0(t) \leq \int_0^1 \frac{\partial}{\partial t} k(t, s) h(s) \, ds \leq M_h w'_0(t).$$

(H8) f is increasing in x, y and $f(t, \lambda x, \lambda y) > \lambda f(t, x, y)$ for all $x > 0, y > 0$ and $t \in [0, 1]$.

Theorem 4.1 *If (H1), (H5), (H6) and (H8) hold, then (1) has exactly one positive solution.*

Proof By Theorem 3.2, (1) has at least one positive solution. It remains to prove the uniqueness of positive solutions. Indeed, if $u_1 \in C^2[0, 1] \cap P$ and $u_2 \in C^2[0, 1] \cap P$ are two positive solutions, then

$$u_i(t) > 0, \quad u'_i(t) > 0, \quad \forall t \in]0, 1[(i = 1, 2).$$

By Lemma 4.1, there are four constants m_i, M_i such that

$$m_i w_0(t) \leq u_i(t) \leq M_i w_0(t), \quad m_i w'_0(t) \leq u'_i(t) \leq M_i w'_0(t) \quad (i = 1, 2).$$

Therefore

$$u_1(t) \geq \frac{m_1}{M_2} w_0(t), \quad u'_1(t) \geq \frac{m_1}{M_2} w'_0(t).$$

Let

$$\mu := \left\{ \lambda > 0 : u_1(t) \geq \lambda u_2(t), u'_1(t) \geq \lambda u'_2(t) \right\}.$$

We claim that $\mu \geq 1$. Indeed, if the claim were false, then $0 < \mu < 1$ and $u_1(t) \geq \mu u_2(t)$ and $u'_1(t) \geq \mu u'_2(t)$. Let

$$\omega(t) := f(t, \mu u_2(t), \mu u'_2(t)) - \mu f(t, u_2(t), u'_2(t)).$$

(H8) implies $\omega(t) > 0$, $t \in]0, 1[$. By Lemma 4.1 again, there is a constant $\varepsilon > 0$ such that

$$\int_0^1 k(t, s) \omega(s) \, ds \geq \varepsilon w_0(t), \quad t \in [0, 1].$$

Therefore, we have

$$\begin{aligned} u_1(t) &= \int_0^1 k(t, s) f(s, u_1(s), u'_1(s)) \, ds \\ &\geq \int_0^1 k(t, s) f(s, \mu u_2(s), \mu u'_2(s)) \, ds \\ &= \int_0^1 k(t, s) \omega(s) \, ds + \mu \int_0^1 k(t, s) f(s, u_2(s), u'_2(s)) \, ds \\ &\geq (\varepsilon + \mu) u_2(t), \\ u'_1(t) &= \int_0^1 \frac{\partial}{\partial t} k(t, s) f(s, u_1(s), u'_1(s)) \, ds \\ &\geq \int_0^1 \frac{\partial}{\partial t} k(t, s) f(s, \mu u_2(s), \mu u'_2(s)) \, ds \\ &= \int_0^1 \frac{\partial}{\partial t} k(t, s) \omega(s) \, ds + \mu \int_0^1 \frac{\partial}{\partial t} k(t, s) f(s, u_2(s), u'_2(s)) \, ds \\ &\geq (\varepsilon + \mu) u'_2(t). \end{aligned}$$

The preceding two inequalities contradict the very definition of μ . As a result, we have proved that $\mu \geq 1$ and thus $u_1(t) \geq u_2(t)$. Similarly, $u_1(t) \geq u_1(t)$. Therefore $u_1(t) \equiv u_2(t)$. This proves the uniqueness of positive solutions of (1). \square

Recall that we have defined, in the proof of Theorem 3.1, the function $\varphi \in P \setminus \{0\}$ by

$$\varphi(t) := t e^{-\beta t}.$$

It is easy to see that

$$\int_0^1 k(t, s)(\beta\varphi(s) + 2\varphi'(s)) \, ds = \frac{\varphi(t)}{\beta}. \quad (11)$$

By Lemma 4.1, we obtain the result below.

Lemma 4.2 *Given every $h \in C([0, 1], \mathbb{R}_+)$, $h \not\equiv 0$, there exist two positive solutions $M_h \geq m_h$ such that*

$$m_h\varphi(t) \leq \int_0^1 k(t, s)h(s) \, ds \leq M_h\varphi(t), \quad m_h\varphi'(t) \leq \int_0^1 \frac{\partial}{\partial t} k(t, s)h(s) \, ds \leq M_h\varphi'(t).$$

Notice that the cone P defines a partial ordering \lesssim in E :

$$u \lesssim v \iff u(t) \leq v(t), \quad u'(t) \leq v'(t), \quad t \in [0, 1].$$

Lemma 4.3 *If (H1) and (H5) hold, then there is a constant $\delta > 0$ such that for every $\varepsilon \in]0, \delta]$, $u_\varepsilon(t) := \varepsilon\varphi(t)$ is a subsolution of the operator $Au = u$, i.e. $Au_\varepsilon \gtrsim u_\varepsilon$.*

Proof Let

$$\delta := r = \frac{r}{\|\varphi'\|_0} < \frac{r}{\|\varphi\|_0} = e^\beta r.$$

Then, by (H5) and (11), for every $\varepsilon \in]0, \delta]$, we have

$$\begin{aligned} (Au_\varepsilon)(t) &\geq d \int_0^1 k(t, s)(\beta u_\varepsilon(s) + 2u'_\varepsilon(s)) \, ds = \frac{d}{\beta} u_\varepsilon(t) \geq u_\varepsilon(t), \\ (Au_\varepsilon)'(t) &\geq d \int_0^1 \frac{\partial}{\partial t} k(t, s)(\beta u_\varepsilon(s) + 2u'_\varepsilon(s)) \, ds = \frac{d}{\beta} u'_\varepsilon(t) \geq u'_\varepsilon(t), \end{aligned}$$

or, equivalently, $Au_\varepsilon \gtrsim u_\varepsilon$. □

Lemma 4.4 *If (H1) and (H6) hold, then there is a constant $m_0 > 0$ such that for all $m \geq m_0$, $v_m(t) := m\varphi(t)$ is a supersolution of the operator equation $Au = u$, i.e. $Av_m \lesssim v_m$.*

Proof By (H6) and Lemma 4.2, for every $m > 0$,

$$\begin{aligned} (Av_m)(t) &= \int_0^1 k(t, s)f(s, v_m(s), v'_m(s)) \, ds \\ &\leq \int_0^1 k(t, s)(c\beta v_m(s) + 2cv'_m(s) + c_2) \, ds \\ &= \frac{cv_m(t)}{\beta} + c_2w_0(t) \\ &\leq \frac{cm\varphi(t)}{\beta} + \frac{c_2e^\beta}{2\beta}\varphi(t) \end{aligned}$$

and

$$\begin{aligned}(Av_m)'(t) &= \int_0^1 \frac{\partial}{\partial t} k(t,s) f(s, v_m(s), v_m'(s)) \, ds \\ &\leq \int_0^1 \frac{\partial}{\partial t} k(t,s) (c\beta v_m(s) + 2cv_m'(s) + c_2) \, ds \\ &= \frac{cm\varphi'(t)}{\beta} + c_2\varphi_0'(t) \\ &\leq \frac{cm\varphi'(t)}{\beta} + \frac{c_2e^\beta}{2\beta} \varphi'(t),\end{aligned}$$

where

$$\frac{e^\beta}{2\beta} = \sup_{0 < t < 1} \frac{w_0(t)}{\varphi(t)} = \max_{0 \leq t \leq 1} \frac{w_0'(t)}{\varphi'(t)}.$$

Let

$$m_0 := \frac{c_2e^\beta}{2(\beta - c)}.$$

Then, for every $m \geq m_0$, $(Av_m)(t) \leq v_m(t)$, $(Av_m)'(t) \leq v_m'(t)$, or equivalently, $Av_m \lesssim v_m$, i.e. v_m is a supersolution of the operator equation $Au = u$. \square

Theorem 4.2 *If (H1), (H5), (H6) and (H8) hold, then, for every $u \in P \setminus \{0\}$, the iterative sequence $\{A^n u\}$ converges to the unique positive solution of (1) by the norm of $C^1[0, 1]$.*

Proof By (H6) and (H8), we have $A(P \setminus \{0\}) \subset P \setminus \{0\}$. For every $u \in P \setminus \{0\}$, Au is increasing in $[0, 1]$, $(Au)(0) = (Au)'(1) - \alpha(Au)(1) = 0$, and

$$(Au)'(t) = \frac{1}{\beta} \int_0^t (1 - \alpha(1-s)) f(s, u(s), u'(s)) \, ds + \frac{\alpha}{\beta} \int_t^1 sf(s, u(s), u'(s)) \, ds > 0.$$

Therefore there are positive constants $c_1(u)$, $c_2(u)$ such that

$$c_1(u)(1 - \beta t)e^{-\beta t} \leq (Au)'(t) \leq c_2(u)(1 - \beta t)e^{-\beta t}, \quad t \in [0, 1].$$

Therefore

$$c_1(u)\varphi(t) \leq (Au)(t) \leq c_2(u)\varphi(t), \quad t \in [0, 1],$$

and, in turn,

$$c_1(u)\varphi \lesssim Au \lesssim c_2(u)\varphi, \quad \forall u \in P \setminus \{0\}. \quad (12)$$

This, along with Lemmas 4.3 and 4.4, implies that there are two positive constants constant ε and m such that $u_0 := \varepsilon\varphi$ and $v_0 := m\varphi$ are a subsolution and a supersolution of

the operator equation $Au = u$, respectively, and $u_0 \lesssim Au \lesssim v_0$. By (H8), $A : P \rightarrow P$ is an increasing operator. Therefore, for every positive integer n ,

$$A^{n-1}u_0 \lesssim A^n u \lesssim A^{n-1}v_0.$$

As is well known, $\{A^n u_0\}$ and $\{A^n v_0\}$ converge to the maximum fixed point and the minimum fixed point in the ordering interval $[u_0, v_0] (= \{u \in P : u_0 \lesssim u \lesssim v_0\})$, respectively. By Theorem 4.1, (1) has exactly one positive solution u^* , and thus A has exactly one positive fixed point u^* . As a result of that, $\{A^n u_0\}$ and $\{A^n v_0\}$ has the same limit u^* . Thus the normality of P implies that $\{A^n u\}$ converges to u^* by the norm of $C^1[0, 1]$. \square

An example for the uniqueness of positive solutions Let

$$f(t, x, y) := x^p + y^q,$$

where $0 < p, q < 1$. Then (H1), (H5), (H6) and (H8) hold. By Theorem 4.2, for every $u_0 \in P \setminus \{0\}$, the iterative sequence

$$u_{n+1}(t) := \int_0^1 k(t, s) (u_n(s)^p + u'_n(s)^q) ds, \quad n = 0, 1, 2, \dots,$$

converges, by the norm of $E := C^1[0, 1]$, to the unique positive solution of Problem (1) with f defined above.

Acknowledgements

The author is really grateful to the anonymous reviewers for their excellent comments and suggestions that improve the paper considerably.

Availability of data and materials

This paper focuses on theoretical analysis, not involving experiments and data.

Competing interests

The author declares that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 8 April 2021 Accepted: 14 June 2021 Published online: 26 June 2021

References

1. Agarwal, R.P., O'Regan, D., Lakshmikantham, V.: An upper and lower solution approach for nonlinear singular boundary value problems with y' dependence. *Arch. Inequal. Appl.* **1**, 119–135 (2003)
2. Agarwal, R.P., O'Regan, D., Yan, B.: Positive solutions for singular three-point boundary-value problems. *Electron. J. Differ. Equ.* **2008**, 116 (2008)
3. Avery, R., Davis, J.M., Henderson, J.: Three symmetric positive solutions for Lidstone problems by a generalization of the Leggett–Williams theorem. *Electron. J. Differ. Equ.* **2000**, 40 (2000)
4. Avery, R.I.: A generalization of the Leggett–Williams fixed point theorem. *MSR Hot-Line* **3**, 9–14 (1999)
5. Avery, R.I., Henderson, J.: An extension of the five functionals fixed point theorem. *Int. J. Differ. Equ. Appl.* **1**, 275–290 (2000)
6. Avery, R.I., Henderson, J.: Two positive fixed points of nonlinear operators on ordered Banach spaces. *Commun. Appl. Nonlinear Anal.* **8**, 27–36 (2001)
7. Bernstein, S.N.: Sur les équations du calcul des variations. *Ann. Sci. Éc. Norm. Supér.* **29**, 431–485 (1912)
8. Du, Z., Xue, C., Ge, W.: Multiple solutions for three-point boundary-value problem with nonlinear terms depending on the first derivative. *Arch. Math.* **84**, 341–349 (2005)

9. Dugundji, J., Granas, A.: *Fixed Point Theory I*. Polish Sci., Warszawa (1982)
10. Erbe, L.H., Hu, S., Wang, H.: Multiple positive solutions of some boundary value problems. *J. Math. Anal. Appl.* **184**, 640–680 (1994)
11. Gaines, R.E., Mawhin, J.L.: *Coincidence Degree and Nonlinear Differential Equations*. Springer, Berlin (1977)
12. Graef, J.R., Kong, L.: Necessary and sufficient conditions for the existence of symmetric positive solutions of singular boundary value problems. *J. Math. Anal. Appl.* **331**, 1467–1484 (2007)
13. Guo, D., Lakshmikantham, V.: *Nonlinear Problems in Abstract Cones*. Academic Press, Boston (1988)
14. Guo, Y., Ge, W.: Positive solutions for three-point boundary value problems with dependence on the first order derivative. *J. Math. Anal. Appl.* **290**, 291–301 (2004)
15. Henderson, J., Thompson, H.B.: Multiple symmetric positive solutions for a second order boundary value problem. *Proc. Am. Math. Soc.* **128**, 2373–2379 (2000)
16. Infante, G.: Positive and increasing solutions of perturbed Hammerstein integral equations with derivative dependence. *Discrete Contin. Dyn. Syst., Ser. B* **25**, 691–699 (2020)
17. Jia, M., Liu, X.: The method of upper and lower solutions for second-order non-homogeneous two-point boundary-value problem. *Electron. J. Differ. Equ.* **2007**, 116 (2007)
18. Jiang, D., Zhang, L.: Positive solutions for boundary value problems of second-order delay differential equations. *Acta Math. Sinica (Chin. Ser.)* **46**, 739–746 (2003)
19. Leggett, R.W., Williams, L.R.: Multiple positive fixed points of a nonlinear operators on ordered Banach spaces. *Indiana Univ. Math. J.* **28**, 673–688 (1979)
20. Li, F., Liu, Z.: Multiple positive solutions of some operators and applications. *Acta Math. Sinica (Chin. Ser.)* **41**, 97–102 (1998)
21. Li, F., Zhang, Y.: Multiple symmetric nonnegative solutions of second-order ordinary differential equations. *Appl. Math. Lett.* **17**, 261–267 (2004)
22. Li, H., Sun, J., Cui, Y.: Positive solutions of nonlinear differential equations on a measure chain. *Chin. Ann. Math.* **30**(A), 1–6 (2009) (in Chinese)
23. Liu, Z., Li, F.: Multiple positive solutions of nonlinear two-point value problems. *J. Math. Anal. Appl.* **203**, 610–625 (1996)
24. Mawhin, J., Przeradzki, P., Szymańska-Dębowska, K.: Second order systems with nonlinear nonlocal boundary conditions. *Electron. J. Qual. Theory Differ. Equ.* **2018**, Article ID 56 (2018)
25. Nagumo, M.: Über die Differentialgleichung $y'' = f(t, y, y')$. *Proc. Phys. Math. Soc. Jpn.* **19**, 861–866 (1937)
26. Naito, Y., Tanaka, S.: Sharp conditions for the existence of sign-changing solutions to equations involving the one-dimensional p -Laplacian. *Nonlinear Anal.* **69**, 3070–3083 (2008)
27. Tanaka, S.: Morse index and symmetry-breaking for positive solutions of one-dimensional Hénon type. *J. Differ. Equ.* **255**, 1709–1733 (2013)
28. Yang, Z.: Existence and nonexistence results for positive solutions of an integral boundary value problem. *Nonlinear Anal.* **65**, 1489–1511 (2006)
29. Yang, Z.: Existence and uniqueness of positive solutions for a higher order boundary value problem. *Comput. Math. Appl.* **54**, 220–228 (2007)
30. Yang, Z., Kong, L.: Positive solutions of a system of second order boundary value problems involving first order derivatives via \mathbb{R}_+^n -monotone matrices. *Nonlinear Anal.* **75**, 2037–2046 (2012)
31. Yang, Z., O'Regan, D.: Positive solutions for a $2n$ -order boundary value problem involving all derivatives of odd orders. *Topol. Methods Nonlinear Anal.* **37**, 87–101 (2011)
32. Yang, Z., O'Regan, D., Agarwal, R.P.: Positive solutions of a second-order boundary value problem via integro-differential equation arguments. *Appl. Anal.* **88**, 1197–1211 (2009)
33. Yang, Z., Wei, G.: Positive solutions of a functional integral equations. *Acta Math. Sinica (Chin. Ser.)* **50**, 363–372 (2007)
34. Zhang, G., Sun, J.: Positive solutions of m -point boundary value problems. *J. Math. Anal. Appl.* **291**, 406–418 (2004)
35. Zhang, G., Sun, J.: Existence of positive solutions for singular second-order m -point boundary value problems. *Acta Math. Appl. Sin. Engl. Ser.* **20**(4), 655–664 (2004)

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► [springeropen.com](https://www.springeropen.com)