Anupama Choudhary*, Devendra Kumar, and Jagdev Singh

Analytical Solution of Fractional Differential Equations arising in Fluid Mechanics by using Sumudu transform method

Abstract: The aim of the present paper is to present analytical solution of fractional differential equations. The fractional derivative is considered in Caputo sense. The results are derived by the application of Sumudu transform in term of Mittag-Leffler function, which are suitable for numerical computation. The results obtain by the Sumudu transform method indicate that the approach is easy to implement and computationally very attractive.

Keywords: Fractional-order differential equations; Riemann-Liouvile fractional integrals; Mittag-Leffler function; Wright function; Binomial series; Gamma function; Sumudu transform of the fractional derivative

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1 Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. Fractional differential equations have gained importance and popularity, mainly due to its demonstrated applications in science and engineering. For example, these equations are increasingly used to model problems in fluid flow, theology, diffusion, relaxation, oscillation, anomalous diffusion, reaction-diffusion, turbulence, diffusive transport akin to diffusion, electric networks, polymer physics, chemical physics, electrochemistry of corrosion, relaxation processes in complex systems, propagation of seismic waves, dynamical processes in self-similar and porous structures and many other physical processes. The most important advantage of using fractional differential equations in these and other applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic and it is one reason why fractional calculus has become more and more popular [1–9]. In view of great importance of fractional differential equations many authors have paid attention for handling linear and non-linear fractional differential equations. Among these is differential transform method [10, 11], homotopy analysis method [12–14], homotopy perturbation method [15], Laplace decomposition method [16–21], homotopy analysis transform method [22–25], homotopy perturbation transform method [26–28] and variational iteration method [29]. In the early 90's, Watugala [30] introduced a new integral transforms named the Sumudu transform and further applied it to the solution of ordinary differential equation in control engineering problems. For further detail and properties about Sumudu transform, see References [31–35]. Recently, Kilicman et al. [36] applied this transform to solve the system of differential equations.

In this paper, we put on the Sumudu transform of the fractional derivative and the expansion coefficients of binomial series to deduct the evident the solutions to homogeneous fractional differential equations. The Sumudu transform has scale and unit preserving properties, so it can be used to solve problems without resorting to a new frequency domain. It is worth mentioning that the Sumudu transform method provides the solution in closed form and it is capable of reducing the volume of computational work as compared to the classical methods.

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2 Preliminary results

The fractional derivative of a function $f(t)$ is defined by [9, 37]

$$\frac{d^\mu}{dt^\mu} f(t) = \begin{cases} f^{(n)}(t) & \text{if } \mu = n \epsilon N, \\ \frac{1}{\Gamma(n-\mu)} \int_0^t \frac{f^{(n)}(t)}{(t-x)^{\mu-n+1}} & \text{if } n-1 < \mu < n, \end{cases} \tag{1}$$

where the gamma function can be defined as a definite integral for $Re(z) > 0$ (Euler's integral form)

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{2}$$

The Mittag-Leffler function [38, 39] is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\alpha+\beta)}, \ (\alpha, \beta \epsilon C, \ Re(\alpha) > 0). \tag{3}$$

The Riemann-Liouville fractional derivatives $D^\alpha_{a+}\varphi$ and $D^\alpha_{a-}\varphi$ of order $\alpha \epsilon C$, $Re(\alpha) \geq 0$ are defined by

$$(D^\alpha_{a+}\varphi)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{\varphi(t)\,dt}{(x-t)^{\alpha-n+1}} \tag{4}$$

$(n = [Re(\alpha)] + 1; x > a)$

and

$$(D^\alpha_{a-}\varphi)(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int_x^b \frac{\varphi(t)\,dt}{(t-x)^{\alpha-n+1}} \tag{5}$$

$(n = [Re(\alpha)] + 1; x < b)$

where $[Re(\alpha)]$ means the integral part of $Re(\alpha)$.

The simplest Wright function is defined by [40, 41]

$$\phi(\alpha, \beta; z) = \sum_{k=0}^\infty \frac{1}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!} \quad \text{where } \alpha, \beta, z \epsilon C \tag{6}$$

and the general Wright function is defined as

$${}_p\psi_q(z) = {}_p\psi_q \left[\begin{array}{c} (a_i, \alpha_i)_{(1,p)} \\ (b_j, \beta_j)_{(1,q)} \end{array} \middle| z \right] = \sum_{k=0}^\infty \frac{\prod_{i=1}^p \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^q \Gamma(b_j + \beta_j k)} \frac{z^k}{k!} \tag{7}$$

where $z, a_i, b_j \epsilon C$ and $\alpha_i, \beta_j \epsilon R$ $(i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q)$.

The Sumudu transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F_s(u)$, defined by [30, 42]

$$S\{f(t)\} = \overline{F}(u) = G(u) = \int_0^\infty (1/u)\, e^{-t/u} f(t)\, dt \quad . \tag{8}$$

We will also use the following result [43]:

$$S^{-1}\left[u^{\gamma-1}(1-\omega u^\beta)^{-\delta}\right] = t^{\gamma-1} E^\delta_{\beta,\gamma}\left(\omega t^\beta\right)$$

and

$$S^{-1}\left[\frac{1}{u\,(u^{-2} + a u^{-\alpha} + b)}\right] = $$
$$= \sum_{k=0}^\infty (-b)^k\, t^{\alpha(k+1)-1} E^{k+1}_{2-\alpha,\ \alpha(k+1)}\left(-a t^{2-\alpha}\right) \tag{9}$$

The Sumudu transform of the Riemann-Liouville fractional derivative is given by [44]

$$S\left[D^\alpha f(t)\right] = \overline{F}_\alpha(u) = u^{-\alpha}\left[G(u) - \sum_{k=0}^n u^{\alpha-k}\left[D^{\alpha-k} f(t)\right]_{t=0}\right] \tag{10}$$

where $-1 < n-1 \leq \alpha < n$ and $n \epsilon N$.

Let $f(t)$ and $g(t)$ be causal functions with Laplace transforms $F(s)$ and $G(s)$ respectively, and $M(u)$ and $N(u)$ respectively. Then the Sumudu transform of the convolution of f and g,

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau, \tag{11}$$

is given by

$$S[(f * g)(t)] = uM(u)\,N(u) \tag{12}$$

Remark: If $G(u)$ is the Sumudu transform of $f(t)$, one can take into consideration the Sumudu transform of the Riemann-Liouville fractional derivative as follows [44]:

$$S\left[D^\alpha f(t)\right] = \overline{F}_\alpha(u) = u^{-\alpha}\left[G(u) - \sum_{k=0}^n u^{\alpha-k}\left[D^{\alpha-k} f(t)\right]_{t=0}\right] \tag{13}$$

Let us take the Laplace transform of $f'(t) = \frac{df}{dt}$ as follows:

$$L\left[D^\alpha f(t)\right](s) = \int_0^\infty e^{-st}\left[D^\alpha f(t)\right] dt$$
$$= \int_0^\infty e^{-st} \frac{1}{\Gamma(n-\alpha)} \left(\int_0^t \frac{f^{(n)}(\xi)}{(t-\xi)^{\alpha-n+1}} d\xi\right) dt$$
$$= \frac{1}{\Gamma(n-\alpha)} \int_0^\infty \left(\int_\xi^\infty e^{-st} \frac{f^{(n)}(\xi)}{(t-\xi)^{\alpha-n+1}} dt\right) d\xi$$
$$= \frac{1}{\Gamma(n-\alpha)} \int_0^\infty f^{(n)}(\xi) \left(\int_0^\infty e^{-s(u+\xi)} u^{n-\alpha-1} du\right) d\xi$$
$$= \frac{1}{\Gamma(n-\alpha)} \int_0^\infty e^{-s\xi} f^{(n)}(\xi) \int_0^\infty e^{-su} u^{n-\alpha-1} du\, d\xi$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_0^\infty e^{-s\xi} f^{(n)}(\xi) \frac{\Gamma(n-\alpha)}{s^{n-\alpha}} d\xi$$

$$= s^{\alpha-n} \int_0^\infty e^{-s\xi} f^{(n)}(\xi) d\xi = s^{\alpha-n} L\left[f^{(n)}(t)\right](s)$$

$$= s^\alpha \overline{F}(s) - \sum_{k=1}^{n} s^{\alpha-k} f^{(k-1)}(0)$$

$$= s^\alpha \overline{F}(s) - \sum_{k=1}^{n} s^{k-1} \left[D^{\alpha-k} f(t)\right]_{t=0}$$

Therefore, when we putting $1/u$ for the variable s, we get the Sumudu transform of fractional order of $f(t)$ as follows

$$S\left[D^\alpha f(t)\right] = \overline{F}_\alpha(u) = u^{-\alpha}\left[G(u) - \sum_{k=0}^{n} u^{\alpha-k}\left[D^{\alpha-k} f(t)\right]_{t=0}\right]$$

3 Solution of the fractional differential equations

In this section, we apply the Sumudu transform method for solving various fractional differential equations arising in fluid mechanics.

Theorem 1. Let $1 < \alpha < 2$ and $a,\ b \,\epsilon\, R$ Then the fractional differential equation

$$y''(t) + a y^{(\alpha)}(t) + b y(t) = f(t) \tag{14}$$

with the initial conditions $y(0) = c_0$ and $y''(0) = c_1$ has its solution given by

$$\begin{aligned} y(t) = {} & c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(\mathrm{r}+\mathrm{k}+1)}{r!} \frac{\left(-\mathrm{at}^{2-\alpha}\right)^{\mathrm{r}}}{\Gamma[(2-\alpha)\mathrm{r}+2\mathrm{k}+1]} \\ & + c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(\mathrm{r}+\mathrm{k}+1)}{r!} \frac{\left(-\mathrm{at}^{2-\alpha}\right)^{\mathrm{r}}}{\Gamma[(2-\alpha)\mathrm{r}+2\mathrm{k}+2]} \\ & + a c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+2-\alpha}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(\mathrm{r}+\mathrm{k}+1)}{r!} \frac{\left(-\mathrm{at}^{2-\alpha}\right)^{\mathrm{r}}}{\Gamma[(2-\alpha)\mathrm{r}+2\mathrm{k}+3-\alpha]} \\ & + a c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+3-\alpha}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(\mathrm{r}+\mathrm{k}+1)}{r!} \frac{\left(-\mathrm{at}^{2-\alpha}\right)^{\mathrm{r}}}{\Gamma[(2-\alpha)\mathrm{r}+2\mathrm{k}+4-\alpha]} \\ & + \sum_{k=0}^{\infty} (-b)^k \int_0^t \tau^{\alpha(k+1)-1} E^{k+1}_{2-\alpha,\alpha(k+1)}\left(-a\tau^{2-\alpha}\right) f(t-\tau)\, d\tau \end{aligned} \tag{15}$$

Proof: Applying the Sumudu transform both sides in Eq. (14), we get

$$\frac{1}{u^2} S[y] - \frac{1}{u^2} y(0) - \frac{1}{u} y'(0) + a \frac{1}{u^{(\alpha)}} S[y] - \frac{a}{u^{(\alpha)}} y(0) - \frac{a}{u^{(\alpha-1)}} y'(0) + bS[y] = \overline{F}(u)$$

$$\left(\frac{1}{u^2} + \frac{a}{u^\alpha} + b\right) S[y] = \frac{c_0}{u^2} + \frac{c_1}{u} + \frac{a c_0}{u^{(\alpha)}} + \frac{a c_1}{u^{(\alpha-1)}} + \overline{F}(u)$$

$$S[y] = \frac{\frac{c_0}{u^2} + \frac{c_1}{u} + \frac{a c_0}{u^{(\alpha)}} + \frac{a c_1}{u^{(\alpha-1)}} + \overline{F}(u)}{\frac{1}{u^2} + \frac{a}{u^\alpha} + b}$$

$$\begin{aligned} S[y] = {} & \frac{c_0}{u^2} \sum_{k=0}^{\infty} (-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r} (-a)^r u^{(2-\alpha)r+2k+2} \\ & + \frac{c_1}{u^2} \sum_{k=0}^{\infty} (-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r} (-a)^r u^{(2-\alpha)r+2k+2} \\ & + \frac{a c_0}{u^\alpha} \sum_{k=0}^{\infty} (-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r} (-a)^r u^{(2-\alpha)r+2k+2} \\ & + \frac{a c_1}{u^{\alpha-1}} \sum_{k=0}^{\infty} (-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r} (-a)^r u^{(2-\alpha)r+2k+2} \\ & + \frac{u\overline{F}(u)}{u\left(u^{-2} + a u^{-\alpha} + b\right)} \end{aligned}$$

$$\begin{aligned} S[y] = {} & c_0 \sum_{k=0}^{\infty} (-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r} (-a)^r u^{(2-\alpha)r+2k} \\ & + c_1 \sum_{k=0}^{\infty} (-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r} (-a)^r u^{(2-\alpha)r+2k+1} \\ & + a c_0 \sum_{k=0}^{\infty} (-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r} (-a)^r u^{(2-\alpha)r+2k-\alpha+2} \\ & + a c_1 \sum_{k=0}^{\infty} (-b)^k \sum_{r=0}^{\infty} \binom{k+r}{r} (-a)^r u^{(2-\alpha)r+2k-\alpha+3} \\ & + u\overline{F}(u) \frac{1}{u\left(u^{-2} + a u^{-\alpha} + b\right)} \end{aligned} \tag{16}$$

Since

$$\begin{aligned} \frac{1}{\frac{1}{u^2} + \frac{a}{u^\alpha} + b} &= \frac{u^\alpha}{u^{\alpha-2} + a + b u^\alpha} = \frac{u^\alpha}{\left(a + u^{\alpha-2}\right)\left(1 + \frac{b u^\alpha}{a + u^{\alpha-2}}\right)} \\ &= \frac{u^\alpha}{\left(a + u^{\alpha-2}\right)} \left(1 + \frac{b u^\alpha}{a + u^{\alpha-2}}\right)^{-1} \\ &= \frac{u^\alpha}{\left(a + u^{\alpha-2}\right)} \left[\sum_{k=0}^{\infty} \left(-\frac{b u^\alpha}{a + u^{\alpha-2}}\right)^k\right] \\ &= \frac{u^\alpha}{\left(a + u^{\alpha-2}\right)} \sum_{k=0}^{\infty} (-b)^k \left(\frac{u^\alpha}{a + u^{\alpha-2}}\right)^k \\ &= \sum_{k=0}^{\infty} (-b)^k \frac{u^{\alpha+\alpha k}}{\left(a + u^{\alpha-2}\right)^{k+1}} \\ &= \sum_{k=0}^{\infty} (-b)^k \frac{u^{2k+2}}{\left(1 + a u^{2-\alpha}\right)^{k+1}} \\ &= \sum_{k=0}^{\infty} (-b)^k u^{2k+2} \sum_{r=0}^{\infty} \binom{k+r}{r} \left(-a u^{2-\alpha}\right)^r \end{aligned}$$

$$= \sum_{k=0}^{\infty} (-b)^k \sum_{r=0}^{\infty} \left(\frac{k+r}{r}\right) (-a)^r u^{(2-\alpha)r+2k+2} \tag{17}$$

Now, taking the inverse Sumudu transform both sides of the Eq. (16), and using results from Eq. (9), we obtain the desired result (15).

If we set $f(t) = 0$ in theorem 1, then our result reduces in the following interesting results given in the form of corollary.

Corollary 1.1. Consider the fractional differential equation is [45]

$$y''(t) + ay^{(\alpha)}(t) + by(t) = 0 \tag{18}$$

where $1 < \alpha < 2$; $a, b \in R$ with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ is given by

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)}{r!} \frac{\left(-at^{2-\alpha}\right)^r}{\Gamma[(2-\alpha)r+2k+1]} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)}{r!} \frac{\left(-at^{2-\alpha}\right)^r}{\Gamma[(2-\alpha)r+2k+2]} \\ &+ ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+2-\alpha}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)}{r!} \frac{\left(-at^{2-\alpha}\right)^r}{\Gamma[(2-\alpha)r+2k+3-\alpha]} \\ &+ ac_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+3-\alpha}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)}{r!} \frac{\left(-at^{2-\alpha}\right)^r}{\Gamma[(2-\alpha)r+2k+4-\alpha]}. \end{aligned} \tag{19}$$

If we set $\alpha = \frac{3}{2}$ in the Eq. (18), then we get the following result derived by [46].

Corollary 1.2. Consider the fractional differential equation of a generalized viscoelastic free damping oscillation is

$$y''(t) + ay^{\left(\frac{3}{2}\right)}(t) + by(t) = 0 \tag{20}$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ has its solution given as

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)}{r!} \frac{\left(-at^{1/2}\right)^r}{\Gamma\left[\frac{1}{2}r+2k+1\right]} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)}{r!} \frac{\left(-at^{1/2}\right)^r}{\Gamma\left[\frac{1}{2}r+2k+2\right]} \\ &+ ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+2-\alpha}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)}{r!} \frac{\left(-at^{1/2}\right)^r}{\Gamma\left[\frac{1}{2}r+2k+\frac{3}{2}\right]} \\ &+ ac_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+3-\alpha}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)}{r!} \frac{\left(-at^{1/2}\right)^r}{\Gamma\left[\frac{1}{2}r+2k+\frac{5}{2}\right]}. \end{aligned} \tag{21}$$

If we put $a = \sqrt{3}$ and $b = 8$ in the Eq. (20), then we get the following result.

Corollary 1.3. The fractional differential equation

$$y''(t) + \sqrt{3}y^{\left(\frac{3}{2}\right)}(t) + 8y(t) = 0 \tag{22}$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ has its solution given as

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)}{r!} \frac{\left(-\sqrt{3}t^{1/2}\right)^r}{\Gamma\left[\frac{1}{2}r+2k+1\right]} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)}{r!} \frac{\left(-\sqrt{3}t^{1/2}\right)^r}{\Gamma\left[\frac{1}{2}r+2k+2\right]} \\ &+ \sqrt{3}c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+2-\alpha}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)}{r!} \frac{\left(-\sqrt{3}t^{1/2}\right)^r}{\Gamma\left[\frac{1}{2}r+2k+\frac{3}{2}\right]} \\ &+ \sqrt{3}c_1 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+3-\alpha}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)}{r!} \frac{\left(-\sqrt{3}t^{1/2}\right)^r}{\Gamma\left[\frac{1}{2}r+2k+\frac{5}{2}\right]}. \end{aligned} \tag{23}$$

Theorem 2. Let $1 < \alpha < 2$ *and* $ab \in R$. then the fractional differential equation

$$y^{(\alpha)}(t) + ay'(t) + by(t) = f(t) \tag{24}$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ has its solution given as

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{\alpha k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)}{r!} \frac{\left(-at^{\alpha-1}\right)^r}{\Gamma[(\alpha-1)r+\alpha k+1]} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{\alpha k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)}{r!} \frac{\left(-at^{\alpha-1}\right)^r}{\Gamma[(\alpha-1)r+\alpha k+2]} \\ &+ ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{\alpha k+\alpha-1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(r+k+1)}{r!} \frac{\left(-at^{\alpha-1}\right)^r}{\Gamma[(\alpha-1)r+\alpha k+\alpha]} \\ &+ \sum_{k=0}^{\infty} (-b)^k \int_0^t \tau^k E_{\alpha-1,\ k+1}^{k+1}\left(-a\tau^{\alpha-1}\right) f(t-\tau)\, d\tau \end{aligned} \tag{25}$$

Proof: Operating with the Sumudu transform of both sides in the Eq. (24), we get

$$S\left[y^{(\alpha)}(t)\right] + aS\left[y'(t)\right] + bS[y(t)] = S[f(t)]$$

$$\left(\frac{1}{u^\alpha} + \frac{a}{u} + b\right) S[y(t)] = \frac{c_0}{u^\alpha} + \frac{c_1}{u^{\alpha-1}} + \frac{ac_0}{u} + \overline{F}(u)$$

That is,

$$S[y(t)] = \frac{\frac{c_0}{u^\alpha} + \frac{c_1}{u^{\alpha-1}} + \frac{ac_0}{u} + \overline{F}(u)}{\left(\frac{1}{u^\alpha} + \frac{a}{u} + b\right)}$$

$$S[y(t)] = \frac{c_0}{u^\alpha} \sum_{k=0}^{\infty} (-b)^k \sum_{r=0}^{\infty} \begin{pmatrix} k+r \\ r \end{pmatrix} (-a)^r u^{\alpha k+\alpha+\alpha r-r}$$

$$+\frac{c_1}{u^{\alpha-1}}\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r u^{\alpha k+\alpha+\alpha r-r}$$
$$+\frac{ac_0}{u}\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r u^{\alpha k+\alpha+\alpha r-r}$$
$$+u\overline{F}(u)\frac{1}{u\left(u^{-\alpha}+au^{-1}+b\right)}$$

$$S[y(t)]=c_0\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r u^{\alpha k+\alpha r-r}$$
$$+c_1\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r u^{\alpha k+\alpha r-r+1}$$
$$+ac_0\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r u^{\alpha k+\alpha+\alpha r-r-1}$$
$$+u\overline{F}(u)\frac{1}{u\left(u^{-\alpha}+au^{-1}+b\right)} \tag{26}$$

since

$$\frac{1}{\left(\frac{1}{u^\alpha}+\frac{a}{u}+b\right)}=\frac{1}{{}^1\!/_u\left({}^1\!/_{u^{\alpha-1}}+a+bu\right)}$$
$$=\frac{u}{\left(u^{1-\alpha}+a+bu\right)}=\frac{u}{\left(a+u^{1-\alpha}\right)\left(1+\frac{bu}{a+u^{1-\alpha}}\right)}$$
$$=\frac{u}{\left(a+u^{1-\alpha}\right)}\left(1+\frac{bu}{a+u^{1-\alpha}}\right)^{-1}$$
$$=\frac{u}{\left(a+u^{1-\alpha}\right)}\sum_{k=0}^{\infty}\left(\frac{-bu}{a+u^{1-\alpha}}\right)^k$$
$$=\frac{u}{\left(a+u^{1-\alpha}\right)}\sum_{k=0}^{\infty}(-b)^k\frac{u^k}{\left(a+u^{1-\alpha}\right)^k}$$
$$=\sum_{k=0}^{\infty}(-b)^k\frac{u^{\alpha k+\alpha}}{\left(a+u^{1-\alpha}\right)^{k+1}}$$
$$=\sum_{k=0}^{\infty}(-b)^k u^{\alpha k+\alpha}\left(a+u^{1-\alpha}\right)^{-(k+1)}$$
$$\frac{1}{\left(\frac{1}{u^\alpha}+\frac{a}{u}+b\right)}=\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r u^{\alpha k+\alpha+\alpha r-r} \tag{27}$$

Now, taking the inverse Sumudu transform both sides of the Eq. (26), and using results from Eq. (9), we obtain the desired result (25).

If we set $f(t)=0$ in theorem 2, then our result reduces in the following interesting results given in the form of corollary.

Corollary 2.1. Consider the fractional differential equation is [45]

$$y^{(\alpha)}(t)+ay'(t)+by(t)=0 \tag{28}$$

with the initial conditions $y(0)=c_0$ and $y'(0)=c_1$ has its solution given as

$$y(t)=c_0\sum_{k=0}^{\infty}\frac{(-b)^k t^{\alpha k}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(\mathrm{r}+\mathrm{k}+1)}{\mathrm{r}!}\frac{\left(-\mathrm{a}t^{\alpha-1}\right)^{\mathrm{r}}}{\Gamma[(\alpha-1)\mathrm{r}+\alpha\mathrm{k}+1]}$$
$$+c_1\sum_{k=0}^{\infty}\frac{(-b)^k t^{\alpha k+1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(\mathrm{r}+\mathrm{k}+1)}{\mathrm{r}!}\frac{\left(-\mathrm{a}t^{\alpha-1}\right)^{\mathrm{r}}}{\Gamma[(\alpha-1)\mathrm{r}+\alpha\mathrm{k}+2]}$$
$$+ac_0\sum_{k=0}^{\infty}\frac{(-b)^k t^{\alpha k+\alpha-1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(\mathrm{r}+\mathrm{k}+1)}{\mathrm{r}!}\frac{\left(-\mathrm{a}t^{\alpha-1}\right)^{\mathrm{r}}}{\Gamma[(\alpha-1)\mathrm{r}+\alpha\mathrm{k}+\alpha]}. \tag{29}$$

If we set $\alpha=\frac{3}{2}$, $a=-1$ and $b=-2$ in Eq. (28), then our result reduces in the following interesting results given in the form of corollary.

Corollary2.2. The fractional differential equation is

$$y^{\left(\frac{3}{2}\right)}(t)-y^{'(t)}-2y(t)=0 \tag{30}$$

with the initial conditions $y(0)=c_0$ and $y'(0)=c_1$ has its solution given as

$$y(t)=c_0\sum_{k=0}^{\infty}\frac{2^k t^{\frac{3}{2}k}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(\mathrm{r}+\mathrm{k}+1)}{\mathrm{r}!}\frac{\mathrm{t}^{\frac{\mathrm{r}}{2}}}{\Gamma\left[\frac{\mathrm{r}}{2}+\frac{3}{2}\mathrm{k}+1\right]}$$
$$+c_1\sum_{k=0}^{\infty}\frac{2^k t^{\frac{3}{2}k+1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(\mathrm{r}+\mathrm{k}+1)}{\mathrm{r}!}\frac{\mathrm{t}^{\frac{\mathrm{r}}{2}}}{\Gamma\left[\frac{\mathrm{r}}{2}+\frac{3}{2}\mathrm{k}+2\right]}$$
$$-c_0\sum_{k=0}^{\infty}\frac{2^k t^{\frac{3}{2}k+\frac{1}{2}}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(\mathrm{r}+\mathrm{k}+1)}{\mathrm{r}!}\frac{\mathrm{t}^{\frac{\mathrm{r}}{2}}}{\Gamma\left[\frac{\mathrm{r}}{2}+\frac{3}{2}\mathrm{k}+\frac{3}{2}\right]} \tag{31}$$

Theorem 3. Let $0<\alpha<1$ and $b\,\epsilon R$. then the fractional differential equation

$$y^{(\alpha)}(t)-by(t)=f(t) \tag{32}$$

with the initial condition $y(0)=c_0$ has its solution given as

$$y(t)=c_0E_{\alpha,1}\left(bt^\alpha\right)+\sum_{k=0}^{\infty}\int_0^t\tau^{\alpha-1}E_{\alpha,\alpha}\left(b\tau^\alpha\right)f(t-\tau)d\tau. \tag{33}$$

Proof: Applying Sumudu transform of both sides in the Eq. (32), we get

$$S\left[y^{(\alpha)}(t)\right]-bS\left[y(t)\right]=S\left[f(t)\right]$$
$$\left(\frac{1}{u^\alpha}-b\right)S\left[y(t)\right]=\frac{c_0}{u^\alpha}+\overline{F}(u)$$
$$S\left[y(t)\right]=\frac{c_0}{u^\alpha\left(\frac{1}{u^\alpha}-b\right)}+\frac{u\overline{F}(u)}{u\left(\frac{1}{u^\alpha}-b\right)}$$
$$=c_0\sum_{k=0}^{\infty}b^k u^{\alpha k}+u\overline{F}(u)\sum_{k=0}^{\infty}b^k u^{\alpha k+(\alpha-1)}. \tag{34}$$

Now, taking the inverse Sumudu transform both sides of the Eq. (34), and using results from equation (3), we obtain the desired result (33).

If we set $f(t) = 0$ in theorem 3, then our result reduces in the following interesting results given in the form of corollary.

Corollary 3.1. Consider the fractional differential equation is [45]

$$y^{(\alpha)}(t) - by(t) = 0 \tag{35}$$

where $0 < \alpha < 1$ and $b \epsilon R$ with the initial condition $y(0) = c_0$ has its solution given as

$$y(t) = c_0 E_{\alpha,1}\left(bt^{\alpha}\right). \tag{36}$$

If we set $a = 0$ in Eq. (28), then we arrive at the following result given as

Corollary 3.2. Consider the fractional differential equation is

$$y^{(\alpha)}(t) + by(t) = 0 \tag{37}$$

where $1 < \alpha \leq 2$ and $b \epsilon R$ with the initial condition $y(0) = c_0 \, and \, y'(0) = c_1$ has its solution given as

$$y(t) = c_0 E_{\alpha,1}\left(-bt^{\alpha}\right) + c_1 t E_{\alpha,2}\left(-bt^{\alpha}\right). \tag{38}$$

If we set $b = \omega^2$ Eq. (37), then we get the following result derived by [46].

Corollary 3.3 A nearly simple harmonic vibration equation by [46]

$$y^{(\alpha)}(t) + \omega^2 y(t) = 0 \tag{39}$$

where $1 < \alpha \leq 2$ and $b \epsilon R$ with the initial condition $y(0) = c_0 \, and \, y'(0) = c_1$ has its solution given as

$$y(t) = c_0 E_{\alpha,1}\left(-\omega^2 t^{\alpha}\right) + c_1 t E_{\alpha,2}\left(-\omega^2 t^{\alpha}\right). \tag{40}$$

4 Conclusions

In this paper, we have presented a solution of a fractional differential equation. The solution has been developed in terms of the generalized Mittag-Leffler form with the help of Sumudu transform and its inverse after deriving the related formulae for fractional integrals, and derivatives. The modifications to Coimbra's proposals carried out here by developing and discussing an alternate mathematical model to psychoanalyse the behaviour of a viscous viscoelastic damping system represents just an example of what needs to be done to increase our general understanding and use of these concepts.

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