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On the Piecewise Linear Exact Solution

Abstract: In this paper we will show how the exact frequency response of the piecewise linear vibration isolator can be reduced to be found by solving only two transcendental equations. Adopting a new nondimensionalization method, the mathematical modeling of the system is presented and the mathematics to determine the exact steady-state response of the system is explained.

Keywords: Piecewise linear systems; Nonlinear vibration isolator; Frequency response; Exact methods; Numerical methods.

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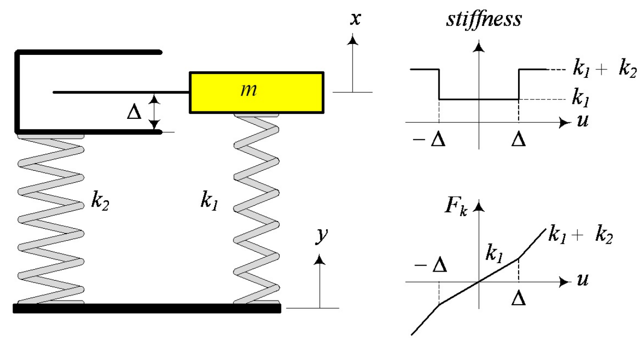

Fig. 1. A dual rate biliear stiffness vibrating system.

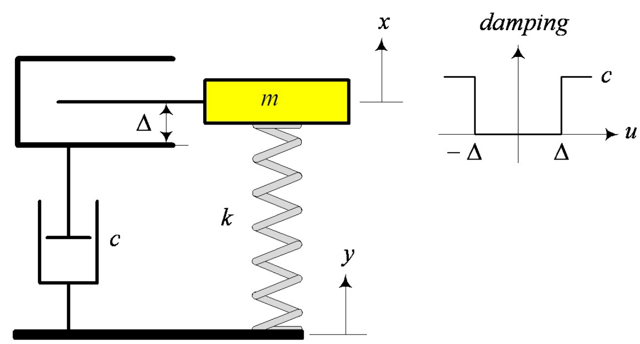

Fig. 2. A vibrating system with gap-damper engagement.

1 Introduction

It is a known fact that to provide a good vibration isolator, the isolator suspension must be as soft as possible, [1]; however, soft suspension provides high relative displacement, [2]. To use a soft suspension while protecting the system from high relative displacement, two passive smart systems including hydraulic engine mounts [3–10] and piecewise linear vibration isolators [11–16] were introduced to provide a tough suspension at high amplitudes, both with a variety of designs, [9, 10, 16]. Piecewise linear is a vibration isolator whose system has nonlinear geometric characteristics of stiffness and damping, where nonlinearity is a result of moving among finite number of liner segments, [15]. These systems potentially can model and a variety of practical applications [17–19]. The piecewise linear systems are difficult to analyze as sudden changes in system parameters display unprecedented complex behavior.

The first analytic investigation of the piecewise linear vibration isolator was based on an undamped bilinear stiffness system as is shown in Figure 1, [20, 21]. Gurtin was the first investigator who analyzed a damped bilinear systems [22]. He presented an approximate method to solve the frequency response of the piecewise linear vibration isolator with dual behavior damping, as is shown in Figure 2, using an equivalent viscous damping.

Although the closed form solution of piecewise linear suspensions are not easy to determine, the simplicity of the system made experiments easy and affordable. Therefore, it is usually easy to validate analytic calculations. Besides the validation of any mathematical analysis, scientists were also successful to discover some interesting behaviors in the system which were not possible to be determined analytically [22–29].

Natsiavas was the scientist who systematically applied the required conditions to the analytic solutions of the system in different domains [30, 31]. Employing perturbation analysis, some investigators developed analytical

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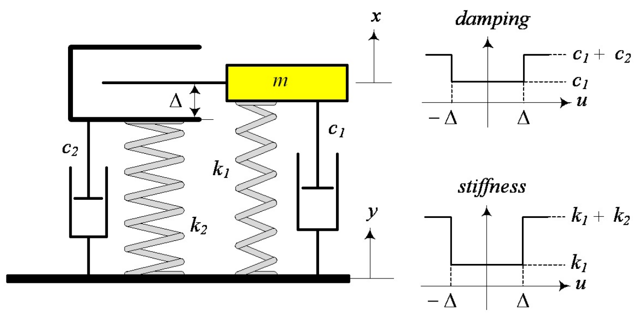

Fig. 3. Mechanical model of the piecewise linear system with symmetric constraints.

equations to calculate approximate frequency response of piecewise linear system [12–14].

The simplest practical model of a piecewise linear vibration isolator is a system with bilinear stiffness and dumping characteristics as is shown in Figure 3. The first spring and damper that are directly attached to the mass m is the Primary Suspension and the second stage, which is effective beyond the clearance amplitude Δ, is the Secondary Suspension. The clearance Δ acts as a switch to engage and disengage the secondary suspension. The clearance represents a sudden change in the system properties providing a hard nonlinearity. In this investigation the examined system is subjected to a periodic base excitation $y(t)$ with period $T > 0$.

$$y = Y \sin(\omega t - \varphi) \tag{1.1}$$

The equations governing the motion of the system could be written as

$$m\ddot{x} + g_1(x, \dot{x}) = f_1(y, \dot{y}) \tag{1.2}$$

where $g_1(x, \dot{x})$ and $f_1(y, \dot{y})$ are piecewise linear functions presenting sudden changing characteristics of the system and sudden changing excitation respectively:

$$g_1(x, \dot{x}) = \begin{cases} (c_1 + c_2)\dot{x} + (k_1 + k_2)x - k_2\Delta & x - y > \Delta \\ c_1\dot{x} + k_1 x & |x - y| < \Delta \\ (c_1 + c_2)\dot{x} + (k_1 + k_2)x + k_2\Delta & x - y < -\Delta \end{cases} \tag{1.3}$$

$$f_1(y, \dot{y}) = \begin{cases} (c_1 + c_2)\dot{y} + (k_1 + k_2)y & x - y > \Delta \\ c_1\dot{y} + k_1 y & |x - y| < \Delta \\ (c_1 + c_2)\dot{y} + (k_1 + k_2)y & x - y < -\Delta \end{cases} \tag{1.4}$$

The equation of motion for the system shown in Figure 3 may also be written in a nondimensional form:

$$w'' + 2\xi_1 w' + w = -v'' + f\left(w, w'\right) \tag{1.5}$$

$$f\left(w, w'\right) = \begin{cases} -2\rho\xi_2 w' - \rho^2 w + \rho^2\delta & w > \delta \\ 0 & |w| < \delta \\ -2\rho\xi_2 w' - \rho^2 w - \rho^2\delta & w < -\delta \end{cases} \tag{1.6}$$

$$v = \frac{y}{Y} = \sin(r\tau - \varphi) \tag{1.7}$$

where,

$$w = u - v = \frac{z}{Y} \qquad z = x - y \qquad u = \frac{x}{Y} \qquad v = \frac{y}{Y}$$
$$\delta = \frac{\Delta}{Y} \qquad \omega_1^2 = \frac{k_1}{m} \qquad \omega_2^2 = \frac{k_2}{m}$$
$$\frac{k_1 + k_2}{m} = \omega_1^2 + \omega_2^2 \qquad 2\xi_1\omega_1 = \frac{c_1}{m} \qquad 2\xi_2\omega_2 = \frac{c_2}{m}$$
$$\frac{c_1 + c_2}{m} = 2\xi_1\omega_1 + 2\xi_2\omega_2 \qquad w' = \frac{dw}{d\tau} \qquad \dot{w} = \frac{dw}{dt}$$
$$\tau = \omega_1 t \qquad r = \frac{\omega}{\omega_1}\rho = \frac{\omega_2}{\omega_1} \qquad \xi_1 < 1 \qquad \xi_2 < 1$$

and therefore,

$$w'' + 2\xi_1 w' + w = r^2 \sin(r\tau - \varphi) + f\left(w, w'\right) \tag{1.8}$$

We seek the frequency response of the system (1.5) by developing and detecting its exact steady state time response.

2 Exact solution

While the relative displacement of m is less than Δ, the system is a linear single valued parameters one degree-of-freedom base exciting system with well known time and frequency responses [2]. However, there is no closed form solution for the system when the relative displacement exceeds Δ. The reason for not having a closed form solution is that when $x > \Delta$ then in every cycle of the motion the mass m will be supported by k_1 and c_1 for a part of the the motion and be supported by $k_1 + k_2$ and $c_1 + c_2$ for the the other part. The four times sudden changes of the stiffness and damping of the system in every cycle makes it impossible to generate a closed form solution [16].

To determine the frequency response of the system, we must search for a possible steady state time response of the system and detect its maximum amplitude for a given excitation frequency ω. Repeating this method ends up with a series of amplitudes for different values of excitation frequencies.

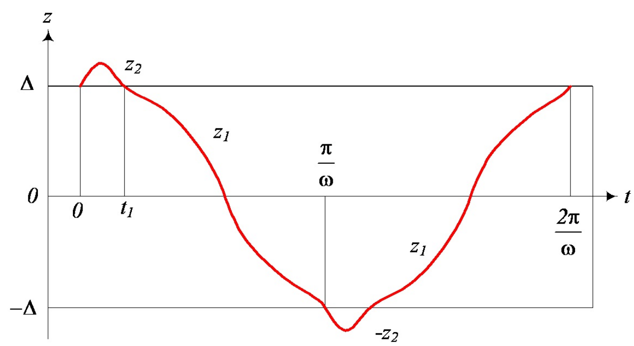

Fig. 4. A steady state periodic response of the system.

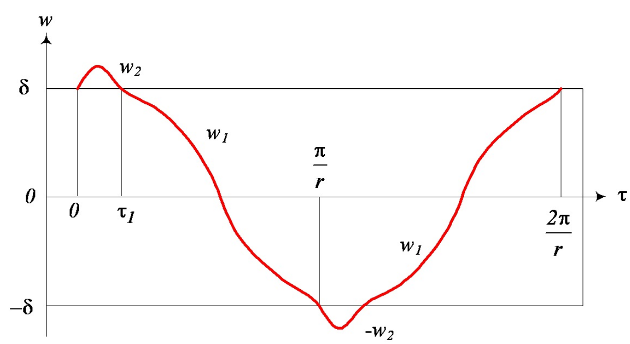

Fig. 5. A steady state periodic response of the system in nondimensional space.

A steady-state periodic response of the system for $x > \Delta$ will be similar to Figure 4. To ensure that the secondary suspension is engaged, we seek a periodic solution of the system (1.5) with the following initial conditions:

$$z(0) = \Delta \qquad \dot{z}(0) > 0 \tag{2.1}$$

To set the time axis to begin at this condition, we introduce a phase lag φ in the excitation function y.

$$y = Y \sin(\omega t - \varphi) \tag{2.2}$$

We also assume that there is a steady state periodic response with exactly the same frequency ω as the the excitation frequency, the in the first half of the period passes through $z = \Delta$ with $\dot{z}(0) < 0$ at a time $t = t_1$, $0 \leqslant t_1 \leqslant 2\pi/\omega$. Therefore, the solution will be $|z(t)| = \Delta$ at $t = 0$, $t = t_1$, $t = \pi/\omega$, $t = \pi/\omega + t_1$ and $t = 2\pi/\omega$ in the period starting at $t = 0$. Let us call the solution in domain $z > \Delta$ as $z_2(t)$ and the solution in domain $|z| < \Delta$ as $z_1(t)$ as are shown in 4. The continuity and compatibility of $z_1(t)$ and $z_2(t)$ need the following conditions:

$$z_2(0) = \Delta \qquad \dot{z}_2(0) = p_2 \qquad z_2(t_1) = \Delta \qquad \dot{z}_2(t_1) = p_1$$
$$z_1(t_1) = \Delta \quad \dot{z}_1(t_1) = p_1 \quad z_1\left(\frac{\pi}{\omega}\right) = -\Delta \quad \dot{z}_1\left(\frac{\pi}{\omega}\right) = -p_2$$
$$p = \frac{dz}{dt} \tag{2.3}$$

Employing the same assumptions, we would have $|w(\tau)| = \delta$ at $\tau = 0$, $\tau = \tau_1$, $\tau = \pi/r$, $\tau = \pi/r + \tau_1$ and $\tau = 2\pi/r$ in the period starting at $\tau = 0$. In the nondimensional form, we name the solution in domain $w > \delta$ as $w_2(\tau)$ and the solution in domain $|w| < \delta$ as $w_1(\tau)$ as are shown in 5. The continuity and compatibility of $w_1(\tau)$ and $w_2(\tau)$ need the following conditions:

$$w_2(0) = \delta \quad \dot{w}_2(0) = q_2 \quad w_2(\tau_1) = \delta \quad \dot{w}_2(\tau_1) = q_1$$
$$w_1(\tau_1) = \delta \quad \dot{w}_1(\tau_1) = q_1 \quad w_1\left(\frac{\pi}{r}\right) = -\delta \quad \dot{w}_1\left(\frac{\pi}{r}\right) = -q_2$$
$$q = \frac{dw}{d\tau} = \frac{1}{\omega_1 Y}\frac{dz}{dt} \tag{2.4}$$

Assuming underdamped solutions, $w_2(\tau)$ and $w_1(\tau)$ are:

$$\begin{aligned} w_1(\tau) &= v\left(A_1 \sin\left(r_{d_1}\tau\right) + B_1 \cos\left(r_{d_1}\tau\right)\right) \\ &\quad + Q_1 \sin(r\tau - \varphi) + Q_2 \cos(r\tau - \varphi) \\ &= e^{-\xi_1\tau}\left(A_1 \sin\left(r_{d_1}\tau\right) + B_1 \cos\left(r_{d_1}\tau\right)\right) \\ &\quad + C_1 \sin r\tau + D_1 \cos r\tau \\ r_{d_1} &= \sqrt{1 - \xi_1^2} \end{aligned} \tag{2.5}$$

$$\begin{aligned} w_2(\tau) &= e^{-(\xi_1+\rho\xi_2)\tau}\left(A_2 \sin\left(r_{d_2}\tau\right) + B_2 \cos\left(r_{d_2}\tau\right)\right) \\ &\quad + Q_3 \sin(r\tau - \varphi) + Q_4 \cos(r\tau - \varphi) + \frac{\rho^2}{1+\rho^2}\delta \\ &= e^{-(\xi_1+\rho\xi_2)\tau}\left(A_2 \sin\left(r_{d_2}\tau\right) + B_2 \cos\left(r_{d_2}\tau\right)\right) \\ &\quad + C_2 \sin(r\tau) + D_2 \cos(r\tau) + \frac{\rho^2}{1+\rho^2}\delta \\ r_{d_2} &= \sqrt{(1+\rho^2) - (\xi_1 + \rho\xi_2)^2} \end{aligned} \tag{2.6}$$

and therefore,

$$\begin{aligned} \dot{w}_1(t) &= -\xi_1 e^{-\xi_1\tau}\left(A_1 \sin\left(r_{d_1}\tau\right) + B_1 \cos\left(r_{d_1}\tau\right)\right) \\ &\quad + r_{d_1} e^{-\xi_1\tau}\left(A_1 \cos\left(r_{d_1}\tau\right) - B_1 \sin\left(r_{d_1}\tau\right)\right) \\ &\quad + r\left(C_1 \cos(r\tau) - D_1 \sin(r\tau)\right) \end{aligned} \tag{2.7}$$

$$\begin{aligned} \dot{w}_2(t) &= -(\xi_1 + \rho\xi_2)\, e^{-(\xi_1+\rho\xi_2)\tau}\left(A_2 \sin\left(r_{d_2}\tau\right) + B_2 \cos\left(r_{d_2}\tau\right)\right) \\ &\quad + r_{d_2} e^{-(\xi_1+\rho\xi_2)\tau}\left(A_2 \cos\left(r_{d_2}\tau\right) - B_2 \sin\left(r_{d_2}\tau\right)\right) \\ &\quad + r\left(C_2 \cos(r\tau) - D_2 \sin(r\tau)\right) \end{aligned} \tag{2.8}$$

The coefficients A_1, B_1, A_2, B_2 will be found by imposing the initial and compatibility conditions (2.4). The coefficients C_1, D_1, C_2, D_2 depend on the forcing function and will be found by collecting the coefficients of $\sin(r\tau)$ and $\cos(r\tau)$ in $w_1(\tau)$ and $w_2(\tau)$

$$C_1 = Q_1 \cos\varphi + Q_2 \sin\varphi \tag{2.9}$$
$$D_1 = Q_2 \cos\varphi - Q_1 \sin\varphi \tag{2.10}$$

$$C_2 = Q_3 \cos\varphi + Q_4 \sin\varphi \tag{2.11}$$
$$D_2 = Q_4 \cos\varphi - Q_3 \sin\varphi \tag{2.12}$$

where,

$$Q_1 = \frac{r^2\left(1-r^2\right)}{\left(1-r^2\right)^2 + \left(2\xi_1 r\right)^2} \tag{2.13}$$
$$Q_2 = \frac{-2\xi_1 r^3}{\left(1-r^2\right)^2 + \left(2\xi_1 r\right)^2} \tag{2.14}$$

$$Q_3 = \frac{r^2\left(1-r^2+\rho^2\right)}{\left(1+\rho^2-r^2\right)^2 + \left(2\left(\xi_1+\rho\xi_2\right)r\right)^2} \tag{2.15}$$
$$Q_4 = \frac{-2\left(\xi_1+\rho\xi_2\right)r^3}{\left(1+\rho^2-r^2\right)^2 + \left(2\left(\xi_1+\rho\xi_2\right)r\right)^2} \tag{2.16}$$

Imposing the eight boundary conditions (2.4) on equations (2.5)-(2.8) produces eight transcendental equations with eight unknowns $A_2, B_2, A_1, B_1, t_1, q_1, q_2$ and φ:

$$\begin{aligned} &w_2(0) = \delta \quad \dot{w}_2(0) = q_2 \quad w_2(\tau_1) = \delta \quad \dot{w}_2(\tau_1) = q_1 \\ &w_1(\tau_1) = \delta \quad \dot{w}_1(\tau_1) = q_1 \quad w_1\left(\frac{\pi}{r}\right) = -\delta \quad \dot{w}_1\left(\frac{\pi}{r}\right) = -q_2 \end{aligned} \tag{2.17}$$

$$q = \frac{dw}{d\tau} = \frac{1}{\omega_1 Y}\frac{dz}{dt} \tag{2.18}$$

$$B_2 + D_2 = \frac{1}{1+\rho^2}\delta \tag{2.19}$$

$$-\left(\xi_1+\rho\xi_2\right)B_2 + r_{d_2}A_2 + rC_2 = q_2 \tag{2.20}$$

$$\begin{aligned} &e^{-(\xi_1+\rho\xi_2)\tau_1}\left(A_2 \sin\left(r_{d_2}\tau_1\right) + B_2\cos\left(r_{d_2}\tau_1\right)\right) \\ &+ C_2\sin(r\tau_1) + D_2\cos(r\tau_1) = \frac{1}{1+\rho^2}\delta \end{aligned} \tag{2.21}$$

$$\begin{aligned} &-\left(\xi_1+\rho\xi_2\right)e^{-(\xi_1+\rho\xi_2)\tau_1}\left(A_2 \sin\left(r_{d_2}\tau_1\right) + B_2\cos\left(r_{d_2}\tau_1\right)\right) \\ &+r_{d_2}e^{-(\xi_1+\rho\xi_2)\tau_1}\left(A_2 \cos\left(r_{d_2}\tau_1\right) - B_2\sin\left(r_{d_2}\tau_1\right)\right) \\ &+r\left(C_2\cos(r\tau_1) - D_2\sin(r\tau_1)\right) = q_1 \end{aligned} \tag{2.22}$$

$$\begin{aligned} &e^{-\xi_1\tau_1}\left(A_1 \sin\left(r_{d_1}\tau_1\right) + B_1\cos\left(r_{d_1}\tau_1\right)\right) \\ &+C_1\sin r\tau_1 + D_1\cos r\tau_1 = \delta \end{aligned} \tag{2.23}$$

$$\begin{aligned} &-\xi_1 e^{-\xi_1\tau_1}\left(A_1 \sin\left(r_{d_1}\tau_1\right) + B_1\cos\left(r_{d_1}\tau_1\right)\right) \\ &+r_{d_1}e^{-\xi_1\tau_1}\left(A_1 \cos\left(r_{d_1}\tau_1\right) - B_1\sin\left(r_{d_1}\tau_1\right)\right) \\ &+r\left(C_1\cos(r\tau_1) - D_1\sin(r\tau_1)\right) = q_1 \end{aligned} \tag{2.24}$$

$$e^{-\xi_1\frac{\pi}{r}}\left(A_1\sin\left(r_{d_1}\frac{\pi}{r}\right) + B_1\cos\left(r_{d_1}\frac{\pi}{r}\right)\right) - D_1 = -\delta \tag{2.25}$$

$$\begin{aligned} &-\xi_1 e^{-\xi_1\frac{\pi}{r}}\left(A_1\sin\left(r_{d_1}\frac{\pi}{r}\right) + B_1\cos\left(r_{d_1}\frac{\pi}{r}\right)\right) \\ &+r_{d_1}e^{-\xi_1\frac{\pi}{r}}\left(A_1\cos\left(r_{d_1}\frac{\pi}{r}\right) - B_1\sin\left(r_{d_1}\frac{\pi}{r}\right)\right) - C_1 r = -q_2 \end{aligned} \tag{2.26}$$

The eight unknowns A_2, B_2, A_1, B_1, τ_1, q_1, q_2 and φ can be found for a set of given system and excitation by solving the following two transcendental couple equations for τ_1 and φ.

$$\begin{aligned} &\left(Q_7\sin\left(r_{d_2}\tau_1\right) + Q_8\cos\left(r_{d_2}\tau_1\right)\right) \\ &+C_2\sin(r\tau_1) + D_2\cos(r\tau_1) = \frac{1}{1+\rho^2}\delta \end{aligned} \tag{2.27}$$

$$\begin{aligned} &-Q_9\sin\left(r_{d_2}\tau_1\right) + Q_{10}\cos\left(r_{d_2}\tau_1\right) + Q_{13}\sin\left(r_{d_1}\tau_1\right) \\ &-Q_{14}\cos\left(r_{d_1}\tau_1\right) + r\left(\left(C_2 - C_1\right)\cos(r\tau_1)\right. \\ &\left.-\left(D_2 - D_1\right)\sin(r\tau_1)\right) = 0 \end{aligned} \tag{2.28}$$

Proof. For a given system parameters and a given excitation function, the value of the parameters Q_1, Q_2, Q_3, Q_4 are known. Having Q_1, Q_2, Q_3, Q_4, the coefficients C_1, D_1, C_2, D_2 are only functions of the phase angle φ. The parameters B_2 and A_2 may be found from the equations (2.19) and (2.20),

$$B_2 = \frac{1}{1+\rho^2}\delta - D_2 \tag{2.29}$$

$$A_2 = \frac{q_2 - rC_2 + B_2\left(\xi_1+\rho\xi_2\right)}{r_{d_2}} = Q_5 - \frac{1}{r_{d_2}}\frac{\delta\rho^2}{1+\rho^2}\left(\xi_1+\rho\xi_2\right) \tag{2.30}$$

and the parameters B_1 and A_1 from the equations (2.25) and (2.26)

$$B_1 = e^{\xi_1\frac{\pi}{r}}\left(Q_6\sin\left(\frac{r_{d_1}\pi}{r}\right) + \left(D_1-\delta\right)\cos\left(\frac{r_{d_1}\pi}{r}\right)\right) \tag{2.31}$$

$$A_1 = e^{\xi_1\frac{\pi}{r}}\left(-Q_6\cos\left(\frac{r_{d_1}\pi}{r}\right) + \left(D_1-\delta\right)\sin\left(\frac{r_{d_1}\pi}{r}\right)\right) \tag{2.32}$$

where,

$$Q_5 = \frac{\left(\delta - D_2\right)\left(\xi_1+\rho\xi_2\right) - C_2 r + q_2}{r_{d_2}} \tag{2.33}$$

$$Q_6 = \frac{(\delta - D_1)\,\xi_1 - C_1 r + q_2}{r_{d_1}} \tag{2.34}$$

Now, substitution of A_1, B_1, A_2 and B_2 in (2.21)-(2.24) produces four equations for q_1, q_2, τ_1 and φ

$$\begin{aligned}&\left(Q_7 \sin\left(r_{d_2}\tau_1\right) + Q_8 \cos\left(r_{d_2}\tau_1\right)\right)\\ &\quad + C_2 \sin(r\tau_1) + D_2 \cos(r\tau_1) = \frac{1}{1+\rho^2}\delta\end{aligned} \tag{2.35}$$

$$\begin{aligned}&-Q_9 \sin\left(r_{d_2}\tau_1\right) + Q_{10} \cos\left(r_{d_2}\tau_1\right)\\ &\quad + r\left(C_2 \cos(r\tau_1) - D_2 \sin(r\tau_1)\right) = q_1\end{aligned} \tag{2.36}$$

$$\begin{aligned}&Q_{11} \sin\left(r_{d_1}\tau_1\right) + Q_{12} \cos\left(r_{d_1}\tau_1\right)\\ &\quad + C_1 \sin(r\tau_1) + D_1 \cos(r\tau_1) = \delta\end{aligned} \tag{2.37}$$

$$\begin{aligned}&-Q_{13} \sin\left(r_{d_1}\tau_1\right) + Q_{14} \cos\left(r_{d_1}\tau_1\right)\\ &\quad + r\left(C_1 \cos(r\tau_1) - D_1 \sin(r\tau_1)\right) = q_1\end{aligned} \tag{2.38}$$

where,

$$Q_7 = A_2 e^{-(\xi_1+\rho\xi_2)\tau_1} \qquad Q_8 = B_2 e^{-(\xi_1+\rho\xi_2)\tau_1} \tag{2.39}$$

$$Q_9 = e^{-(\xi_1+\rho\xi_2)\tau_1}\left((\xi_1+\rho\xi_2)\,A_2 + r_{d_2} B_2\right) \tag{2.40}$$

$$Q_{10} = e^{-(\xi_1+\rho\xi_2)\tau_1}\left(r_{d_2} A_2 - (\xi_1+\rho\xi_2)\,B_2\right) \tag{2.41}$$

$$Q_{11} = A_1 e^{-\xi_1\tau_1} \qquad Q_{12} = B_1 e^{-\xi_1\tau_1} \tag{2.42}$$

$$Q_{13} = e^{-\xi_1\tau_1}\left(\xi_1 A_1 + r_{d_1} B_1\right) \tag{2.43}$$

$$Q_{14} = e^{-\xi_1\tau_1}\left(r_{d_1} A_1 - \xi_1 B_1\right) \tag{2.44}$$

The parameter q_1 can be eliminated between Equations (2.36) and (2.38) by

$$\dot{w}_1(\tau_1) = \dot{w}_2(\tau_1) \tag{2.45}$$

to reduce the number of equations to three to find q_2, τ_1 and φ.

$$\begin{aligned}&\left(Q_7 \sin\left(r_{d_2}\tau_1\right) + Q_8 \cos\left(r_{d_2}\tau_1\right)\right)\\ &\quad + C_2 \sin(r\tau_1) + D_2 \cos(r\tau_1) = \frac{1}{1+\rho^2}\delta\end{aligned} \tag{2.46}$$

$$\begin{aligned}&-Q_9 \sin\left(r_{d_2}\tau_1\right) + Q_{10} \cos\left(r_{d_2}\tau_1\right) + Q_{13} \sin\left(r_{d_1}\tau_1\right)\\ &\quad - Q_{14} \cos\left(r_{d_1}\tau_1\right) + r\left((C_2 - C_1)\cos(r\tau_1)\right.\\ &\quad \left. - (D_2 - D_1)\sin(r\tau_1)\right) = 0\end{aligned} \tag{2.47}$$

$$\begin{aligned}&Q_{11} \sin\left(r_{d_1}\tau_1\right) + Q_{12} \cos\left(r_{d_1}\tau_1\right)\\ &\quad + C_1 \sin(r\tau_1) + D_1 \cos(r\tau_1) = \delta\end{aligned} \tag{2.48}$$

The parameter q_2 is embedded in A_2, B_1, and A_1. Therefore, Q_5 to Q_7 and Q_9 to Q_{14} are functions of q_2. Substituting Q_{11} and Q_{12} in Equation (2.48) produces a linear equation of Q_6

$$Q_{15} Q_6 + Q_{16} = \delta \tag{2.49}$$

therefore,

$$Q_6 = \frac{\delta - Q_{16}}{Q_{15}} \tag{2.50}$$

where

$$Q_{15} = e^{\left(\frac{\pi}{r}-\tau_1\right)\xi_1} \sin\left(\frac{\pi}{r} - \tau_1\right) r_{d_1} \tag{2.51}$$

$$\begin{aligned}Q_{16} &= e^{\left(\frac{\pi}{r}-\tau_1\right)\xi_1}\left(C_1 \sin(r\tau_1) + D_1 \cos(r\tau_1)\right)\\ &\quad + e^{\left(\frac{\pi}{r}-\tau_1\right)\xi_1}(D_1 - \delta)\cos\left(\frac{\pi}{r} - \tau_1\right) r_{d_1}\end{aligned} \tag{2.52}$$

Q_6 is solvable for q_2

$$\begin{aligned}q_2 &= rC_1 + (D_1 - \delta)\,\xi_1 - Q_6 r_{d_1}\\ &= rC_1 + (D_1 - \delta)\,\xi_1 - \frac{\delta - Q_{16}}{Q_{15}} r_{d_1}\end{aligned} \tag{2.53}$$

and therefore, eliminating q_2 in Equations (2.46) and (2.47) reduces the number of transcendental equations to two to determine t_1 and φ for a set of given system and excitation.

$$\begin{aligned}&\left(Q_7 \sin\left(r_{d_2}\tau_1\right) + Q_8 \cos\left(r_{d_2}\tau_1\right)\right)\\ &\quad + C_2 \sin(r\tau_1) + D_2 \cos(r\tau_1) = \frac{1}{1+\rho^2}\delta\end{aligned} \tag{2.54}$$

$$\begin{aligned}&-Q_9 \sin\left(r_{d_2}\tau_1\right) + Q_{10} \cos\left(r_{d_2}\tau_1\right) + Q_{13} \sin\left(r_{d_1}\tau_1\right)\\ &\quad - Q_{14} \cos\left(r_{d_1}\tau_1\right) + r\left((C_2 - C_1)\cos(r\tau_1)\right.\\ &\quad \left. - (D_2 - D_1)\sin(r\tau_1)\right) = 0\end{aligned} \tag{2.55}$$

□

3 Steady-state time response

The primary suspension's frequency response is

$$Z_1 = \sqrt{C_1^2 + D_1^2} = \frac{r^2}{\sqrt{\left(1 - r^2\right)^2 + \left(2\xi_1 r\right)^2}} \tag{3.1}$$

The peak value of Z_1 is Z_P

$$Z_P = \frac{1}{2\sqrt{\xi_1^2 - \xi_1^4}} \tag{3.2}$$

that happens at

$$r_P = \frac{1}{\sqrt{1 - 2\xi_1^2}} \tag{3.3}$$

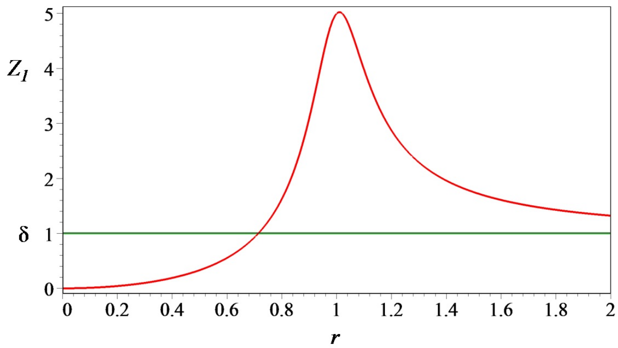

Fig. 6. The frequency response curve of the primary or $\omega_1 = 1$, $\delta = 1$, $\xi_1 = 0.11$.

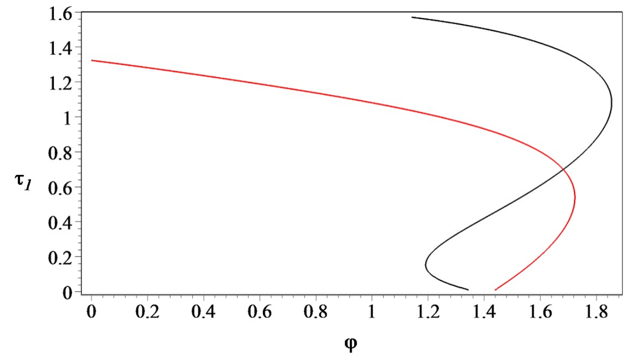

Fig. 7. The implicit plot of the two transendental equations (2.46) and (2.47) as functions of τ_1 and φ for $\omega_2 = 2$, $\xi_2 = 0.1$, $\omega_1 = 1$, $\delta = 1$, $\xi_1 = 0.1$, $r = 1$.

While $\delta < Z_P$, there is a frequency span in which the secondary suspension engages. No engagement is expected outside the frequency span. If $\delta > 1$, then the engagement will begin after a frequency and never ends.

Without loosing generality we may assume

$$\omega_1 = 1 \tag{3.4}$$

and set

$$\delta = 1 \qquad \xi_1 = 0.1 \tag{3.5}$$

and plot the associated frequency response curve of the primary in Figure 6. The amplitude of the primary will greater than $Z_1 = \delta$ when the excitation frequency is

$$r > 0.7142857143 \tag{3.6}$$

To express the method and determine a sample of steady state time response, let us set the secondary suspension parameters as

$$\omega_2 = 2 \qquad \xi_2 = 0.1 \tag{3.7}$$

Substituting the system characteristics we should solve the two equations of (2.46) and (2.47) t_1 and φ. Figure 7 illustrates the implicit plot of the equations, with a unique solution at

$$\varphi = -0.4856710316 \qquad t_1 = 1.383067511 \tag{3.8}$$

Employing t_1 and φ we calculate

$$q_1 = -0.7345298322 \qquad q_2 = 0.9020194017$$

and determine $w_2(t)$ and $w_1(t)$ to plot the the steady state solution as shown in Figure 8. The steady state response of the system is as shown in Figure 9 for

$$r = 1.4 \tag{3.9}$$

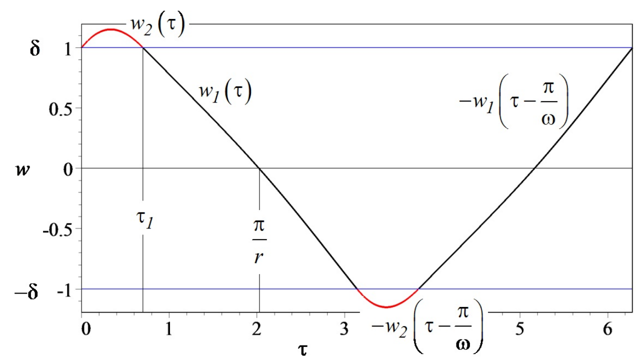

Fig. 8. A sample of the steady response of the system for $\omega_2 = 2$, $\xi_2 = 0.1$, $\omega_1 = 1$, $\delta = 1$, $\xi_1 = 0.1$, $r = 1$.

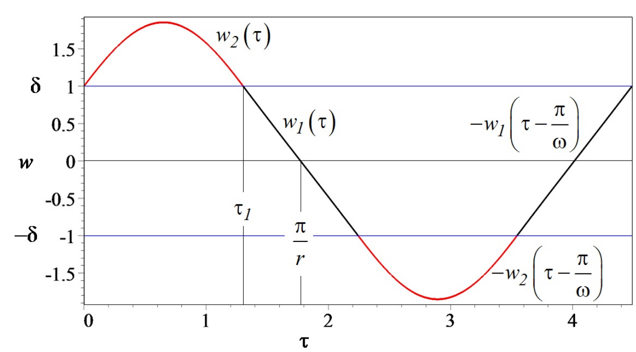

Fig. 9. A sample of the steady response of the system for $\omega_2 = 2$, $\xi_2 = 0.1$, $\omega_1 = 1$, $\delta = 1$, $\xi_1 = 0.1$, $r = 1.4$.

4 Conclusion

Dynamic systems with a sudden change in any parameters are among highly nonlinear system with time and frequency responses hard to calculate. Around 1990, a method has been introduced to calculate the steady-state time response of piecewise linear suspension. The method was based on a set of eight transcendental equations that appeared as a result of imposing eight boundary conditions to synchronize and make compatible the exact solutions for primary and secondary. In this article we have shown how the eight equations can systematically be reduced to two transcendental equations. The equations, that are based on a new nondimensionalization method, have been solved for a sample system and the steady-state time response examined. Having only two equations simplifies many of the numerical problems, narrows down the domain of the possible solutions, reduces the number of multiple solutions, and reduces the calculation time.

5 Key Symbols

A_i	sin coefficient in $w_i(t)$
B_i	cos coefficient in $w_i(t)$
c	damping coefficient [N s / m]
c_1	damping coefficient of the primary suspension [N s / m]
c_2	damping coefficient of the secondary suspension [N s / m]
C_i	sine coefficient in $w_i(t)$
D_i	cosine coefficient in $w_i(t)$
k	stiffness [N / m]
k_1	stiffness of the primary suspension [N / m]
k_2	stiffness of the secondary suspension [N / m]
m	mass [kg]
q_1	speed of m when $w_1 = \delta$
q_2	speed of m when $w_2 = \delta$
Q_i	short notation of mathematical expression
r	$= \omega/\omega_1$,excitation frequency ratio
r_{d_1}	damped natural frequency of the primary suspension
r_{d_2}	damped natural frequency of the secondary suspension
t	time [s]
t_1	time when $w_2 = w_1 = +\delta$ [s]
v	dimensionless base excitation amplitude
w	dimensionless relative displacement
w_1	time response of the system when $w < \delta$
w_2	time response of the system when $w > \delta$
x	absolute displacement of m [m]
y	base excitation function [m]
Y	base excitation amplitude [m]
z	relative displacement [m]
z_1	relative displacement when $z < \Delta$
z_2	relative displacement when $z > \Delta$
Z_1	frequency response of the primary suspension
δ	dimensionless gap distance
Δ	gap distance
q_1	velocity in t_1
q_2	velocity in $z_2(t) = 0$
ξ	damping ratio
ξ_1	damping ratio of the primary suspension
ξ_2	damping ratio of the secondary suspension
ρ	$= \omega_2/\omega_1$, natural frequency frequency ratio
φ	phase lag
τ	$= \omega_1 t$, dimensionless time
τ_1	dimensionless t_1
ω	excitation frequency [1/ s]
ω_1	natural frequency of the primary suspension
ω_2	natural frequency of the secondary suspension
ω_{d_i}	damped natural frequency

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