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On the solution of fuzzy differential equations by Fuzzy Sumudu Transform

Abstract: In this paper, Sumudu transform is advanced and designed for the solution of linear differential models with uncertainty. For this purpose, Sumudu transform is coupled with fuzzy theory and all its fundamental properties are formalized in fuzzy sense. At last, to demonstrate the accuracy of this approach, fuzzy Sumudu transform is employed to some examples of fuzzy linear differential equations considered under generalized H-differentiability and analytical solutions are obtained efficiently.

Keywords: Fuzzy-valued function; H-differentiability; fuzzy Sumudu transform

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1 Introduction

In the literature, there are numerous integral transforms extensively used to solve the differential equations in different fields of physics, engineering and astronomy. As a consequence, there are several works on the theory and application of integral transforms such as Laplace, Fourier, Mellin and Hankel, to name a few. Sequentially, of these transforms, in early 1990, Watugala [1] introduced a new integral transform, named “Sumudu transform”. Sumudu transform was primarily shown to be a theoretical dual of the Laplace transform in [2], eventually, many authors studied its properties, coupled it with other theories and established efficient and straightforward methodologies for treating ordinary and partial differential equations in various mathematical and physical sciences problems. Belgacem et al. [3, 4] established fundamental properties of Sumudu transform, and applied it to Maxwell’s equations [5]. Eltayeb et al. [6, 7] discussed existence and uniqueness of Sumudu transform and applied this transform to solve the system of differential equations. Singh et al. [8], Gupta et al. [9], Katatbeh et al. [10] and many others [11–15] elaborated its applications on different fractional differential models. Solutions of integral equations using this transform are found in the work of Asiru [16, 17]. Similarly, many others used this transformation for investigations of several problems.

The key purpose of the following study is to show the applicability and accuracy of Sumudu transform in solving the fuzzy linear differential equations. Fuzzy theory plays an important role in modeling dynamical systems under uncertainty. Fuzzy linear differential equations are one of the simplest fuzzy differential models which have significant importance in many applications [18–26]. Here fuzzy differential equations are considered under generalized Hukuhara differentiability. Hukuhara derivative was presented by Hukuhara [27]. H-derivative of a fuzzy-valued function is also found in paper of Puri et al. [26].

The present work is organized in sections where some necessary preliminaries of fuzzy and calculus theory are explained in Section 2, basic descriptions of fuzzy Sumudu and fuzzy Laplace transforms and detailed derivation of properties of fuzzy Sumudu transform are elaborated in Section 3. In Section 4, several examples are solved analytically using this approach, and at last its effective conclusion is drawn in the Section 5.

2 Preliminaries

In this section, basic definitions of fuzzy number, fuzzy Riemann-Liouville integral and H-differentiability of fuzzy-valued functions along with fundamental theorems of fuzzy differential equations are demonstrated.

2.1 Fuzzy Number

Fuzzy number is as a mapping $v : R \to [0, 1]$ with the properties of being upper semi continuous, convex, normal, and compactly supported, in a metric space E^1. The r-level of a fuzzy number $v \in E$, for $0 \le r \le 1$, is represented by an ordered pair of lower and upper bound

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interval i.e.$[v]^r = [\underline{v}(r), \overline{v}(r)]$ where lower bound is left continuous non-decreasing and upper bound is left continuous non-increasing functions on the interval $[0,1]$. (see [18, 19])

2.2 Riemann–Liouville Integral

Let $f(x)$ be a fuzzy valued function on $[a,\infty)$ represented by $\left[\underline{f}(x;r), \overline{f}(x;r)\right]$. For any fixed $r \in [0,1]$, let $\underline{f}(x;r)$ and $\overline{f}(x;r)$ are Riemann-integrable functions on $[a,b]$ for every $b \geq a$, if there exists two positive functions $\underline{M}(r)$ and $\overline{M}(r)$ such that $\int_a^b \left|\underline{f}(x;r)\right| dx \leq \underline{M}(r)$ and $\int_a^b \left|\overline{f}(x;r)\right| dx \leq \overline{M}(r)$ for every $b \geq a$, then $f(x)$ is said to be improper fuzzy Riemann-Liouville integrable function on $[a,\infty)$, i.e.

$$\int_a^\infty f(x)\,dx = \left[\int_a^\infty \underline{f}(x;r)\,dx, \int_a^\infty \overline{f}(x;r)\,dx\right], \quad 0 \leq r \leq 1 \tag{1}$$

Also found in Refs. [28, 29].

2.3 H-differentiability

Let $f:(a,b) \to E$, then f is said to be strongly generalized H-differentiable function at $x_0 \in (0,b)$, if there exists an element $f'(x_0) \in E$ such that for $h > 0$ and sufficiently close to zero:

1.

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)\,\Theta f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0)\,\Theta f(x_0-h)}{h} \tag{2}$$

2.

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0)\,\Theta f(x_0+h)}{(-h)} = \lim_{h\to 0} \frac{f(x_0-h)\,\Theta f(x_0)}{(-h)} \tag{3}$$

3.

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)\,\Theta f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0-h)\,\Theta f(x_0)}{(-h)} \tag{4}$$

4.

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0)\,\Theta f(x_0+h)}{(-h)} = \lim_{h\to 0} \frac{f(x_0)\,\Theta f(x_0-h)}{h} \tag{5}$$

Equivalent definition is found in Refs. [30, 31].

Theorem 2.3.1

Let $f:R \to E$ be a fuzzy valued function such that for $r \in [0,1]$, $[f(x)]^r = \left[\underline{f_r}(x), \overline{f_r}(x)\right]$

1. If $f(x)$ is differentiable function in first form i.e. (1)-differentiable, then

$$\left[f'(x)\right]^r = \left[\underline{f_r'}(x), \overline{f_r'}(x)\right] \tag{6}$$

2. If $f(x)$ is a differentiable function in second form i.e. (2)-differentiable, then

$$\left[f'(x)\right]^r = \left[\overline{f_r'}(x), \underline{f_r'}(x)\right] \tag{7}$$

2.4 *n*-th order Fuzzy linear differential equations

Consider the following *nth* order fuzzy linear differential equation under generalized H-differentiability, proposed in Allahviranloo et al. [29], as:

$$\begin{aligned} y^{(n)}(x) + \phi\left(x, y(x), y'(x), \ldots, y^{(n-1)}(x)\right) \\ = \psi\left(x, y(x), y'(x), \ldots, y^{(n-1)}(x)\right) \end{aligned} \tag{8}$$

where $n \in Z$, with initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_0', \ldots y^{(n-1)}(x_0) = y_0^{(n-1)} \tag{9}$$

where $y(x) = \left(\underline{y}(x;r), \overline{y}(x;r)\right)$ is a fuzzy-valued function of x, with linear fuzzy-valued functions $\psi\left(x, y(x), y'(x), \ldots, y^{(n-1)}(x)\right)$, $\phi\left(x, y(x), y'(x), \ldots, y^{(n-1)}(x)\right)$.

Theorem 2.4.1

Let $x_0 \in (a,b)$ and $f : [a,b] \times E \times E \to E$ is continuous fuzzy-valued function. Also, assume that $f(x), f(x), f'(x), \ldots$ are continuous, then Definition 2.3 can be restated for *nth*-order differential of f as:

f is strongly generalized H-differentiable of the *nth*-order at x_0, if there exists an element $f^{(n)}(x_0) \in E$, such that for $h > 0$ and sufficiently near zero:

1.

$$\begin{aligned} f^{(n)}(x_0) &= \lim_{h\to 0} \frac{f^{(n-1)}(x_0+h)\,\Theta f^{(n-1)}(x_0)}{h} \\ &= \lim_{h\to 0} \frac{f^{(n-1)}(x_0)\,\Theta f^{(n-1)}(x_0-h)}{h} \end{aligned} \tag{10}$$

2.

$$\begin{aligned} f^{(n)}(x_0) &= \lim_{h\to 0} \frac{f^{(n-1)}(x_0)\,\Theta f^{(n-1)}(x_0+h)}{(-h)} \\ &= \lim_{h\to 0} \frac{f^{(n-1)}(x_0-h)\,\Theta f^{(n-1)}(x_0)}{(-h)} \end{aligned} \tag{11}$$

3.

$$\begin{aligned} f^{(n)}(x_0) &= \lim_{h\to 0} \frac{f^{(n-1)}(x_0+h)\,\Theta f^{(n-1)}(x_0)}{h} \\ &= \lim_{h\to 0} \frac{f^{(n-1)}(x_0-h)\,\Theta f^{(n-1)}(x_0)}{(-h)} \end{aligned} \tag{12}$$

4.

$$f^{(n)}(x_0) = \lim_{h\to 0} \frac{f^{(n-1)}(x_0)\,\Theta f^{(n-1)}(x_0+h)}{(-h)} = \lim_{h\to 0} \frac{f^{(n-1)}(x_0)\,\Theta f^{(n-1)}(x_0-h)}{h} \tag{13}$$

Similar to the Theorem 2.3.1 we have the following results for *nth*-order strongly generalized H-differentiability of fuzzy-valued function.

Theorem 2.4.2

Let $f(x), f'(x), ..., f^{(n-1)}(x)$ are differentiable fuzzy-valued functions, with r-cut representation $[f(x)]^r = \left[\underline{f_r}(x), \overline{f^r}(x)\right]$:

1. If $f(x)$ and $f'(x), ..., f^{(n-1)}(x)$ are (1)-differentiable, then
$$\left[f^{(n)}(x)\right]^r = \left[\underline{f_r^{(n)}}(x), \overline{f_r^{(n)}}(x)\right] \tag{14}$$
2. If $f(x)$ and $f'(x), ..., f^{(n-1)}(x)$ are (2)-differentiable, then
$$\left[f^{(n)}(x)\right]^r = \left[\underline{f_r^{(n)}}(x), \overline{f_r^{(n)}}(x)\right] \tag{15}$$
3. If $f(x)$ is (1)-differentiable and $f'(x), ..., f^{(n-1)}(x)$ are (2)-differentiable, then
$$\left[f^{(n)}(x)\right]^r = \left[\overline{f_r^{(n)}}(x), \underline{f_r^{(n)}}(x)\right] \tag{16}$$
4. If $f(x)$ is (2)-differentiable and $f'(x), ..., f^{(n-1)}(x)$ are (1)-differentiable, then
$$\left[f^{(n)}(x)\right]^r = \left[\overline{f_r^{(n)}}(x), \underline{f_r^{(n)}}(x)\right] \tag{17}$$

3 Fuzzy Sumudu Transform

Here, we present definitions, notations of Sumudu, fuzzy Sumudu and fuzzy Laplace transforms and fundamental properties of fuzzy Sumudu transformation.

3.1 Sumudu Transformation

Let A be the set of functions defined as:

$$A = \Big\{ f(x) \;\mid\; \exists M, \tau_1, \text{ and/or } \tau_2 > 0, \text{ such that } |f(x)| < Me^{|x|/\tau_j}, \text{ if } x \in (-1)^j \times [0,\infty) \Big\}$$

where constant M must be finite, while τ_1 and τ_2 each may be finite and do not need to exist simultaneously. Using u to factor the variable x in the argument of the function f, Sumudu transform is defined as follows:

$$\mathbf{G}(u) = \mathbf{S}[f(x)] = \begin{cases} \int_0^\infty f(ux)\,e^{-x}dx & 0 \le u < \tau_2 \\ \int_0^\infty f(ux)\,e^{-x}dx, & -\tau_1 < u \le 0 \end{cases} \tag{18}$$

Here M is taken equal to 1, τ_2 is finite and τ_1 is simply not needed. Both parts define the domain of f, and sign of variable x will remain unchanged. (see [1–6])

3.2 Fuzzy Sumudu Transformation

Let f be a continuous fuzzy-valued function defined in parametric form $[f(x)]^r = \left[\underline{f}(x;r), \overline{f}(x;r)\right]$ for $0 \le r \le 1$. Suppose that $f(ux)\,e^{-x}$ is improper fuzzy Riemann integrable function on $[0,\infty)$, with $u > 0$ a real parameter, then

$$S[f(x)] = \int_0^\infty f(ux)\,e^{-x}dx = G(u) \tag{19}$$

is called fuzzy Sumudu transform. Since f is fuzzy-valued function, therefore parametric representation of Eq. (19) will be, for $0 \le r \le 1$:

$$\int_0^\infty f(ux)\,e^{-x}dx = \left[\int_0^\infty \underline{f}(ux;r)\,e^{-x}dx, \int_0^\infty \overline{f}(ux;r)\,e^{-x}dx\right] \tag{20}$$

or

$$S[f(x)] = \left[S\left[\underline{f}(x;r)\right], S\left[\overline{f}(x;r)\right]\right] = \left[\underline{G}(u), \overline{G}(u)\right] = G(u) \tag{21}$$

where

$$\underline{G}(u) = S\left[\underline{f}(x;r)\right] = \int_0^\infty \underline{f}(ux;r)\,e^{-x}dx,$$

$$\text{and } \overline{G}(u) = S\left[\overline{f}(x;r)\right] = \int_0^\infty \overline{f}(ux;r)\,e^{-x}dx \tag{22}$$

3.3 Fuzzy Laplace Transformation

Let $f(x)$ be the fuzzy-valued function which vanishes for negative values of x. Then fuzzy Laplace transform of $f(x)$ is defined by the following expression: (see [28])

$$L[f(x)] = \int_0^\infty e^{-ux}f(x)\,dx = F(u) \tag{23}$$

With parametric form

$$\int_0^\infty e^{-ux}f(x)\,dx = \left[\int_0^\infty e^{-ux}\underline{f}(x;r)\,dx, \int_0^\infty e^{-ux}\overline{f}(x;r)\,dx\right], \quad 0 \le r \le 1 \tag{24}$$

or

$$L[f(x)] = \left[L\left[\underline{f}(x;r)\right], L\left[\overline{f}(x;r)\right]\right] = \left[\underline{F}(u), \overline{F}(u)\right] = F(u) \tag{25}$$

where

$$\underline{F}(u) = L\left[\underline{f}(x;r)\right] = \int_0^\infty e^{-ux}\underline{f}(x;r)\,dx,$$

$$\overline{F}(u) = L\left[\overline{f}(x;r)\right] = \int_0^\infty e^{-ux}\overline{f}(x;r)\,dx \quad (26)$$

provided that integral exists.

Theorem 3.1

If fuzzy Laplace transform of a fuzzy- valued function $f(x)$ is $L[f(x)] = F(u)$ and fuzzy Sumudu transform of this function is $S[f(x)] = G(u)$, then for $0 \le r \le 1$,

$$G(u) = \frac{1}{u}F\left(\frac{1}{u}\right), \quad (27)$$

or

$$\underline{G}(u) = \frac{1}{u}\underline{F}\left(\frac{1}{u}\right), \qquad \overline{G}(u) = \frac{1}{u}\overline{F}\left(\frac{1}{u}\right) \quad (28)$$

Proof:

By the definition of fuzzy Sumudu transformation,

$$S[f(x)] = \left[S\left[\underline{f}(x;r)\right], S\left[\overline{f}(x;r)\right]\right] = \left[\underline{G}(u), \overline{G}(u)\right] = G(u) \quad (29)$$

Using Eq. (20) and changing variable $ux = x'$ and $dx = \frac{dx'}{u}$ we get:

$$G(u) = \left[\frac{1}{u}\int_0^\infty e^{-\left(\frac{x'}{u}\right)}\underline{f}\left(x';r\right)dx', \frac{1}{u}\int_0^\infty e^{-\left(\frac{x'}{u}\right)}\overline{f}\left(x';r\right)dx'\right] \quad (30)$$

Recalling the definition of fuzzy Laplace transform, above equation reduces to:

$$G(u) = \left[\frac{1}{u}\underline{F}\left(\frac{1}{u}\right), \frac{1}{u}\overline{F}\left(\frac{1}{u}\right)\right] \quad (31)$$

or

$$G(u) = \frac{1}{u}F\left(\frac{1}{u}\right) \quad (32)$$

Theorem 3.2

Let f be fuzzy-valued function, then fuzzy Sumudu transform of $f(ax;r)$, where a is a constant, is given by:

$$S[f(ax)] = S\left[\underline{f}(ax;r), \overline{f}(ax;r)\right] = \left[\underline{G}(au), \overline{G}(au)\right] = G(au), \qquad 0 \le r \le 1 \quad (33)$$

Proof:

Using Definition 3.2, fuzzy Sumudu transform of $\underline{f}(ax;r)$ and $\overline{f}(ax;r)$ are:

$$S\left[\underline{f}(ax;r), \overline{f}(ax;r)\right] = \left[\int_0^\infty e^{-x}\underline{f}(uax;r)\,dx, \int_0^\infty e^{-x}\overline{f}(uax;r)\,dx\right] = \left[\underline{G}(au), \overline{G}(au)\right] \quad (34)$$

proved.

Theorem 3.3

Let f be fuzzy-valued function. Then fuzzy Sumudu transform of $f(x-b)$, where b is a constant, is given by:

$$S[f(x-b)] = S\left[\underline{f}(x-b;r), \overline{f}(x-b;r)\right] = \left[e^{-b}\underline{G}(u),\ e^{-b}\overline{G}(u)\right] = e^{-b}G(u), \quad 0 \le r \le 1 \quad (35)$$

Proof:

Starting with Definition 3.2, fuzzy Sumudu transform of $\underline{f}(x-b;r)$ and $\overline{f}(x-b;r)$ are:

$$S\left[\underline{f}(x-b;r), \overline{f}(x-b;r)\right] = \left[\int_0^\infty e^{-x}\underline{f}(u(x-b);r)\,dx,\ \int_0^\infty e^{-x}\overline{f}(u(x-b);r)\,dx\right] \quad (36)$$

changing variable $x - b = x'$ and $dx = dx'$

$$S\left[\underline{f}\left(x';r\right), \overline{f}\left(x';r\right)\right] = \left[\int_0^\infty e^{-(x'+b)}\underline{f}\left(ux';r\right)dx', \int_0^\infty e^{-(x'+b)}\overline{f}\left(ux';r\right)dx'\right] \quad (37)$$

using exponential property in above equation we get:

$$S\left[\underline{f}\left(x';r\right), \overline{f}\left(x';r\right)\right] = \left[e^{-b}\int_0^\infty e^{-x'}\underline{f}\left(ux';r\right)dx', e^{-b}\int_0^\infty e^{-x'}\overline{f}\left(ux';r\right)dx'\right] \quad (38)$$

again using Definition 3.2

$$S\left[\underline{f}(x-b;r), \overline{f}(x-b;r)\right] = \left[e^{-b}\underline{G}(u),\ e^{-b}\overline{G}(u)\right] \quad (39)$$

or

$$S[f(x-b)] = e^{-b}G(u). \quad (40)$$

Theorem 3.4

Let f be fuzzy-valued function, then fuzzy Sumudu transform of $e^{-cx}f(x)$ is:

$$S\left[e^{-cx}f(x)\right] = \frac{1}{(1+cu)}G\left(\frac{u}{1+cu}\right), \qquad c \in \text{Real} \quad (41)$$

with lower and upper functions

$$S\left[e^{-cx}\underline{f}(x;r), e^{-cx}\overline{f}(x;r)\right] = \left[\frac{1}{(1+cu)}\underline{G}\left(\frac{u}{1+cu}\right), \frac{1}{(1+cu)}\overline{G}\left(\frac{u}{1+cu}\right)\right], \quad 0 \le r \le 1 \quad (42)$$

Proof:

Starting from the equalities in Definition 3.2, fuzzy Sumudu transform of $e^{-cx}\underline{f}(x;r)$ and $e^{-cx}\overline{f}(x;r)$ are:

$$S\left[e^{-cx}\underline{f}(x;r), e^{-cx}\overline{f}(x;r)\right] = \left[\int_0^\infty e^{-x}e^{-c(ux)}\underline{f}(ux;r)\,dx, \int_0^\infty e^{-x}e^{-c(ux)}\overline{f}(ux;r)\,dx\right] \tag{43}$$

yielding:

$$S\left[e^{-cx}\underline{f}(x;r), e^{-cx}\overline{f}(x;r)\right] = \left[\int_0^\infty e^{-(1+cu)x}\underline{f}(ux;r)\,dx, \int_0^\infty e^{-(1+cu)x}\overline{f}(ux;r)\,dx\right] \tag{44}$$

changing variable $(1+cu)x = x'$ and $(1+cu)\,dx = dx'$ we get:

$$S\left[e^{-cx}\underline{f}(x;r), e^{-cx}\overline{f}(x;r)\right] = \left[\frac{1}{1+cu}\int_0^\infty e^{-x'}\underline{f}\left(\frac{u}{1+cu}x';r\right)dx', \frac{1}{1+cu}\int_0^\infty e^{-x'}\overline{f}\left(\frac{u}{1+cu}x';r\right)dx'\right] \tag{45}$$

by the definition of fuzzy Sumudu transform, above equation reduces to:

$$S\left[e^{-cx}\underline{f}(x;r), e^{-cx}\overline{f}(x;r)\right] = \left[\frac{1}{(1+cu)}\underline{G}\left(\frac{u}{1+cu}\right), \frac{1}{(1+cu)}\overline{G}\left(\frac{u}{1+cu}\right)\right] \tag{46}$$

or

$$S\left[e^{-cx}f(x)\right] = \frac{1}{(1+cu)}G\left(\frac{u}{1+cu}\right), \tag{47}$$

Theorem 3.5

Let f be differentiable fuzzy-valued function. If $f(x) = f^{(0)}(x)$, and for $j \geq 1$, jth derivative of $f(x)$ is $f^{(j)}(x)$ then for $m \geq 1$:

$$S\left[f^{(m)}(x)\right] = \frac{S[f(x)]}{u^m} - \sum_{j=0}^{m-1}\frac{f^{(j)}(0)}{u^{m-j}}, \tag{48}$$

with lower and upper functions, for $0 \leq r \leq 1$,

$$S\left[\underline{f}^{(m)}(x;r), \overline{f}^{(m)}(x;r)\right] = \left[\frac{S\left[\underline{f}(x;r)\right]}{u^m} - \sum_{j=0}^{m-1}\frac{\underline{f}^{(j)}(0;r)}{u^{m-j}}, \frac{S\left[\overline{f}(x;r)\right]}{u^m} - \sum_{j=0}^{m-1}\frac{\overline{f}^{(j)}(0;r)}{u^{m-j}}\right] \tag{49}$$

Proof:

Let fuzzy Sumudu transform of $\underline{f}'(x;r)$ and $\overline{f}'(x;r)$, for $0 \leq r \leq 1$, are:

$$S\left[\underline{f}'(x;r)\right] = \int_0^\infty e^{-x}\underline{f}'(ux;r)dx = \frac{1}{u}\left[\frac{d}{dx}\int_0^\infty e^{-x}\underline{f}(ux;r)\,dx + \int_0^\infty e^{-x}\underline{f}(ux;r)\,dx\right] \tag{50}$$

and

$$S\left[\overline{f}'(x;r)\right] = \int_0^\infty e^{-x}\overline{f}'(ux;r)dx = \frac{1}{u}\left[\frac{d}{dx}\int_0^\infty e^{-x}\overline{f}(ux;r)\,dx + \int_0^\infty e^{-x}\overline{f}(ux;r)\,dx\right] \tag{51}$$

using fundamental theorem of calculus

$$S\left[\underline{f}'(x;r), \overline{f}'(x;r)\right] = \left[\frac{1}{u}\left\{S\left[\underline{f}(x;r)\right] + e^{-x}\underline{f}(ux;r)\Big|_0^\infty\right\}, \frac{1}{u}\left\{S\left[\overline{f}(x;r)\right] + e^{-x}\overline{f}(ux;r)\Big|_0^\infty\right\}\right] \tag{52}$$

since $\lim_{x\to\infty} e^{-x}f(ux;r) = 0$ hence,

$$S\left[\underline{f}'(x;r), \overline{f}'(x;r)\right] = \left[\frac{1}{u}\left\{S\left[\underline{f}(x;r)\right] - \underline{f}(0;r)\right\}, \frac{1}{u}\left\{S\left[\overline{f}(x;r)\right] - \overline{f}(0;r)\right\}\right] \tag{53}$$

Now second derivative of lower and upper functions is obtained as:

$$S\left[\left(\underline{f}'(x;r)\right)'\right] = \frac{1}{u}\left[S\left[\underline{f}'(x;r)\right] - \underline{f}'(0;r)\right] = \frac{1}{u}\left[\frac{S\left[\underline{f}(x;r)\right] - \underline{f}(0;r)}{u} - \underline{f}'(0;r)\right] \tag{54}$$

and

$$S\left[\left(\overline{f}'(x;r)\right)'\right] = \frac{1}{u}\left[S\left[\overline{f}'(x;r)\right] - \overline{f}'(0;r)\right] = \frac{1}{u}\left[\frac{S\left[\overline{f}(x;r)\right] - \overline{f}(0;r)}{u} - \overline{f}'(0;r)\right] \tag{55}$$

hence, we get:

$$S\left[\left(\underline{f}'(x;r)\right)', \left(\overline{f}'(x;r)\right)'\right] = \left[\frac{S\left[\underline{f}(x;r)\right]}{u^2} - \frac{\underline{f}(0;r)}{u^2} - \frac{\underline{f}'(0;r)}{u}, \frac{S\left[\overline{f}(x;r)\right]}{u^2} - \frac{\overline{f}(0;r)}{u^2} - \frac{\overline{f}'(0;r)}{u}\right] \tag{56}$$

and so on, for $m+1th$ order we have:

$$S\left[\underline{f}^{(m+1)}(x;r),\overline{f}^{(m+1)}(x;r)\right]=\left[\frac{S\left[\underline{f}^{(m)}(x;r)\right]-\underline{f}^{(m)}(0;r)}{u},\right.$$
$$\left.\frac{S\left[\overline{f}^{(m)}(x;r)\right]-\overline{f}^{(m)}(0;r)}{u}\right]=\left[\frac{S\left[\underline{f}(x;r)\right]}{u^{m+1}}-\sum_{j=0}^{m}\frac{\underline{f}^{(j)}(0;r)}{u^{m-j}},\right.$$
$$\left.\frac{S\left[\overline{f}(x;r)\right]}{u^{m+1}}-\sum_{j=0}^{m}\frac{\overline{f}^{(j)}(0;r)}{u^{m-j}}\right] \tag{57}$$

and

$$S\left[f^{(m+1)}(x)\right]=\frac{S\left[f^{(m)}(x)\right]-f^{(m)}(0)}{u}=\frac{S\left[f(x)\right]}{u^{m+1}}-\sum_{j=0}^{m}\frac{f^{(j)}(0)}{u^{m-j}} \tag{58}$$

or, for any $m\geq 1$:

$$S\left[\underline{f}^{(m)}(x;r),\overline{f}^{(m)}(x;r)\right]=\left[\frac{S\left[\underline{f}(x;r)\right]}{u^{m}}-\sum_{j=0}^{m-1}\frac{\underline{f}^{(j)}(0;r)}{u^{m-j}},\frac{S\left[\overline{f}(x;r)\right]}{u^{m}}-\sum_{j=0}^{m-1}\frac{\overline{f}^{(j)}(0;r)}{u^{m-j}}\right] \tag{59}$$

Theorem 3.6

Suppose that f is continuous fuzzy-valued function on $[0,\infty)$ and that f' is piecewise continuous fuzzy-valued function on $[0,\infty)$, then under strongly generalized H-differentiability:

If f is (1)-differentiable, then

$$S\left[f'(x)\right]=S\left[\underline{f}'(x;r),\overline{f}'(x;r)\right]=\frac{1}{u}\left\{S\left[f(x)\right]\Theta f(0)\right\},\quad 0\leq r\leq 1 \tag{60}$$

and, If f is (2)-differentiable, then

$$S\left[f'(x)\right]=S\left[\overline{f}'(x;r),\underline{f}'(x;r)\right]=\frac{1}{u}\left\{(-f(0))\,\Theta\,(-S\left[f(x)\right])\right\},\quad 0\leq r\leq 1 \tag{61}$$

In order to solve second order fuzzy-differential equations under strongly generalized H-differentiability, we need the fuzzy Sumudu transform of second order derivatives under generalized H-differentiability. In this connection, we prove the following result:

Theorem 3.7

Suppose that f and f' are continuous fuzzy-valued functions on $[0,\infty)$ and that f'' is piecewise continuous fuzzy-valued function on $[0,\infty)$, then from Theorem 3.5 for $0\leq r\leq 1$ and $m=2$:

If f and f' are (1)-differentiable, then

$$S\left[f''(x)\right]=S\left[\underline{f}''(x;r),\overline{f}''(x;r)\right]=\left\{\frac{S\left[f(x)\right]}{u^{2}}\Theta\frac{f(0)}{u^{2}}\Theta\frac{f'(0)}{u}\right\} \tag{62}$$

If f is (1)-differentiable and f' is (2)-differentiable, then

$$S\left[f''(x)\right]=S\left[\underline{f}''(x;r),\overline{f}''(x;r)\right]=\left\{-\frac{f(0)}{u^{2}}-\frac{f'(0)}{u}\Theta\frac{S\left[f(x)\right]}{-u^{2}}\right\} \tag{63}$$

If f is (2)-differentiable and f' is (1)-differentiable, then

$$S\left[f''(x)\right]=S\left[\underline{f}''(x;r),\overline{f}''(x;r)\right]=\left\{-\frac{f(0)}{u^{2}}\Theta\frac{f'(0)}{u}\Theta\frac{S\left[f(x)\right]}{-u^{2}}\right\} \tag{64}$$

If f and f' are (2)-differentiable, then

$$S\left[f''(x)\right]=S\left[\underline{f}''(x;r),\overline{f}''(x;r)\right]=\left\{\frac{S\left[f(x)\right]}{u^{2}}\Theta\frac{f(0)}{u^{2}}-\frac{f'(0)}{u}\right\} \tag{65}$$

Proof:

Let $\underline{f}'$ and $\underline{f}''$ are lower function's derivatives, $\overline{f}'$ and $\overline{f}''$ are upper function's derivatives. Now we prove Eq. (62) as, for arbitrary fixed $r\in[0,1)$:

$$\frac{S\left[f(x)\right]}{u^{2}}\Theta\frac{f(0)}{u^{2}}\Theta\frac{f'(0)}{u}=\left[\left\{\frac{S\left[\underline{f}(x;r)\right]}{u^{2}}-\frac{\underline{f}(0;r)}{u^{2}}-\frac{\underline{f}'(0;r)}{u}\right\},\left\{\frac{S\left[\overline{f}(x;r)\right]}{u^{2}}-\frac{\overline{f}(0;r)}{u^{2}}-\frac{\overline{f}'(0;r)}{u}\right\}\right] \tag{66}$$

since

$$\begin{cases} S\left[\underline{f}''(x;r)\right] & =S\left[\underline{f}''(x;r)\right]\\ & =\dfrac{S\left[\underline{f}(x;r)\right]}{u^{2}}-\dfrac{\underline{f}(0;r)}{u^{2}}-\dfrac{\underline{f}'(0;r)}{u}\\ S\left[\overline{f}''(x;r)\right] & =S\left[\overline{f}''(x;r)\right]\\ & =\dfrac{S\left[\overline{f}(x;r)\right]}{u^{2}}-\dfrac{\overline{f}(0;r)}{u^{2}}-\dfrac{\overline{f}'(0;r)}{u}\end{cases} \tag{67}$$

and, also since

$$\underline{f}'(0;r)=\underline{f}'(0;r),\qquad 0\leq r\leq 1 \tag{68}$$

$$\overline{f}'(0;r)=\overline{f}'(0;r),\qquad 0\leq r\leq 1 \tag{69}$$

we get,

$$\frac{S\left[f(x)\right]}{u^{2}}\Theta\frac{f(0)}{u^{2}}\Theta\frac{f'(0)}{u}=\left[S\left[\underline{f}''(x;r)\right],S\left[\overline{f}''(x;r)\right]\right] \tag{70}$$

$$\frac{S[f(x)]}{u^2}\Theta\frac{f(0)}{u^2}\Theta\frac{f'(0)}{u}=S\left[\underline{f}''(x;r),\overline{f}''(x;r)\right] \tag{71}$$

$$\frac{S[f(x)]}{u^2}\Theta\frac{f(0)}{u^2}\Theta\frac{f'(0)}{u}=S\left[f''(x)\right] \tag{72}$$

Now proving Eq. (63) for arbitrary fixed $r \in [0, 1)$,

$$\begin{aligned}&-\frac{f(0)}{u^2}-\frac{f'(0)}{u}\Theta\frac{S[f(x)]}{-u^2}\\ &=\left[\left\{-\frac{\overline{f}(0;r)}{u^2}-\frac{\overline{f}'(0;r)}{u}+\frac{S\left[\overline{f}(x;r)\right]}{u^2}\right\},\right.\\ &\left.\left\{-\frac{\underline{f}(0;r)}{u^2}-\frac{\underline{f}'(0;r)}{u}+\frac{S\left[\underline{f}(x;r)\right]}{u^2}\right\}\right]\end{aligned} \tag{73}$$

since,

$$\begin{cases} S\left[\overline{f}''(x;r)\right] & = S\left[\underline{f}''(x;r)\right] \\ & = \dfrac{S\left[\underline{f}(x;r)\right]}{u^2}-\dfrac{\underline{f}(0;r)}{u^2}-\dfrac{\underline{f}'(0;r)}{u} \\ S\left[\underline{f}''(x;r)\right] & = S\left[\overline{f}''(x;r)\right] \\ & = \dfrac{S\left[\overline{f}(x;r)\right]}{u^2}-\dfrac{\overline{f}(0;r)}{u^2}-\dfrac{\overline{f}'(0;r)}{u} \end{cases} \tag{74}$$

and

$$\underline{f}'(0;r)=\underline{f}'(0;r), \qquad 0\le r\le 1 \tag{75}$$

$$\overline{f}'(0;r)=\overline{f}'(0;r), \qquad 0\le r\le 1 \tag{76}$$

we get

$$-\frac{f(0)}{u^2}-\frac{f'(0)}{u}\Theta\frac{S[f(x)]}{-u^2}=\left[S\left[\underline{f}''(x;r)\right],S\left[\overline{f}''(x;r)\right]\right] \tag{77}$$

$$-\frac{f(0)}{u^2}-\frac{f'(0)}{u}\Theta\frac{S[f(x)]}{-u^2}=S\left[\underline{f}''(x;r),\overline{f}''(x;r)\right] \tag{78}$$

$$-\frac{f(0)}{u^2}-\frac{f'(0)}{u}\Theta\frac{S[f(x)]}{-u^2}=S\left[f''(x)\right] \tag{79}$$

Next proving Eq. (64) for arbitrary fixed$r \in [0, 1)$ we have

$$\begin{aligned}&-\frac{f(0)}{u^2}\Theta\frac{f'(0)}{u}\Theta\frac{S[f(x)]}{-u^2}\\ &=\left[\left\{-\frac{\overline{f}(0;r)}{u^2}-\frac{\overline{f}'(0;r)}{u}+\frac{S\left[\overline{f}(x;r)\right]}{u^2}\right\},\right.\\ &\left.\left\{-\frac{\underline{f}(0;r)}{u^2}-\frac{\underline{f}'(0;r)}{u}+\frac{S\left[\underline{f}(x;r)\right]}{u^2}\right\}\right]\end{aligned} \tag{80}$$

since

$$\begin{cases} S\left[\overline{f}''(x;r)\right] & = S\left[\underline{f}''(x;r)\right] \\ & = \dfrac{S\left[\underline{f}(x;r)\right]}{u^2}-\dfrac{\underline{f}(0;r)}{u^2}-\dfrac{\underline{f}'(0;r)}{u} \\ S\left[\underline{f}''(x;r)\right] & = S\left[\overline{f}''(x;r)\right] \\ & = \dfrac{S\left[\overline{f}(x;r)\right]}{u^2}-\dfrac{\overline{f}(0;r)}{u^2}-\dfrac{\overline{f}'(0;r)}{u} \end{cases} \tag{81}$$

and,

$$\overline{f}'(0;r)=\underline{f}'(0;r), \qquad 0\le r\le 1 \tag{82}$$

$$\underline{f}'(0;r)=\overline{f}'(0;r), \qquad 0\le r\le 1 \tag{83}$$

we get

$$-\frac{f(0)}{u^2}\Theta\frac{f'(0)}{u}\Theta\frac{S[f(x)]}{-u^2}=\left[S\left[\underline{f}''(x;r)\right],S\left[\overline{f}''(x;r)\right]\right] \tag{84}$$

$$-\frac{f(0)}{u^2}\Theta\frac{f'(0)}{u}\Theta\frac{S[f(x)]}{-u^2}=S\left[\underline{f}''(x;r),\overline{f}''(x;r)\right] \tag{85}$$

$$-\frac{f(0)}{u^2}\Theta\frac{f'(0)}{u}\Theta\frac{S[f(x)]}{-u^2}=S\left[f''(x)\right] \tag{86}$$

Now proving Eq. (65) for arbitrary fixed $r \in [0, 1)$ we have

$$\begin{aligned}&\frac{S[f(x)]}{u^2}\Theta\frac{f(0)}{u^2}-\frac{f'(0)}{u}\\ &=\left[\left\{\frac{S\left[\overline{f}(x;r)\right]}{u^2}-\frac{\overline{f}(0;r)}{u^2}-\frac{\overline{f}'(0;r)}{u}\right\},\right.\\ &\left.\left\{\frac{S\left[\underline{f}(x;r)\right]}{u^2}-\frac{\underline{f}(0;r)}{u^2}-\frac{\underline{f}'(0;r)}{u}\right\}\right]\end{aligned} \tag{87}$$

since

$$\begin{cases} S\left[\underline{f}''(x;r)\right]=S\left[\underline{f}''(x;r)\right]=\dfrac{S\left[\underline{f}(x;r)\right]}{u^2}-\dfrac{\underline{f}(0;r)}{u^2}-\dfrac{\underline{f}'(0;r)}{u} \\ S\left[\overline{f}''(x;r)\right]=S\left[\overline{f}''(x;r)\right]=\dfrac{S\left[\overline{f}(x;r)\right]}{u^2}-\dfrac{\overline{f}(0;r)}{u^2}-\dfrac{\overline{f}'(0;r)}{u} \end{cases} \tag{88}$$

and,

$$\overline{f}'(0;r)=\underline{f}'(0;r), \qquad 0\le r\le 1 \tag{89}$$

$$\underline{f}'(0;r)=\overline{f}'(0;r), \qquad 0\le r\le 1 \tag{90}$$

then, we get

$$\frac{S[f(x)]}{u^2}\Theta\frac{f(0)}{u^2}-\frac{f'(0)}{u}=\left[S\left[\underline{f}''(x;r)\right],S\left[\overline{f}''(x;r)\right]\right] \tag{91}$$

$$\frac{S[f(x)]}{u^2}\Theta\frac{f(0)}{u^2}-\frac{f'(0)}{u}=S\left[\underline{f}''(x;r),\overline{f}''(x;r)\right] \quad (92)$$

$$\frac{S[f(x)]}{u^2}\Theta\frac{f(0)}{u^2}-\frac{f'(0)}{u}=S\left[f''(x)\right] \quad (93)$$

prove completed.

3.4 Convolution of Fuzzy-valued Functions

The convolution, $f*g$ of two fuzzy-valued functions f and g is defined by,

$$(f*g)(x)=\int_0^x f(\tau)\,g(x-\tau)\,d\tau \quad (94)$$

Theorem 3.4.1

Let f and g be fuzzy-valued functions, having fuzzy Sumudu transforms $G(u)$ and $N(u)$, respectively, then the fuzzy Sumudu transform of the convolution of f and g, $f*g$, is given by:

$$S[(f*g)(x)]=uG(u)N(u) \quad (95)$$

with lower and upper functions

$$S\left[\left(\underline{f}*\underline{g}\right)(x;r)\right]=u\underline{G}(u)\underline{N}(u)$$
$$\text{and } S\left[\left(\overline{f}*\overline{g}\right)(x;r)\right]=u\overline{G}(u)\overline{N}(u), \qquad 0\le r\le 1 \quad (96)$$

Proof:

Starting the prove with the product of fuzzy Sumudu transforms of f and g:

$$u\underline{G}(u)\underline{N}(u)=u\int_0^\infty e^{-\eta}\underline{f}(u\eta;r)\,d\eta\int_0^\infty e^{-\phi}\underline{g}(u\phi;r)\,d\phi,$$
$$0\le r\le 1 \quad (97)$$

and

$$u\overline{G}(u)\overline{N}(u)=u\int_0^\infty e^{-\eta}\overline{f}(u\eta;r)\,d\eta\int_0^\infty e^{-\phi}\overline{g}(u\phi;r)\,d\phi,$$
$$0\le r\le 1 \quad (98)$$

Using exponential property we obtain:

$$u\underline{G}(u)\underline{N}(u)=\int_0^\infty\int_0^\infty e^{-(\eta+\phi)}\underline{f}(u\eta;r)\,\underline{g}(u\phi;r)\,u\,d\eta\,d\phi,$$
$$0\le r\le 1 \quad (99)$$

and

$$u\overline{G}(u)\overline{N}(u)=\int_0^\infty\int_0^\infty e^{-(\eta+\phi)}\overline{f}(u\eta;r)\,\overline{g}(u\phi;r)\,u\,d\eta\,d\phi,$$
$$0\le r\le 1 \quad (100)$$

let $x=\eta+\phi$, since η is fixed in the interior integrals so $dx=d\phi$, hence above equations will reduce to:

$$u\underline{G}(u)\underline{N}(u)=\int_0^\infty\int_0^x e^{-x}\underline{f}(u\eta;r)\,\underline{g}(u(x-\eta);r)\,u\,d\eta\,dx,$$
$$0\le r\le 1 \quad (101)$$

and

$$u\overline{G}(u)\overline{N}(u)=\int_0^\infty\int_0^x e^{-x}\overline{f}(u\eta;r)\,\overline{g}(u(x-\eta);r)\,u\,d\eta\,dx,$$
$$0\le r\le 1 \quad (102)$$

or

$$u\underline{G}(u)\underline{N}(u)=\int_0^\infty e^{-x}\int_0^x \underline{f}(u\eta;r)\,\underline{g}(u(x-\eta);r)\,u\,d\eta\,dx,$$
$$0\le r\le 1 \quad (103)$$

and

$$u\overline{G}(u)\overline{N}(u)=\int_0^\infty e^{-x}\int_0^x \overline{f}(u\eta;r)\,\overline{g}(u(x-\eta);r)\,u\,d\eta\,dx,$$
$$0\le r\le 1 \quad (104)$$

taking, $\tau=u\eta$ and $d\tau=u\,d\eta$, $u\eta\in[0,ux]$ when $\eta\in[0,x]$, therefore:

$$u\underline{G}(u)\underline{N}(u)=\int_0^\infty e^{-x}\left[\int_0^x\underline{f}(\tau;r)\,\underline{g}((ux-\tau);r)\,d\tau\right]dx$$
$$=S\left[\left(\underline{f}*\underline{g}\right)(x;r)\right] \quad (105)$$

and

$$u\overline{G}(u)\overline{N}(u)=\int_0^\infty e^{-x}\left[\int_0^{ux}\overline{f}(\tau;r)\,\overline{g}((ux-\tau);r)\,d\tau\right]dx$$
$$=S\left[\left(\overline{f}*\overline{g}\right)(x;r)\right] \quad (106)$$

For $0\le r\le 1$.

Lemma 3.4.1.1

Let f and g be fuzzy-valued function and $g(x)=1$, then convolution theorem yields the fuzzy Sumudu transform of the anti-derivative of $f(x)$. i.e.

$$S[f(x)]=S\left[\int_0^x f(\tau)\,d\tau\right]=uG(u). \quad (107)$$

with lower and upper functions are:

$$S\left[\underline{f}(x;r)\right] = S\left[\int_0^x \underline{f}(\tau;r)\,d\tau\right] = u\underline{G}(u),$$
$$S\left[\overline{f}(x;r)\right] = S\left[\int_0^x \overline{f}(\tau;r)\,d\tau\right] = u\overline{G}(u), \quad 0 \le r \le 1 \tag{108}$$

4 Applications of Fuzzy Sumudu transformation on Fuzzy Differential Equations

In this section, applications of fuzzy Sumudu transform is elaborated by solving some fuzzy differential equations, taken from [28].

4.1 Example

Let

$$y''(x) = \sigma_0, \qquad \sigma_0 = (r-1, 1-r) \tag{109}$$

be second order fuzzy differential equation with initial conditions

$$y(0;r) = (r-1, 1-r), y'(0;r) = (r-1, 1-r) \tag{110}$$

To solve this equation we consider the following four cases of strongly generalized H-differentiability:

Case I:

Let $y(x)$ and $y'(x)$ are (1)-differentiable. Then applying fuzzy Sumudu transform on both sides of Eq. (109) we obtain:

$$S\left[y''(x)\right] = S[\sigma_0], \tag{111}$$

with lower and upper functions,

$$S\left[\underline{y}''(x;r)\right] = S[\sigma_0], \quad S\left[\overline{y}''(x;r)\right] = S[\sigma_0], \ 0 \le r \le 1 \tag{112}$$

Using Theorem 3.7, we get:

$$\frac{S\left[\underline{y}(x;r)\right]}{u^2} \Theta \frac{\underline{y}(0;r)}{u^2} \Theta \frac{\underline{y}'(0;r)}{u} = \sigma_0,$$
$$\frac{S\left[\overline{y}(x;r)\right]}{u^2} \Theta \frac{\overline{y}(0;r)}{u^2} \Theta \frac{\overline{y}'(0;r)}{u} = \sigma_0 \tag{113}$$

After some simplification:

$$S\left[\underline{y}(x;r)\right] = \underline{y}(0;r) + u\underline{y}'(0;r) + u^2\sigma_0,$$
$$S\left[\overline{y}(x;r)\right] = \overline{y}(0;r) + u\overline{y}'(0;r) + u^2\sigma_0 \tag{114}$$

Taking inverse fuzzy Sumudu transform on both sides of Eq. (114), we get

$$\underline{y}(x;r) = \underline{y}(0;r)S^{-1}[1] + \underline{y}'(0;r)S^{-1}[u] + \sigma_0 S^{-1}\left[u^2\right], \tag{115}$$

$$\overline{y}(x;r) = \overline{y}(0;r)S^{-1}[1] + \overline{y}'(0;r)S^{-1}[u] + \sigma_0 S^{-1}\left[u^2\right]. \tag{116}$$

Using inverse values from Table A.1 in Belgacem et al. [2], we obtain

$$\underline{y}(x;r) = \underline{y}(0;r) + \underline{y}'(0;r)x + \sigma_0\frac{x^2}{2}, \tag{117}$$

$$\overline{y}(x;r) = \overline{y}(0;r) + \overline{y}'(0;r)x + \sigma_0\frac{x^2}{2}. \tag{118}$$

Using Eq. (110) we obtain solutions as:

$$\underline{y}(x;r) = (r-1)\left[1 + x + \frac{x^2}{2}\right],$$
$$\overline{y}(x;r) = (1-r)\left[1 + x + \frac{x^2}{2}\right] \tag{119}$$

Case II:

Let $y(x)$ and $y'(x)$ be (1)-differentiable and (2)-differentiable, respectively. Considering lower and upper functions of Eq. (109) and applying fuzzy Sumudu transform on both sides we get:

$$-\frac{\underline{y}(0;r)}{u^2} - \frac{\underline{y}'(0;r)}{u} \Theta \frac{S\left[\underline{y}(x;r)\right]}{-u^2} = \sigma_0,$$
$$-\frac{\overline{y}(0;r)}{u^2} - \frac{\overline{y}'(0;r)}{u} \Theta \frac{S\left[\overline{y}(x;r)\right]}{-u^2} = \sigma_0 \tag{120}$$

Following similar manipulation done in Case I, we get:

$$\underline{y}(x;r) = (r-1)\left[1 + x - \frac{x^2}{2}\right],$$
$$\overline{y}(x;r) = (1-r)\left[1 + x - \frac{x^2}{2}\right] \tag{121}$$

Case III:

Let $y(x)$ is (2)-differentiable and $y'(x)$ is (1)-differentiable. Taking lower and upper functions of Eq. (109) and applying fuzzy Sumudu transform on both sides,

$$-\frac{\underline{y}(0;r)}{u^2} \Theta \frac{\underline{y}'(0;r)}{u} \Theta \frac{S\left[\underline{y}(x;r)\right]}{-u^2} = \sigma_0,$$
$$-\frac{\overline{y}(0;r)}{u^2} \Theta \frac{\overline{y}'(0;r)}{u} \Theta \frac{S\left[\overline{y}(x;r)\right]}{-u^2} = \sigma_0 \tag{122}$$

Similarly, following Case I, after some manipulation we obtain the solutions of lower and upper functions as:

$$\underline{y}(x;r) = (r-1)\left[1 - x - \frac{x^2}{2}\right],$$
$$\overline{y}(x;r) = (1-r)\left[1 - x - \frac{x^2}{2}\right] \tag{123}$$

Case IV:

Consider $y(x)$ and $y'(x)$ of Eq. (109) are (2)-differentiable. Applying fuzzy Sumudu transform on both sides of its lower and upper functions we get:

$$\frac{S\left[\underline{y}(x;r)\right]}{u^2}\Theta\frac{\underline{y}(0;r)}{u^2}-\frac{\underline{y}'(0;r)}{u}=\sigma_0,$$
$$\frac{S\left[\overline{y}(x;r)\right]}{u^2}\Theta\frac{\overline{y}(0;r)}{u^2}+\frac{\overline{y}'(0;r)}{u}=\sigma_0 \tag{124}$$

Similar to Case I, after some simplification we get the solutions of lower and upper functions as:

$$\underline{y}(x;r)=(r-1)\left[1-x+\frac{x^2}{2}\right],$$
$$\overline{y}(x;r)=(1-r)\left[1-x+\frac{x^2}{2}\right] \tag{125}$$

4.2 Example

Considering another example of fuzzy differential equation with fuzzy initial conditions as

$$y''(x)+y(x)=\sigma_0,\qquad \sigma_0=(r,2-r) \tag{126}$$

$$y(0;r)=(r-1,1-r),\,y'(0;r)=(r-1,1-r) \tag{127}$$

As in Example 4.1, we have the following four cases:

Case I:

Let $y(x)$ and $y'(x)$ are (1)-differentiable. Then on applying fuzzy Sumudu transform on both sides of Eq. (126) we get:

$$S\left[y''(x)\right]+S\left[y(x)\right]=S\left[\sigma_0\right], \tag{128}$$

with lower and upper functions

$$S\left[\underline{y}''(x;r)\right]+S\left[\underline{y}(x;r)\right]=S\left[\sigma_0\right],$$
$$S\left[\overline{y}''(x;r)\right]+S\left[\overline{y}(x;r)\right]=S\left[\sigma_0\right],\qquad 0\le r\le 1 \tag{129}$$

Using Theorem 3.7

$$\frac{S\left[\underline{y}(x;r)\right]}{u^2}\Theta\frac{\underline{y}(0;r)}{u^2}\Theta\frac{\underline{y}'(0;r)}{u}+S\left[\underline{y}(x;r)\right]=\sigma_0 \tag{130}$$

and

$$\frac{S\left[\overline{y}(x;r)\right]}{u^2}\Theta\frac{\overline{y}(0;r)}{u^2}\Theta\frac{\overline{y}'(0;r)}{u}+S\left[\overline{y}(x;r)\right]=\sigma_0 \tag{131}$$

After some simplification:

$$S\left[\underline{y}(x;r)\right]\left(1+u^2\right)=\underline{y}(0;r)+u\underline{y}'(0;r)+u^2\sigma_0,$$
$$S\left[\overline{y}(x;r)\right]\left(1+u^2\right)=\overline{y}(0;r)+u\overline{y}'(0;r)+u^2\sigma_0 \tag{132}$$

Taking inverse fuzzy Sumudu transform on both sides of Eq. (132):

$$\underline{y}(x;r)=\underline{y}(0;r)S^{-1}\left[\frac{1}{(1+u^2)}\right]+\underline{y}'(0;r)S^{-1}\left[\frac{u}{(1+u^2)}\right]+\sigma_0 S^{-1}\left[\frac{u^2}{(1+u^2)}\right], \tag{133}$$

$$\overline{y}(x;r)=\overline{y}(0;r)S^{-1}\left[\frac{1}{(1+u^2)}\right]+\overline{y}'(0;r)S^{-1}\left[\frac{u}{(1+u^2)}\right]+\sigma_0 S^{-1}\left[\frac{u^2}{(1+u^2)}\right] \tag{134}$$

Using inverse values from Table A.1 in Belgacem et al. [2]:

$$\underline{y}(x;r)=\underline{y}(0;r)\left[\cos x\right]+\underline{y}'(0;r)\left[\sin x\right]+\sigma_0\left[1-\cos x\right], \tag{135}$$

$$\overline{y}(x;r)=\overline{y}(0;r)\left[\cos x\right]+\overline{y}'(0;r)\left[\sin x\right]+\sigma_0\left[1-\cos x\right]. \tag{136}$$

Using initial conditions from Eq. (127), solutions obtained are:

$$\underline{y}(x;r)=r(1+\sin x)-\cos x-\sin x,$$
$$\overline{y}(x;r)=(2-r)(1+\sin x)-\cos x-\sin x \tag{137}$$

Case II:

Let $y(x)$ is (1)-differentiable and $y'(x)$ is (2)-differentiable. Then applying fuzzy Sumudu transform on both sides of lower and upper functions of Eq. (126) we get:

$$-\frac{\underline{y}(0;r)}{u^2}-\frac{\underline{y}'(0;r)}{u}\Theta\frac{S\left[\underline{y}(x;r)\right]}{-u^2}+S\left[\underline{y}(x;r)\right]=\sigma_0 \tag{138}$$

$$-\frac{\overline{y}(0;r)}{u^2}-\frac{\overline{y}'(0;r)}{u}\Theta\frac{S\left[\overline{y}(x;r)\right]}{-u^2}+S\left[\overline{y}(x;r)\right]=\sigma_0 \tag{139}$$

After doing manipulation as done in Case I, we obtain the solutions of lower and upper functions as:

$$\underline{y}(x;r)=r(1+\sinh x)-\cos x-\sinh x,$$
$$\overline{y}(x;r)=(2-r)(1+\sinh x)-\cos x-\sinh x \tag{140}$$

Case III:

Let $y(x)$ and $y'(x)$ in Eq. (126) be (2)-differentiable and (1)-differentiable, respectively. On applying fuzzy Sumudu transform on both sides of its lower and upper functions, we obtain:

$$-\frac{\underline{y}(0;r)}{u^2}\Theta\frac{\underline{y}'(0;r)}{u}\Theta\frac{S\left[\underline{y}(x;r)\right]}{-u^2}+S\left[\underline{y}(x;r)\right]=\sigma_0 \tag{141}$$

$$-\frac{\overline{y}(0;r)}{u^2}\Theta\frac{\overline{y'}(0;r)}{u}\Theta\frac{S[\overline{y}(x;r)]}{-u^2}+S[\overline{y}(x;r)]=\sigma_0 \quad (142)$$

Similarly, following manipulation in Case I, we obtain the solutions of lower and upper functions as:

$$\underline{y}(x;r)=r(1-\sinh x)-\cos x+\sinh x,$$
$$\overline{y}(x;r)=(2-r)(1-\sinh x)-\cos x+\sinh x \quad (143)$$

Case IV:

Let $y(x)$ and $y'(x)$ are (2)-differentiable. Taking lower and upper functions of Eq. (126) and applying fuzzy Sumudu transform on both sides we obtain:

$$\frac{S\left[\underline{y}(x;r)\right]}{u^2}\Theta\frac{\underline{y}(0;r)}{u^2}-\frac{\underline{y'}(0;r)}{u}+S\left[\underline{y}(x;r)\right]=\sigma_0 \quad (144)$$

$$\frac{S[\overline{y}(x;r)]}{u^2}\Theta\frac{\overline{y}(0;r)}{u^2}+\frac{\overline{y'}(0;r)}{u}+S\left[\underline{y}(x;r)\right]=\sigma_0 \quad (145)$$

Similar to Case I, after some manipulation we obtain lower and upper solutions as:

$$\underline{y}(x;r)=r(1-\sin x)-\cos x+\sin x,$$
$$\overline{y}(x;r)=(2-r)(1-\sin x)-\cos x+\sin x \quad (146)$$

5 Conclusions

In this document, we extended Sumudu transform to fuzzy Sumudu transform for the solution of linear differential models with uncertainty. Presented illustration of its fundamental properties and its application on some second order fuzzy linear differential equations considered under strongly generalized Hukuhara differentiability. Thus it is concluded that:

- The proposed transformation has unit and scale preserving property, which is advantageous for uncertain physical models.
- Due to analytical duality of fuzzy Sumudu transform with fuzzy Laplace transform as discussed in Theorem 3.1 each problem solved by fuzzy Laplace transform can also be solved by fuzzy Sumudu transform method.
- From examples it is depicted that fuzzy Sumudu transform is very effective and reliable tool with less computation in obtaining exact solutions of fuzzy differential equations.

Further, to study its applications in future we will generalize fuzzy Sumudu transformation to fractional order for solving fuzzy fractional linear differential equations.

Authors' contributions

All the authors read and approved the whole manuscript and have equal contribution to each part of this article.

Competing interest

The authors have no competing interest.

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