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Block pulse transform method for linearization of nonlinear SDOF systems

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Abstract: This paper addresses a linearization technique for a linearizable nonlinear SDOF system based on block pulse (BP) transform. BP transform method is a useful tool for giving a solution of difference equations with less computational costs. The main goal of this work is on reducing the costs of linearizing computation. It is necessary to compare the results obtained using this method with other traditional methods to verify the effectiveness of the proposed method. Therefore, the efficiency of BP transform method is compared with the traditional equivalent linearization (EL) method, used to linearize nonlinear systems. Two numerical simulations are applied to the Duffing oscillator system to demonstrate the feasibility of the proposed method based on BP transform. Finally, the results of comparison between existing approaches that have applied to the considered problem depicted the proposed method base on orthogonal functions is able to approximate the nonlinear system's behavior and shows the superiority of the proposed approach in the sense that it is more accurate by computational advantageous.

Keywords: Linearization, block pulse transform, nonlinearity, Duffing oscillator

1 Introduction

Study of nonlinearity in dynamic systems has been a prominent area of research in last decades. This inherent phenomenon inevitably occurs in physical systems. In mechanical and structural systems nonlinearities can arise in various forms and usually becomes progressively more significant as the motion amplitude increases. The major sources of nonlinearity arise from misalignment, looseness, temperature effects, impedance mismatching, preload, exciter problems and overloads [1]. Nonlinear systems may show complicated behavior, such as limit cycles, bifurcations and even chaos which are difficult to predict. Besides in practical applications, due to the high intensity nature and often complex nature of non-stationary environmental loads such as wind loads, sea waves and earthquakes, the systems subjected to these loadings may experience excessive stress or displacements that results inelastic behavior [2]. To describe real observed processes in field of engineering, researchers often use mathematical models. For dynamic processes, these models contain many different types of equations such as ordinary or partial differential equations, difference equations, and algebraic equations. In general case, due to the nature of the considered problems are nonlinear, there is no exact solution for such equations. Except for some special cases, the solutions are approximate [3]. There are diverse approaches which have been developed over the years to treat the nonlinear problems. Most commonly used methods include Perturbation, Monte Carlo simulation and well known harmonic balance method which is one of the main techniques for obtaining approximate analytic solutions to nonlinear ordinary differential equations. Also there are the semi-analytical methods, such as the high order harmonic balance (HOHB) method which has been developed to avoid the tedious algebraic calculations involved in the classical harmonic balance method in processing the nonlinear term in the nonlinear dynamical system [4], the high dimensional harmonic balance (HDHB) method and the time domain collocation (TDC) method [5]. Meanwhile, among many methods dealing with a nonlinear system, linearization methods are the oldest and the most popular methods of approximation. To approximate the nonlinear problems a powerful linearization technique is required to analyze and predict nonlinear system's behavior in order to design an accurate and desirable scheme of a system during its operation under any excitations. One of the linearization methods is the Lyapunov linearization technique used to approximate a nonlinear system by a linear one that is around the equilibrium point, and it is expected the behavior of the linear system will be the same as that of nonlinear one [6]. The other one is feedback linearization

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method which is a common approach used in nonlinear control systems. This method is based on the idea of transforming original system model into equivalent linear one which is changed to the state variables and a suitable input instead [7, 8]. Describing Function is another method for system linearization which is an extension of the frequency response method of linear control can be used as the approximate analysis to predict some important characteristics of nonlinear system including systems with hard nonlinearities [9]. Statistical equivalent linearization (EL) technique is commonly used approach in nonlinear problems. The statistical EL method is based on the idea that the nonlinear system is replaced by an equivalent linear equation by minimizing the difference between the two systems in some appropriate sense. In order to predict the response of this kind of system or to get an approximation solution of nonlinear equation this method is applied to estimate the accurate equivalent linear parameters. This method was proposed by Caughey [10] as a way to solve nonlinear vibration problems. Statistical EL method has proven to be very useful approximation technique in structural dynamics and earthquake engineering. All methods of statistical and EL can be considered in different fields such as state space, frequency domain, distribution space and characteristic functions space. Usually they consist of two main steps. In the first step, deal to find explicit or implicit analytical formulas for linearization coefficients based on the linearization criterion which is depend on unknown response characteristics such as mean value, variance, and higher-order moments. In the second step, deal to replace the unknown characteristics by the corresponding ones determined for linearized systems. It is worth mentioning that accuracy and feasibility of these solutions dependent on the type of nonlinearity and amplitude of external excitation forces. This method has been widely used [11, 12]. Discrete-time (data sampled) systems have resulted in corresponding demand for designing and understanding these systems. These systems are governed by difference equations in which members are coupled to each other. One source of difference equations is the numerical evaluation of integrals. Also we could use the conventional Laplace transform to solve these difference equations [13].

Block pulse (BP) transform provides a useful tool to solve difference equations of any order with less computational costs. The BP transform originated from BP functions. The BP functions are a set of orthogonal functions with piecewise constant values and are usually applied as a useful tool in the analysis, synthesis, identification and other problems of control and systems science. This set of functions was first introduced to electrical engineers by Harmuth [14]. Some papers discussed the BP functions and their operational matrix for integration in order to reduce the complexity of expressions in solving certain control problems via Walsh functions. PurnachandraRao and RanganathaRao [15] used BP functions to determine the piecewise constant feedback controls for a finite linear optimal control problem of a power system that the proposed method is simple and computationally advantageous. Sannuti showed that the application of BP functions results in an enormous reduction of computational effort over Walsh functions in control system applications [16]. In active control problem a new method proposed based on BP functions evolves minimizing computational costs of analytical approaches [17, 18].

The main objective of this study is the using of BP transform in linearization procedure through its easy and simple operation. The input - output relationships for a linearized system and nonlinear system are obtained using the BP transform. Following the basic procedure of the traditional EL approach, one can find the least mean square error between the linearized and nonlinear equations. The effectiveness of the proposed method is validated on nonlinear Duffing oscillator system. Different simulations used to verify the accuracy and feasibility of the proposed method, the traditional EL results of displacement have been compared with those obtained by this method. Results from this study presented that this method can approximate the nonlinear systems behavior for stationary excitation better than the traditional EL method.

Frequency response function (FRF) summarizes essential information to specify the dynamics of a structure. The FRF of linear and nonlinear system that is linearized by existing methods have been compared. This comparison confirmed the accuracy of proposed method.

The remaining of this paper is organized as follows. Section 2 presents the BP transform formulation. In section 3 linearization method based on BP transform is proposed. The simulations have been carried out to compare with the traditional EL method in Section 4 and followed by the conclusion in Section 5.

2 Block pulse transform

The block pulse transform method provides a technique for transforming a difference equation into an algebraic equation. The BP transform method is very similar to z-transform method that can greatly facilitate the analysis [19]. The BP transform of function $f(t)$ is defined by:

$$F(z) = \sum_{i=0}^{\infty} f_i^{-i} \tag{1}$$

where $f_\mathbf{i}$ are the coefficients of the terms $z^{-\mathbf{i}}$ ($i = 0, 1, 2, \ldots$) in the power series which are the values of the sampled signal $f(t)$ at the corresponding time instants h. Where h is the sampling period. The summery of BP transform properties are following [20]:

For addition and subtraction of function $x(t) = f(t) \pm g(t)$, we have:

$$X(z) = F(z) \pm G(z) \tag{2}$$

For multiplied by a scalar $x(t) = kf(t)$, we have:

$$X(z) = kF(z) \tag{3}$$

For multiplication of function $x(t) = f(t)g(t)$, we have:

$$X(z) = \sum_{i=0}^{\infty} f_i g_i z^{-i} \tag{4}$$

For division of function $x(t) = f(t)/g(t)$ with $g(t) \neq 0$, we have:

$$X(z) = \sum_{i=0}^{\infty} (f_i/g_i) z^{-i} \tag{5}$$

For derivation of function $x(t) = \frac{df(t)}{dt}$, we have:

$$X(z) = \frac{2}{h}\frac{1 - z^{-1}}{1 + z^{-1}} F(z) - f_0^{(0)} \frac{2}{h}\frac{1}{1 + z^{-1}} \tag{6}$$

where $f_0^{(0)}$ is the initial value of $f(t)$.

3 Linearization based on BP transform

Consider a SDOF nonlinear system subject to external signal $f(t)$ with the equation motion:

$$m\ddot{x}(t) + g(x, \dot{x}) = f(t) \tag{7}$$

Where m is mass of system and $g(x, \dot{x})$ is the nonlinear function. Performing the BP transform both sides of (7) yields:

$$m\ddot{X}(z) + G(z) = F(z) \tag{8}$$

Where, $X''(z)$, $F(z)$ and $G(z)$ are BP transform of , $x''(t)$, $g(x, x')$ and $f(t)$ respectively. Regarding to equation (7), the equation of an equivalent linearized system can be found as follows:

$$m\ddot{x}(t) + c_{eq}\dot{x}(t) + k_{eq}x(t) = f(t) \tag{9}$$

where c_{eq} and k_{eq} are unknown damping and stiffness coefficients to be determined by EL to replace the nonlinear term. By using the BP transform on both sides of the Eq. (9), the following form can be obtained:

$$m\ddot{X}(z) + c_{eq}\dot{X}(z) + k_{eq}X(z) = F(z) \tag{10}$$

In which $X'(z)$ and $X(z)$ are BP transform of $x'(t)$ and $x(t)$, respectively. The replacement of a nonlinear system by a linear system is in some probabilistic sense and it will yield the error. The error may be defined as:

$$e = c_{eq}\dot{X}(z) + k_{eq}X(z) - G(z) \tag{11}$$

By using the basic procedure of the EL approach the solution of the nonlinear system is approximated. Equivalent parameters should be selected such that error e would be as small as possible by finding the mean square least error between the original equation and equivalent one [21].

$$e^2 = [c_{eq}\dot{X}(z) + k_{eq}X(z) - G(z)]^2 \tag{12}$$

To determine the unknown coefficients from equation (12), the expected value of the error is derived and its derivations in respect of unknown coefficients are written. It should be noted that the expected value of the error has been considered for having generality. This is true where the error to be assumed as a random variable. The coefficients c_{eq} and k_{eq} are determined by the following equation:

$$\frac{\partial E[e^2]}{\partial c_{eq}} = \frac{\partial E[e^2]}{\partial k_{eq}} = 0 \tag{13}$$

Noting $E[x\dot{x}] = 0$ due to displacement and the velocity are uncorrelated and by using BP transform properties from section (2), the Eq. (13) leads to the following equations:

$$c_{eq} = \frac{\partial E\left[\left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right) X(z)G(z)\right]}{E\left[\left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right) X(z)\right]} \tag{14}$$

and

$$k_{eq} = \frac{\partial E[X(z)G(z)]}{E[X(z)]^2} \tag{15}$$

In practical applications involving numerical algorithms all parameters are computing at discrete grid points. In discrete domain the minimization conditions are:

$$\frac{\partial}{\partial c_{eq}} \sum_{i=0}^{T} E[e^2] = 0 \tag{16}$$

and

$$\frac{\partial}{\partial k_{eq}} \sum_{i=0}^{T} E[e^2] = 0 \tag{17}$$

where $T$is the final time-instants in simulation.

In fact, the equivalent coefficients are calculated numerically. The parameters are to be discretized over all the time steps. In order to find the equivalent parameters the Eqs. (16) and (17) can be defined as follow:

$$c_{eq} = \frac{\sum_{i=0}^{T} \left[\left(\frac{2}{h} \frac{1-z^{-1}}{1+z^{-1}} \right) x_i g_i z^{-i} \right]}{\sum_{i=0}^{T} \left[\left(\frac{2}{h} \frac{1-z^{-1}}{1+z^{-1}} \right) x_i z^{-i} \right]} \tag{18}$$

and

$$k_{eq} = \frac{\partial \sum_{i=0}^{T} x_i g_i z^{-i}}{\sum_{i=0}^{T} (x_i z^{-i})^2} \tag{19}$$

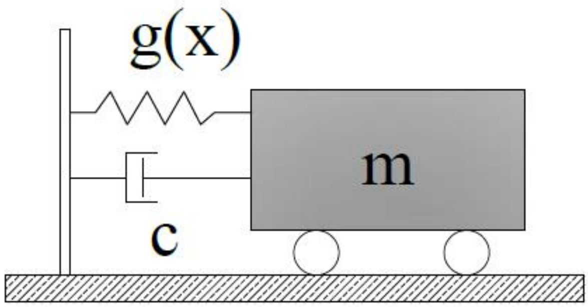

Fig. 1. Nonlinear SDOF system.

4 Linearization results

To implement the proposed methodology for finding equivalent damping and stiffness coefficients of a system we consider a nonlinear SDOF system as shown in Figure 1. The nonlinear equation is Duffing equation i.e. an oscillator with nonlinear stiffness subjected to external signal $f(t)$:

$$m\ddot{x}(t) + c\dot{x}(t) + g(x) = f(t) \tag{20}$$

where m is the mass, c is the damping coefficient, $f(t)$ is the external excitation and $x(t)$ is the displacement response of the system. $g(x)$ is the nonlinear restoring force that could depend on displacement defined as follows [22]:

$$g(x) = kx + \alpha k_3 x^3 \tag{21}$$

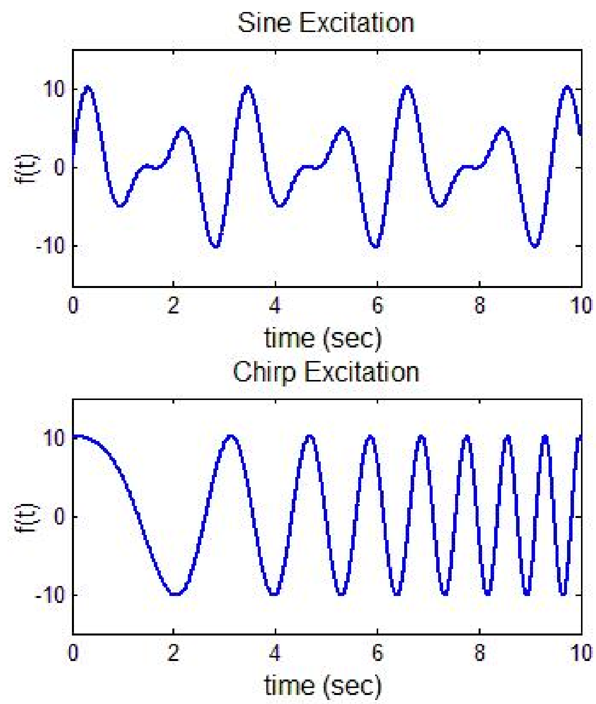

Fig. 2. External excitations.

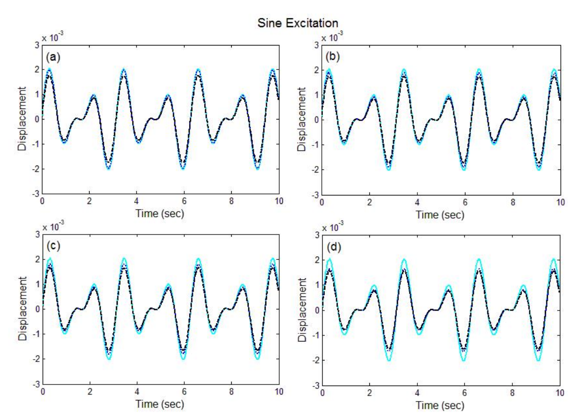

Fig. 3. Displacement response of linearized system; linear system (solid line), linearized system by proposed method (dotted line), linearized system by traditional EL method (dashed line); (a) α =1.0. (b) α =1.05. (c) α =1.1. (d) α =1.2.

where α is the non-linear factor that shows the rate of nonlinearity in the nonlinear system.

The system parameters are chosen as follows, $m = 2$, $c = 20$, $k = 10^4$ and $k_3 = 3.8 \times 10^9$. The system excited by two signal. First test signal is the sine excitation which contain different frequencies $f(t) = \sin(2t) + 6\sin(4t) + 4\sin(6t)$, and the second one is a chirp signal which increases linearly with time from 0.1 Hz to 1.5 Hz with a sampling period of 0.01. Figure 2 shows the schemes of excitation signals. It should be noted that the nonlinear Duffing system subjected to two stationary excitations. Using equation (18) and (19) for the stationary excitation may make some confliction. For the cases in which excitation is not stochastic the computed error is not random variable. It is obvious that in such cases expected value of error is replaced with sum of square errors and derivation of it in respect to unknown parameters is performed.

The nonlinear Duffing system subjected to two chosen stationary excitations. Figures 3 and 4 for $\alpha = 1, 1.05, 1.1, 1.2$, show the displacement response of linearized system by proposed method and traditional EL method. As illustrated in Figures 2 and 3, three curves are plotted in each figures. a) The response of the linear system for the system in which K3=0. This one is illustrated only to show the disparity of the linearized system from linear response in respect of nonlinearity rate. It named "linear system" in figures. b) The response of nonlinear system linearized by the proposed method. c) The response of nonlinear system linearized by EL method.

The numerical results on SDOF nonlinear system reveal that the proposed linearization method based on BP transform is a promising tool in linearizing nonlinear systems and this method by less computational expenses is more accurate than traditional EL method. But the results demonstrate that the errors between linear and linearized systems increase in high rate of nonlinearity. Figure 5 presents this fact for excitations.

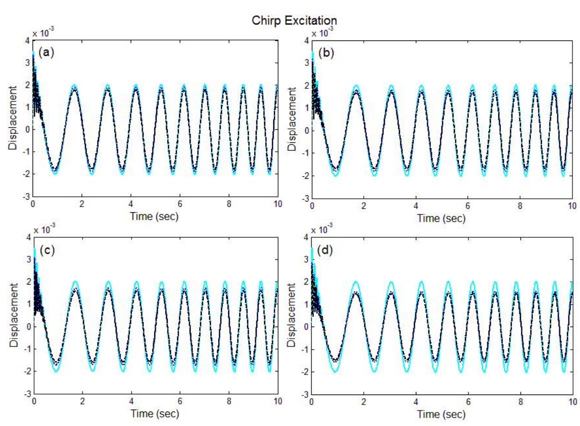

Fig. 4. Displacement response of linearized system; linear system (solid line), linearized system by proposed method (dotted line), linearized system by traditional EL method (dashed line); (a) α =1.0. (b) α =1.05. (c) α =1.1. (d) α =1.2.

In the field of structural dynamics, one of the most widely-used method of visualizing the properties of a system is to build the frequency response function (FRF). The

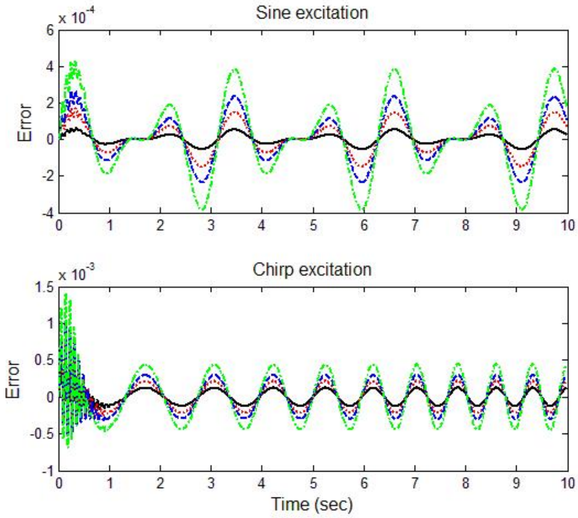

Fig. 5. Displacement errors; α =1.0. (solid line), α =1.05. (dotted line), α =1.1. (dashed line), α =1.2. (dash-dotted line).

FRF summarizes most of the necessary information such as resonances, anti-resonances, modal density and phase are directly visible to specify the dynamics of a structure. In addition, the FRF can rapidly provide an indication of whether a system is linear or nonlinear. Figure 6 presents a comparison of FRF of nonlinear system linearized by two methods.

5 Conclusion

This paper deals with linearization of nonlinear SDOF system. The proposed method based on BP transform compared with traditional EL approach. This technique permits to avoid solving complicated nonlinear algebraic equations. In order to investigate the performance of the proposed linearization method, the Duffing oscillator subjected to two stationary signals and the feasibility of the proposed method is demonstrated. The results and comparisons reveal that for any rate of nonlinearity in nonlinear system, the displacement response of the proposed method is well-approximated than the traditional EL method. The frequency response function of linear and linearized systems have compared and this fact is also observed that the approximated nonlinear system by proposed method is well-behaved than the approximated nonlinear system by traditional EL approach. Lastly it should be pointed out that the error rises with the nonlinearity, but the proposed method errors are smaller than the use of the traditional EL approach.

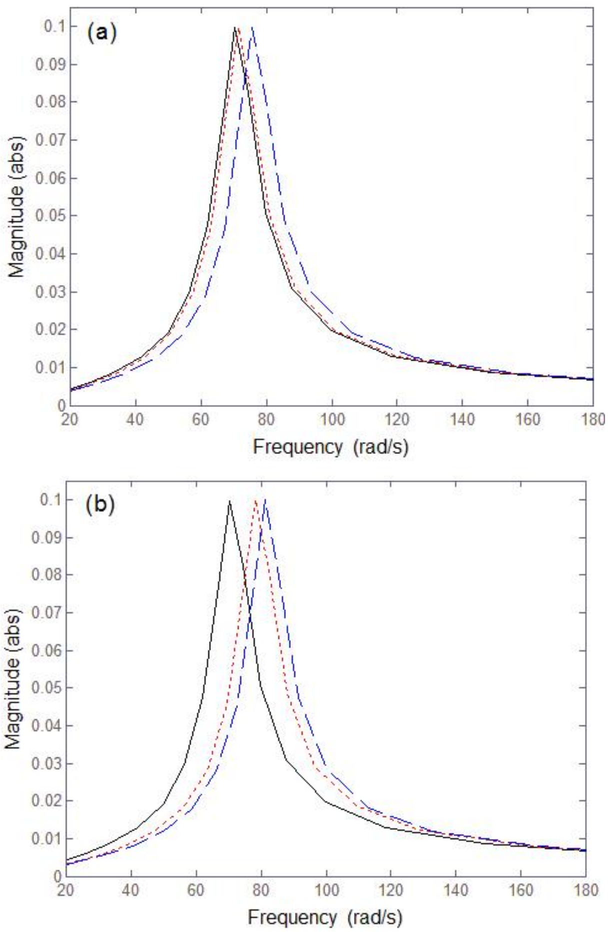

Fig. 6. Linearized FRF of system; linear system (solid line), linearized system by proposed method (dotted line), linearized system by traditional EL method (dashed line); (a) α =1.0. and (b) α =1.2.

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