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Exact traveling wave solutions of partial differential equations with power law nonlinearity

DOI 10.1515/nleng-2015-0005
Received February 18, 2015; revised April 16, 2015; accepted June 9, 2015.

Abstract: In this paper, we applied the functional variable method for four famous partial differential equations with power law nonlinearity. These equations are included the Kadomtsev-Petviashvili, (3+1)-Zakharov-Kuznetsov, Benjamin-Bona-Mahony-Peregrine and Boussinesq equations. Various exact traveling wave solutions of these equations are obtained that include the hyperbolic function solutions and the trigonometric function solutions. The solutions shown that this method provides a very effective, simple and powerful mathematical tool for solving nonlinear equations in various fields of applied sciences.

Keywords: functional variable method; partial differential equation; power-law nonlinearity

1 Introduction

Nonlinear partial differential equations (NLPDEs) are very important in various fields of science and technology, especially in biology, solid state physics, fluid mechanics, plasma physics, optical fibers, chemical kinematics, and chemical physics. In the research of the theory of NLPDEs, searching for more explicit exact solutions to NLPDEs is one of the most fundamental and significant studies in recent years. With the help of computerized symbolic computation, much work has focused on the various extensions and applications of the known algebraic methods to construct the solutions to NLPDEs. A special class of analytical solutions, the so-called traveling waves, for NLPDEs is of fundamental importance because lots of mathematical-physical models are often described by such wave phenomena. Therefore, the investigation of traveling wave solutions is becoming more and more attractive in nonlinear sciences nowadays. In recent years, many approaches have been utilized for finding the traveling wave solutions of nonlinear partial differential equations, for example, the tanh method [1, 2], the extended tanh-function method [3, 4], the generalized hyperbolic function method [5, 6], the first integral method [7], the (G'/G)-expansion method [8, 9], the Exp-function method [10-11] and so on. In [12, 13], Zerarka et al. introduced the so-called functional variable method to find the exact solutions for a wide class of linear and nonlinear wave equations. This method was further developed by many authors [14–16]. The advantage of this method is that one treats nonlinear problems by essentially linear methods, based on which it is easy to construct in full the exact solutions such as soliton-like waves, compacton solutions and noncompacton solutions, trigonometric function solutions, pattern soliton solutions, black solitons or kink solutions, and so on. The aim of this paper is to apply the functional variable method to find the exact solutions of Kadomtsev-Petviashvili, (3+1)-Zakharov-Kuznetsov, Benjamin-Bona-Mahony-Peregrine and Boussinesq equations with power law nonlinearity. We will present a useful remark of the functional variable method for finding traveling wave solutions of nonlinear partial differential equations, namely, Remark 2. Then, by using the Remark 2, two kinds of exact solutions for the equations with power law nonlinearity are obtained in a unified way.

The rest of this paper is organized as follows. In Section 2, a description of the functional variable method is given in detail. In Section 3, the application of our method to the Kadomtsev-Petviashvili equation with power law nonlinearity is illustrated. In section 4 we will use this method to the Zakharov-Kuznetsov equation with power law nonlinearity. In section 5 and 6 we will solve the Benjamin-Bona-Mahony-Peregrine and Boussineq equations with power law nonlinearity sequently with same method. Conclusions are presented in Section 7.

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2 The functional variable method

In this Section we describe the main steps of the functional variable method for finding exact solutions of nonlinear PDEs.

Consider a general nonlinear PDE in the form

$$P(u, D_t u, D_x u, D_y u, D_z u, D_t^2 u, \ldots) = 0, \tag{1}$$

where $u = u(x, y, z, t)$ is the solution of nonlinear PDE (1), the subscript denotes partial derivative and $P$is a polynomial in its arguments. Zerarka et al. in [12] has summarized the functional variable method in the following. Using a wave transformation ξ=$l_1 x + l_2 y + l_3 z - \lambda t$ so that

$$u(x, y, z, t) = U(\xi), \tag{2}$$

where l_1, l_2, l_3 and λ are constant to be determined later. This enables us to use the following changes

$$\begin{aligned} &D_t(.) = -\lambda \tfrac{d}{d\xi}(.),\\ &D_x(.) = l_1 \tfrac{d}{d\xi}(.),\\ &D_y(.) = l_2 \tfrac{d}{d\xi}(.)\\ &D_z(.) = l_3 \tfrac{d}{d\xi}(.),\\ &D_t^2(.) = \lambda^2 \tfrac{d^2}{d\xi^2}(.),\\ &\vdots \end{aligned}$$

Using Eq. (2), the PDE (1) can be converted to a nonlinear ordinary differential equation (ODE) as

$$G\,(U, U_\xi, U_{\xi\xi}, U_{\xi\xi\xi}, \ldots) = 0, \tag{3}$$

where G is a polynomial in $U = U(\xi)$. If all terms contain derivatives, then Eq. (3) is integrated where integration constants are considered zeros. Then we make a transformation in which the unknown function U is considered as a functional variable in the form

$$U_\xi = F(U), \tag{4}$$

and some successive derivatives of U are

$$\begin{aligned} &U_{\xi\xi} = \tfrac{1}{2}(F^2)^{'},\\ &U_{\xi\xi\xi} = \tfrac{1}{2}(F^2)^{''}\sqrt{F^2},\\ &U_{\xi\xi\xi\xi} = \tfrac{1}{2}[(F^2)^{'''}F^2 + (F^2)^{''}(F^2)^{'}],\\ &\vdots \end{aligned} \tag{5}$$

where "$'$" stands for $\frac{d}{dU}$.

The ODE (3) can be reduced in terms of U, F and its derivative upon using the expressions of Eq. (5) into Eq. (3) gives

$$Q(U, F, F^{'}, F^{''}, F^{'''}, \ldots) = 0. \tag{6}$$

The key idea of this particular form Eq. (6) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, Eq. (6) provides the expression of F, and this, together with Eq. (4), give appropriate solutions to the original problem.

Remark 1. The functional variable method definitely can be applied to nonlinear PDEs which can be converted to a second-order ordinary differential equation (ODE) through the travelling wave transformation.

Remark 2. Consider the following second-order ordinary differential equation

$$U_{\xi\xi} = k_1 U - k_2 U^{n+1}, \quad n \neq 0, \tag{7}$$

where k_1 and k_2 are constants and U is a functional variable in the form (4). Then using (5) transformation, the exact solutions of the Eq. (7) are obtained as

Type I. When $k_1 > 0$, the solutions of Eq. (7) are

$$U_1(\xi) = \left\{\frac{(n+2)k_1}{2k_2}\text{cech}^2(\frac{n}{2}\sqrt{k_1}\xi)\right\}^{\frac{1}{n}}, \tag{8}$$

$$U_2(\xi) = \left\{-\frac{(n+2)k_1}{2k_2}\text{sech}^2(\frac{n}{2}\sqrt{k_1}\xi)\right\}^{\frac{1}{n}}, \tag{9}$$

Type II. When $k_1 < 0$, the solutions of Eq. (7) are

$$U_3(\xi) = \left\{\frac{(n+2)k_1}{2k_2}\csc^2(\frac{n}{2}\sqrt{-k_1}\xi)\right\}^{\frac{1}{n}}, \tag{10}$$

$$U_4(\xi) = \left\{\frac{(n+2)k_1}{2k_2}\sec^2(\frac{n}{2}\sqrt{-k_1}\xi)\right\}^{\frac{1}{n}}. \tag{11}$$

Proof. According to Eq. (4), we get from (7) an expression for the function$F(U)$

$$\frac{1}{2}\left(F^2(U)\right)^{'} = k_1 U - k_2 U^{n+1}, \tag{12}$$

where the prime denotes differentiation with respect to ξ. Integrating Eq. (12) with respect to U and after the mathematical manipulations, we have

$$F(U) = \pm U\sqrt{k_1 - \frac{2k_2}{n+2}U^n}, \tag{13}$$

or

$$F(U) = \pm\sqrt{k_1}U\sqrt{1 - \frac{2k_2}{(n+2)k_1}U^n}. \tag{14}$$

After changing the variables

$$Z = \frac{2k_2}{(n+2)k_1}U^n, \tag{15}$$

or

$$\left(\frac{(n+2)k_1}{2k_2}Z\right)^{\frac{1}{n}} = U, \tag{16}$$

with differentiation from Eq. (16)

$$\frac{1}{n}\left(\frac{(n+2)k_1}{2k_2}\right)^{\frac{1}{n}} Z^{\frac{1-n}{n}}dZ = dU(\xi). \tag{17}$$

s We use (17) transformation to the Eq. (14)

$$\frac{dZ}{Z\sqrt{1-Z}} = \pm n\sqrt{k_1}d\xi, \tag{18}$$

with integrating from Eq. (18) and with setting the constant of integration as zero

$$\ln\left|\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}}\right| = \pm n\sqrt{k_1}\xi. \tag{19}$$

In this case we have

$$\left|\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}}\right| = e^{\pm n\sqrt{k_1}\xi}. \tag{20}$$

If $\theta = \pm n\sqrt{k_1}\xi$, two cases will be considered separately.
Case 1. Suppose that $k_1 > 0$, then

$$\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}} = e^{\theta}, \tag{21}$$

thus, according to (21), we have

$$Z = \frac{4}{e^{-\theta}+e^{\theta}+2} = \frac{2}{\cosh\theta+1} = \frac{1}{\cosh^2\left(\frac{\theta}{2}\right)+1}$$
$$= \sec h^2\left(\frac{\theta}{2}\right),$$

so

$$Z = \operatorname{sech}^2(\frac{n}{2}\sqrt{k_1}\xi). \tag{22}$$

Now, suppose that $k_1 < 0$, then

$$\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}} = e^{i\theta}, \tag{23}$$

thus, according to (23), we have

$$Z = \frac{4}{e^{-i\theta}+e^{i\theta}+2} = \frac{2}{\cos\theta+1} = \frac{1}{\cos^2\left(\frac{\theta}{2}\right)+1}$$
$$= \sec^2\left(\frac{\theta}{2}\right),$$

hence

$$Z = \sec^2(\frac{n}{2}\sqrt{-k_1}\xi). \tag{24}$$

Case 2. Suppose that $k_1 > 0$, then

$$\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}} = -e^{\theta}, \tag{25}$$

therefore, according to (25), we have

$$Z = -\frac{4}{e^{-\theta}+e^{\theta}+2} = \frac{2}{\cosh\theta-1} = \frac{1}{\sinh^2\left(\frac{\theta}{2}\right)+1}$$
$$= -\csc h^2\left(\frac{\theta}{2}\right),$$

so

$$Z = -\operatorname{csch}^2(\frac{n}{2}\sqrt{k_1}\xi). \tag{26}$$

Now, assume that $k_1 < 0$, then

$$\frac{1-\sqrt{1-Z}}{1+\sqrt{1-Z}} = -e^{-i\theta}, \tag{27}$$

thus, according to (27), we have

$$Z = -\frac{4}{e^{-i\theta}+e^{i\theta}-2} = \frac{2}{1-\cos\theta} = \frac{1}{\sin^2\left(\frac{\theta}{2}\right)} = \csc^2\left(\frac{\theta}{2}\right),$$

so

$$Z = \csc^2(\frac{n}{2}\sqrt{-k_1}\xi). \tag{28}$$

Now, using the relations (7), (22), (24), (26) and (28), the solutions of Eq. (7) are in the following forms:

When $k_1 > 0$, the solutions of Eq. (7) are

$$U_1(\xi) = \left\{\frac{(n+2)k_1}{2k_2}\operatorname{csch}^2(\frac{n}{2}\sqrt{k_1}\xi)\right\}^{\frac{1}{n}},$$

$$U_2(\xi) = \left\{-\frac{(n+2)k_1}{2k_2}\operatorname{sech}^2(\frac{n}{2}\sqrt{k_1}\xi)\right\}^{\frac{1}{n}}.$$

When $k_1 < 0$, the solutions of Eq. (7) are

$$U_3(\xi) = \left\{\frac{(n+2)k_1}{2k_2}\csc^2(\frac{n}{2}\sqrt{-k_1}\xi)\right\}^{\frac{1}{n}},$$

$$U_4(\xi) = \left\{\frac{(n+2)k_1}{2k_2}\sec^2(\frac{n}{2}\sqrt{-k_1}\xi)\right\}^{\frac{1}{n}}.$$

□

3 The Kadomtsev-Petviashvili equation with power law nonlinearity

In this section, we have applied functional variable method to obtain the exact solutions of the Kadomtsev-Petviashvili equation with power law nonlinearity in the form [17]

$$\frac{\partial}{\partial x}\left(\frac{\partial u(x,y,t)}{\partial t}+au^n(x,y,t)\frac{\partial u(x,y,t)}{\partial x}+\frac{\partial^3 u(x,y,t)}{\partial x^3}\right)+b\frac{\partial^2 u(x,y,t)}{\partial y^2}=0,\ \ t>0. \tag{29}$$

Here in Eq. (29) a and b are real valued constants. The first term represents the evolution term while the coefficient of a is the nonlinear term with the power law dictated by the exponent n while the third term represents the dispersion in the x-direction. Finally, the coefficient of b represents dispersion in the y direction. This equation studied in [17] by the ansatz method for 1-soliton solution. Arbabi [18] used the sine-cosine methods to obtain exact solutions of Eq. (29). The special case $n = 2$ is known as the modified Kadomtsev-Petviashvili equation. It needs to be noted here that if $b < 0$, Eq. (29) is known as the KP-I equation while if $b > 0$, (29) is known as the KP-II equation [19].

Using the traveling wave transformation$u(x, y, t) = U(\xi)$, $\xi = l_1x + l_2y - \lambda t$, Eq. (29) can be reduced to the following nonlinear ODE

$$l_1(-\lambda U_\xi + al_1U^nU_\xi + l_1^3U_{\xi\xi\xi})_\xi + l_2^2bU_{\xi\xi} = 0. \tag{30}$$

Integrating Eq. (30) twice with respect to ξ and setting the integration constants as zero yields

$$-l_1\lambda U+\frac{al_1^2}{n+1}U^{n+1}+l_1^4U_{\xi\xi}+l_2^2bU=0, \tag{31}$$

or

$$U_{\xi\xi}=\frac{l_1\lambda-l_2^2b}{l_1^4}U-\frac{a}{l_1^2(n+1)}U^{n+1}. \tag{32}$$

Substituting Eq. (5) into Eq. (32) we obtain

$$(F(U)^2)'=\frac{2(l_1\lambda-l_2^2b)}{l_1^4}U-\frac{2a}{l_1^2(n+1)}U^{n+1}, \tag{33}$$

where the prime denotes differentiation with respect to ξ. Integrating Eq. (33) and after the mathematical manipulations, we have

$$F(U)=\pm\sqrt{\frac{(l_1\lambda-l_2^2b)}{l_1^4}}U\times\sqrt{1-\frac{2al_1^2}{(n+1)(n+2)(l_1\lambda-l_2^2b)}U^n}. \tag{34}$$

Now, using the relations (5), (8), (9), (10) and (11), we deduce the following exact solutions of the Kadomtsev-Petviashvili equation with power law nonlinearity.

When $\frac{(l_1\lambda-l_2^2b)}{l_1^4}>0$, we have

$$u_1(x,y,t)=\left\{-\frac{(n+1)(n+2)(l_1\lambda-l_2^2b)}{2al_1^2}\times \operatorname{csch}^2\left(\frac{n}{2}\sqrt{\frac{(l_1\lambda-l_2^2b)}{l_1^4}}(l_1x+l_2y-\lambda t)\right)\right\}^{\frac{1}{n}}, \tag{35}$$

$$u_2(x,y,t)=\left\{\frac{(n+1)(n+2)(l_1\lambda-l_2^2b)}{2al_1^2}\times \operatorname{sech}^2\left(\frac{n}{2}\sqrt{\frac{(l_1\lambda-l_2^2b)}{l_1^4}}(l_1x+l_2y-\lambda t)\right)\right\}^{\frac{1}{n}}, \tag{36}$$

and for $\frac{(l_1\lambda-l_2^2b)}{l_1^4}<0$, we obtain the periodic wave solutions

$$u_3(x,y,t)=\left\{\frac{(n+1)(n+2)(l_1\lambda-l_2^2b)}{2al_1^2}\times \csc^2\left(\frac{n}{2}\sqrt{-\frac{(l_1\lambda-l_2^2b)}{l_1^4}}(l_1x+l_2y-\lambda t)\right)\right\}^{\frac{1}{n}}, \tag{37}$$

$$u_4(x,y,t)=\left\{\frac{(n+1)(n+2)(l_1\lambda-l_2^2b)}{2al_1^2}\times \sec^2\left(\frac{n}{2}\sqrt{-\frac{(l_1\lambda-l_2^2b)}{l_1^4}}(l_1x+l_2y-\lambda t)\right)\right\}^{\frac{1}{n}}. \tag{38}$$

Let us compare between our results obtained in the present article with the well-known results obtained by other authors using different methods as follows: our results of the Kadomtsev-Petviashvili equation with power law nonlinearity are a few different from those obtained in [17] and [18].

4 The (1+3)-Zakharov-Kuznetsov equation with power law nonlinearity

In this part we are trying to find exact solution of the (1+3)-Zakharov-Kuzetsov equation with power law nonlinearity in the form of [20]

$$\frac{\partial u(x,y,z,t)}{\partial t}+au^n(x,y,z,t)\frac{\partial u(x,y,z,t)}{\partial x}+b\frac{\partial}{\partial x}\left(\frac{\partial^2 u(x,y,z,t)}{\partial x^2}+\frac{\partial^2 u(x,y,z,t)}{\partial y^2}+\frac{\partial^2 u(x,y,z,t)}{\partial z^2}\right)=0. \tag{39}$$

In Eq. (39), a and b are real valued constants. The first term is the evolution term, while the coefficients of a and b, respectively, are the nonlinearity and dispersion. Also the parameter n is the power law nonlinearity parameter. Solitons are the result of a delicate balance between dispersion and nonlinearity. Eq. (39) typically appears in the study of plasma physics. B. T. Matebese et al. in [21] solved the (3+1)-dimensional Zakharov-Kuzetsov equation by G'/G–expansion method, extended tanh-function method and ansatz metod. The special case where $n = 1$ gives the (3+1)-dimensional Zakharov-Kuzetsov equation [22]. Let

$$U(\xi) = u(x, y, z, t), \ \ \xi = l_1 x + l_2 y + l_3 z - \lambda t, \tag{40}$$

with puting the relation (40) and its derivatives in to the Eq. (39)

$$-\lambda U_\xi + a l_1 U^n U_\xi + b l_1 (l_1^2 U_{\xi\xi} + l_2^2 U_{\xi\xi} + l_3^2 U_{\xi\xi})_\xi = 0. \tag{41}$$

Integrating Eq. (41) once with respect to ξ and setting the integration constants as zero yields

$$-\lambda U + \frac{a l_1}{n+1} U^{n+1} + b l_1 (l_1^2 U_{\xi\xi} + l_2^2 U_{\xi\xi} + l_3^2 U_{\xi\xi}) = 0, \tag{42}$$

or

$$U_{\xi\xi} = \frac{2\lambda}{b l_1 (l_1^2 + l_2^2 + l_3^2)} U - \frac{2a}{(n+1) b (l_1^2 + l_2^2 + l_3^2)} U^{n+1}. \tag{43}$$

Let use transformation (5) for Eq. (43)

$$(F(U)^2)' = \frac{2\lambda}{b l_1 (l_1^2 + l_2^2 + l_3^2)} U - \frac{2a}{(n+1) b (l_1^2 + l_2^2 + l_3^2)} U^{n+1}, \tag{44}$$

where the prime denotes differentiation with respect to ξ.with Integrating Eq. (44) and after the mathematical calculations, we have

$$F(U) = \pm \sqrt{\frac{\lambda}{b l_1 (l_1^2 + l_2^2 + l_3^2)}} U \sqrt{1 - \frac{2 l_1 a}{\lambda (n+1)(n+2)} U^n}. \tag{45}$$

Using the relations (5), (8), (9), (10) and (11), we have the following traveling wave solutions of the (1+3)-Zakharov-Kuzetsov equation with power law nonlinearity which contain traveling wave solutions as follows.

So we can obtain following hyperbolic solution for $\frac{\lambda}{b l_1 (l_1^2 + l_2^2 + l_3^2)} > 0$ as

$$u_1(x, y, z, t) = \left\{ -\frac{\lambda (n+1)(n+2)}{2 l_1 a} \times \operatorname{csc} h^2 \left(\frac{n}{2} \sqrt{\frac{\lambda}{b l_1 (l_1^2 + l_2^2 + l_3^2)}} (l_1 x + l_2 y + l_3 z - \lambda t) \right) \right\}^{\frac{1}{n}}, \tag{46}$$

$$u_2(x, y, z, t) = \left\{ \frac{\lambda (n+1)(n+2)}{2 l_1 a} \times \operatorname{sec} h^2 \left(\frac{n}{2} \sqrt{\frac{\lambda}{b l_1 (l_1^2 + l_2^2 + l_3^2)}} (l_1 x + l_2 y + l_3 z - \lambda t) \right) \right\}^{\frac{1}{n}}, \tag{47}$$

and for $\frac{\lambda}{b l_1 (l_1^2 + l_2^2 + l_3^2)} < 0$,

$$u_3(x, y, z, t) = \left\{ \frac{\lambda (n+1)(n+2)}{2 l_1 a} \times \csc^2 \left(\frac{n}{2} \sqrt{-\frac{\lambda}{b l_1 (l_1^2 + l_2^2 + l_3^2)}} (l_1 x + l_2 y + l_3 z - \lambda t) \right\}^{\frac{1}{n}}, \tag{48}$$

$$u_4(x, y, z, t) = \left\{ \frac{\lambda (n+1)(n+2)}{2 l_1 a} \times \sec^2 \left(\frac{n}{2} \sqrt{-\frac{\lambda}{b l_1 (l_1^2 + l_2^2 + l_3^2)}} (l_1 x + l_2 y + l_3 z - \lambda t) \right) \right\}^{\frac{1}{n}}. \tag{49}$$

If we put $l_1 = \alpha\rho$, $l_2 = -(\alpha\rho + 1)$, $l_3 = 1$ and $\lambda = \alpha$,our solution (47) turn out to solution (4) obtained in [21], but other our solutions of the Eq. (39) are new.

5 Benjamin-Bona-Mahuny-Peregrine equation with power law nonlinearity

Our purpose in this section is trying to use functional variable method to obtain the exact solutions of the Benjamin-Bona-Mahony-Peregrine equation with power law nonlinearity in the form [23]

$$\frac{\partial u(x,t)}{\partial t} + a \frac{\partial u(x,t)}{\partial x} + b u^n(x,t) \frac{\partial u(x,t)}{\partial x} + c \frac{\partial^3 u(x,t)}{\partial^2 x \partial t} = 0, \tag{50}$$

where coefficients a, b, c and n are real constants. The exponent n represents the power law nonlinearity parameter and it is necessary to have $n \neq 0$ since these values will place (50) outside the nonlinear regime. The firs term represents the evolution term, while the last term represents the dispersion term. The third term is the nonlinear term. Khalique in [23] obtained exact solutions of (50) using Lie symmetry approach and simplest equation method. The special case where $n = 2$, the Benjamin-Bona-Mahony equation with power law nonlinearity is called the modified Benjamin-Bona-Mahony equation [24]. Let

$$U(\xi) = u(x,t), \ \ \xi = lx - \lambda t, \tag{51}$$

from relation (51) and its derivatives we have

$$-\lambda U_\xi + a l U_\xi + b l U^n U_\xi - \lambda c l^2 U_{\xi\xi\xi} = 0. \tag{52}$$

Integrating Eq. (52) once with respect to ξ and setting the integration constants as zero yields

$$-\lambda U + a l U + \frac{bl}{n+1} U^{n+1} - \lambda c l^2 U_{\xi\xi} = 0, \tag{53}$$

or

$$U_{\xi\xi} = \frac{2(al-\lambda)}{\lambda cl^2}U + \frac{2b}{(n+1)\lambda cl}U^{n+1}, \tag{54}$$

Let use transformation (5) for Eq. (54)

$$(F(U))' = \frac{2(al-\lambda)}{\lambda cl^2}U + \frac{2b}{(n+1)\lambda cl}U^{n+1}, \tag{55}$$

where the prime denotes differentiation with respect to ξ. Integrating Eq. (55) and after the mathematical calculations, we have

$$F(U) = \pm\sqrt{\frac{al-\lambda}{\lambda cl^2}}U\sqrt{1 + \frac{2bl}{(n+1)(n+2)(al-\lambda)}U^n}. \tag{56}$$

Using the relations (5), (8), (9), (10) and (11), when $\frac{al-\lambda}{\lambda cl^2} > 0$, the solution of Eq. (50) is in the following forms

$$u_1(x,t) = \left\{\frac{(n+1)(n+2)(al-\lambda)}{2bl}\operatorname{csch}^2\left(\frac{n}{2}\sqrt{\frac{al-\lambda}{\lambda cl^2}}(lx-\lambda t)\right)\right\}^{\frac{1}{n}}, \tag{57}$$

$$u_2(x,t) = \left\{-\frac{(n+1)(n+2)(al-\lambda)}{2bl}\operatorname{sech}^2\left(\frac{n}{2}\sqrt{\frac{al-\lambda}{\lambda cl^2}}(lx-\lambda t)\right)\right\}^{\frac{1}{n}}, \tag{58}$$

and for $\frac{al-\lambda}{\lambda cl^2} < 0$,

$$u_3(x,t) = \left\{-\frac{(n+1)(n+2)(al-\lambda)}{2bl}\csc^2\left(\frac{n}{2}\sqrt{-\frac{al-\lambda}{\lambda cl^2}}(lx-\lambda t)\right)\right\}^{\frac{1}{n}}, \tag{59}$$

$$u_4(x,t) = \left\{-\frac{(n+1)(n+2)(al-\lambda)}{2bl}\sec^2\left(\frac{n}{2}\sqrt{-\frac{al-\lambda}{\lambda cl^2}}(lx-\lambda t)\right)\right\}^{\frac{1}{n}}. \tag{60}$$

Equations (57)–(58) and (59)–(60) are new types of exact traveling wave solutions to the Benjamin-Bona-Mahony-Peregrine equation with power law nonlinearity.

6 Boussinesq equation with power law nonlinearity

Finaly, we use functional variable method for solving Boussinesq equation with power law nonlinearity as introduce below [25]

$$\frac{\partial^2 u(x,t)}{\partial t^2} - a\frac{\partial^4 u(x,t)}{\partial x^4} - b\frac{\partial^2 u(x,t)}{\partial x^2} - c\frac{\partial^2(u^{2n}(x,t))}{\partial x^2} = 0, \tag{61}$$

which a, b, c and n are real-valued constants and $n > 1$. The nonlinear term is generalized to an arbitrary exponent n thus making it into power law nonlinearity. By the ansatze method, Anjan Biswas et al. [4] obtained new solitonary solutions for Eq. (61). The special case where $n = 2$, the Boussinesq equation with power law nonlinearity equation is called the (1+1)-dimensional Boussinesq equation that describes the propagation of long waves on the surface of water with a small amplitude and plays a vital part in fluid mechanics [26].

Let

$$u(x,t) = U(\xi), \qquad \xi = lx - \lambda t, \tag{62}$$

with using relation (62) and its derivatives in Eq. (61) we have

$$\lambda^2 U_{\xi\xi} - al^4 U_{\xi\xi\xi\xi} - bl^2 U_{\xi\xi} - cl^2(U^{2n})_{\xi\xi} = 0 \tag{63}$$

Integrating Eq. (63) twice with respect to ξ and setting the integration constants as zero yields

$$U_{\xi\xi} = \frac{(\lambda^2 - bl^2)}{al^4}U - \frac{c}{al^2}U^{2n}. \tag{64}$$

Let use transformation (5) for Eq. (64)

$$(F(U)^2)' = \frac{2(\lambda^2 - bl^2)}{al^4}U - \frac{2c}{al^2}U^{2n}. \tag{65}$$

If integrate once Eq. (65) and do some simple calculates

$$F(U) = \pm\sqrt{\frac{\lambda^2 - bl^2}{al^4}}U\sqrt{1 - \frac{2cl^2}{(\lambda^2 - bl^2)(2n+1)}U^{2n-1}}. \tag{66}$$

Using the relations (5), (8), (9), (10) and (11), when $\frac{\lambda^2 - bl^2}{al^4} > 0$, the solution of Eq. (60) is in the following forms

$$u_1(x,t) = \left\{-\frac{(\lambda^2 - bl^2)(2n+1)}{2cl^2}\csc h^2\left(\frac{2n-1}{2}\sqrt{\frac{\lambda^2 - bl^2}{al^4}}(lx-\lambda t)\right)\right\}^{\frac{1}{2n-1}}, \tag{67}$$

$$u_2(x,t) = \left\{\frac{(\lambda^2 - bl^2)(2n+1)}{2cl^2}\sec h^2\left(\frac{2n-1}{2}\sqrt{\frac{\lambda^2 - bl^2}{al^4}}(lx-\lambda t)\right)\right\}^{\frac{1}{2n-1}}, \tag{68}$$

also for $\frac{\lambda^2-bl^2}{al^4} < 0$,

$u_3(x,t) =$

$$\left\{ \frac{(\lambda^2 - bl^2)(2n+1)}{2cl^2} \csc^2\left(\frac{2n-1}{2}\sqrt{-\frac{\lambda^2 - bl^2}{al^4}}(lx - \lambda t)\right) \right\}^{\frac{1}{2n-1}}, \tag{69}$$

$u_4(x,t) =$

$$\left\{ \frac{(\lambda^2 - bl^2)(2n+1)}{2cl^2} \sec^2\left(\frac{2n-1}{2}\sqrt{-\frac{\lambda^2 - bl^2}{al^4}}(lx - \lambda t)\right) \right\}^{\frac{1}{2n-1}}. \tag{70}$$

Our exact solutions (67) to (70) of Equation (61) are new.

7 Conclusions

In this work, the functional variable method was applied successfully for solving four equations with power law nonlinearity namely the Kadomtsev-Petviashvili equation, the (3+1)-Zakharov-Kuzetsov equation, the Benjamin-Bona-MahonyPeregrine equation and the Boussinesq equation. The performance of this method is reliable and effective and gives the exact solitary wave solutions and periodic wave solutions. This method has more advantages: it is direct and concise. Moreover, we conclude that presented method is reliable, and yields an effective approach for finding solutions of nonlinear equations, arising in applied physics and engineering.

Acknowledgement: We are very grateful to three anonymous referees for their careful reading and valuable comments which led to the improvement of this paper.

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