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An efficient computer based wavelets approximation method to solve Fuzzy boundary value differential equations

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Abstract: In the present work a wavelets approximation method is employed to solve fuzzy boundary value differential equations (FBVDEs). Essentially, a truncated Legendre wavelets series together with the Legendre wavelets operational matrix of derivative are utilized to convert FBVDE into a simple computational problem by reducing it into a system of fuzzy algebraic linear equations. The capability of scheme is investigated on second order FBVDE considered under generalized H-differentiability. Solutions are represented graphically showing competency and accuracy of this method.

Keywords: H-differentiability; Fuzzy boundary value differential equation (FBVDE); Legendre wavelets; shifted Legendre polynomials; operational matrix

1 Introduction

Fuzzy boundary value differential equations have become focus of interest for many researchers in different disciplines of science and technology because of the fact that while modeling a physical phenomenon some quantities for instance, boundary values can be uncertain thus fuzzy numbers or fuzzy functions are then measured to deal with these uncertainties. The numerical solution of FBVDEs is an important problem in numerical analysis, therefore several research works are brought out on the development of numerical and analytical methods for the solutions of FBVDEs [1–4].

In recent years, wavelet analysis has gained special attention for its wide applications in science and engineering. Its conventional applications are not only found in signal and image processing, but also in dealing with integral and differential equations. The main advantageous characteristic of wavelet is the ability of converting the differential and integral equations into a system of linear and nonlinear system of algebraic equations with the aid of its operational matrices of derivatives and integration. Wavelet analysis mainly approximates the functions through family of orthogonal functions. These functions are achieved through the dilation and translation of mother wavelet function. Some of the most frequently used orthogonal functions are sine-cosine functions, Chebyshev, Laguerre and Legendre functions. Recently, different types of wavelets according to their orthogonal functions have been utilized for the solution of integral and differential problems of integer and fractional order, for instance, B-spline [5], Chebyshev wavelets [6–9], Haar wavelets method [10–13] and Legendre wavelets [14–17]. For the reason of simplicity and having good interpolating properties, Legendre wavelets method (LWM) has received considerable attention in dealing with various problems. LWM uses Legendre polynomials as their basis functions and has the remarkable property of giving accurate solution for small number of collocation points. In several research papers [18–21] the operational matrix of integration and derivative of Legendre wavelets have been developed and applied on different problems.

In view of successful and increasing applications of LWM together with its operational matrix of derivative and operational matrix of integration in numerical solution of integral and differential equations, we hold that it should be applicable to solve FBVDEs as well. Thus in this paper our aim is to elaborate applications of LWM in solving linear second order FBVDE under generalized H-differentiability. The main advantage of this approach is that it reduces the differential equations to system of algebraic equations, hence greatly simplifies the problem and makes the computation easy. Here, fuzzy function and its derivatives are approximated by employing truncated se-

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ries of Legendre wavelets and operational matrix of derivative of Legendre wavelets, respectively. Further, the paper is structured as follows: In Section 2, some significant definitions of fuzzy theory and mathematical notations of FBVDE are described. Section 3, demonstrates formulation of Legendre wavelets and Legendre polynomials, details of function approximation by Legendre wavelets and the Legendre operational matrix of derivative. In Section 4, scheme of the proposed method is illustrated in detail. Solutions of a second order FBVDE for all the cases of H-differentiability are discussed graphically in Section 5. Finally, conclusion is drawn in Section 6 on the basis of facts and figures from Section 5.

2 Preliminaries

Basic definition of fuzzy number and its detailed properties are found in many research papers such as [22, 23]. Here, we introduce definitions and some notations which are prerequisite of this paper.

Let $\mathbf{R}$ be set of all real numbers, a mapping $w : \mathbf{R} \to [0, 1]$ is said to be a fuzzy number if w is upper semi continuous, fuzzy convex, normal and compact. The parametric form of a fuzzy number w is an ordered pair of functions $\underline{w}(\rho)$ and $\overline{w}(\rho)$ called lower and upper branch of w, respectively. It can also be represented as $[w]^{\rho} = [\underline{w}(\rho) ; \overline{w}(\rho)]$ for $0 \le \rho \le 1$, with the properties that, $\underline{w}(\rho) \le \overline{w}(\rho)$ where $\underline{w}(\rho)$ and $\overline{w}(\rho)$ are bounded non-decreasing and non-increasing functions, respectively, left continuous in $\rho \in (0, 1]$ and right continuous at $\rho = 0$. Any functiong (ξ) is said to be a fuzzy valued function, if for all $\xi \in \mathbf{R}$, $g : \mathbf{R} \to \mathbf{E}$ where $\mathbf{E}$ is the space of all fuzzy numbers on $\mathbf{R}$.

2.1 Generalized H-differentiability of FBVDE

In this paper, we propose boundary value problem for second order FDE considered under strongly generalized Hukuhara differentiability. Hukuhara derivative has been followed by many authors for ordinary as well as fractional fuzzy differential equations of initial and boundary value problems. Theorems related to generalized H-differentiability of fuzzy-valued functions are found in many papers [4, 24, 25]. Consider the following FBVDE,

$$\kappa''(\xi) = \mathrm{H}\left(\xi, \kappa(\xi), \kappa'(\xi)\right), \quad \kappa(0) = \mu, \quad \kappa(b) = \nu. \tag{1}$$

Where $\xi \in [0, b)$, $\kappa(\xi)$ is fuzzy valued function with parametric representation $[\kappa(\xi;\rho)] = [\underline{\kappa}(\xi;\rho), \overline{\kappa}(\xi;\rho)]$ for $0 \le \rho \le 1$, $H\left(\xi, \kappa(\xi), \kappa'(\xi)\right)$is linear fuzzy function, μ and ν are fuzzy numbers such that $[\mu(\rho)] = \left[\underline{\mu}(\rho), \overline{\mu}(\rho)\right]$ and $[\nu(\rho)] = [\underline{\nu}(\rho), \overline{\nu}(\rho)]$.

Let Eq. (1) be strongly generalized H-differentiable then we have following four cases of FBVDE.

Case 1: When $\kappa(\xi)$ and$\kappa'(\xi)$ are (1)-differentiable. Then

$$\begin{aligned}\kappa''(\xi) &= \left[\underline{\kappa''}(\xi;\rho), \overline{\kappa''}(\xi;\rho)\right] \\ &= \left[\mathrm{H}\left(\xi, \underline{\kappa}(\xi;\rho), \underline{\kappa'}(\xi;\rho)\right), \mathrm{H}\left(\xi, \overline{\kappa}(\xi;\rho), \overline{\kappa'}(\xi;\rho)\right)\right]\end{aligned} \tag{2}$$

$$\begin{aligned}&\text{with } [\underline{\kappa}(0;\rho), \overline{\kappa}(0;\rho)] = \left[\underline{\mu}(\rho), \overline{\mu}(\rho)\right] \\ &\text{and } [\underline{\kappa}(b;\rho), \overline{\kappa}(b;\rho)] = [\underline{\nu}(\rho), \overline{\nu}(\rho)]\end{aligned} \tag{3}$$

Case 2: When $\kappa(\xi)$ and $\kappa'(\xi)$ are (2)-differentiable. Then

$$\begin{aligned}\kappa''(\xi) &= \left[\underline{\kappa''}(\xi;\rho), \overline{\kappa''}(\xi;\rho)\right] \\ &= \left[\mathrm{H}\left(\xi, \underline{\kappa}(\xi;\rho), \overline{\kappa'}(\xi;\rho)\right), \mathrm{H}\left(\xi, \overline{\kappa}(\xi;\rho), \underline{\kappa'}(\xi;\rho)\right)\right]\end{aligned} \tag{4}$$

$$\begin{aligned}&\text{with } [\underline{\kappa}(0;\rho), \overline{\kappa}(0;\rho)] = \left[\underline{\mu}(\rho), \overline{\mu}(\rho)\right] \\ &\text{and } [\underline{\kappa}(b;\rho), \overline{\kappa}(b;\rho)] = [\underline{\nu}(\rho), \overline{\nu}(\rho)]\end{aligned} \tag{5}$$

Case 3: When $\kappa(\xi)$ is (1)-differentiable and $\kappa'(\xi)$ is (2)-differentiable. Then

$$\begin{aligned}\kappa''(\xi) &= \left[\overline{\kappa''}(\xi;\rho), \underline{\kappa''}(\xi;\rho)\right] \\ &= \left[\mathrm{H}\left(\xi, \underline{\kappa}(\xi;\rho), \underline{\kappa'}(\xi;\rho)\right), \mathrm{H}\left(\xi, \overline{\kappa}(\xi;\rho), \overline{\kappa'}(\xi;\rho)\right)\right]\end{aligned} \tag{6}$$

$$\begin{aligned}&\text{with } [\underline{\kappa}(0;\rho), \overline{\kappa}(0;\rho)] = \left[\underline{\mu}(\rho), \overline{\mu}(\rho)\right] \\ &\text{and } [\underline{\kappa}(b;\rho), \overline{\kappa}(b;\rho)] = [\underline{\nu}(\rho), \overline{\nu}(\rho)]\end{aligned} \tag{7}$$

Case 4: When $\kappa(\xi)$ is (2)-differentiable and $\kappa'(\xi)$ is (1)-differentiable. Then

$$\begin{aligned}\kappa''(\xi) &= \left[\overline{\kappa''}(\xi;\rho), \underline{\kappa''}(\xi;\rho)\right] \\ &= \left[\mathrm{H}\left(\xi, \underline{\kappa}(\xi;\rho), \overline{\kappa'}(\xi;\rho)\right), \mathrm{H}\left(\xi, \overline{\kappa}(\xi;\rho), \underline{\kappa'}(\xi;\rho)\right)\right]\end{aligned} \tag{8}$$

$$\begin{aligned}&\text{with } [\underline{\kappa}(0;\rho), \overline{\kappa}(0;\rho)] = \left[\underline{\mu}(\rho), \overline{\mu}(\rho)\right] \\ &\text{and } [\underline{\kappa}(b;\rho), \overline{\kappa}(b;\rho)] = [\underline{\nu}(\rho), \overline{\nu}(\rho)]\end{aligned} \tag{9}$$

3 Legendre Wavelets

Wavelets constitute a family of functions constructed from dilation parameter α and the translation parameter β of a single function called the mother wavelet and on continuous variation of α and β the family of continuous wavelets are obtained, that is

$$\eta_{\alpha,\beta}(\xi) = |\alpha|^{-k} \eta\left(\frac{\xi - \beta}{\alpha}\right), \quad \alpha, \beta \in \mathbf{R}, \alpha \neq 0. \tag{10}$$

If parameters α and β are restricted to discrete values as $\alpha = \alpha_0^{-k}$ and $\beta = n\beta_0\alpha_0^{-k}$, $\alpha_0 > 1$, $\beta_0 > 0$ where n and k are positive integers we get the following family of discrete wavelets:

$$\eta_{k,n}(\xi) = |\alpha|^{-k}\,\eta\left(\alpha_0^k\xi - n\beta_0\right) \tag{11}$$

Where $\eta_{k,n}(\xi)$ form a basis of $\mathbf{L}^2(\mathbf{R})$. In particular, when $\alpha_0 = 2$ and $\beta_0 = 1$, $\eta_{k,n}(\xi)$ forms an orthonormal basis.

Legendre wavelets $\eta_{nm}(\xi) = \eta(k, \hat{n}, m, \xi)$ are defined on the interval $[0, 1)$ and have four arguments: $\hat{n} = 2n-1$, $n = 1, 2, 3\ldots, 2^{k-1}$, k is any positive integer, m is order for Legendre polynomials and ξ is the normalized time, which are formulated as:

$$\eta_{nm}(\xi) = \begin{cases}\sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}}L_m\left(2^k\xi - \hat{n}\right), & \text{for } \frac{\hat{n}-1}{2^k} \le \xi \le \frac{\hat{n}+1}{2^k}\\ 0, & \text{otherwise}\end{cases} \tag{12}$$

where $m = 0, 1, \ldots M-1$, M is a positive integer. The coefficient $\sqrt{m+1/2}$ is for orthonormality, the dilation parameter $\alpha = 2^{-k}$, and the translation parameter $\beta = \hat{n}2^{-k}$. Here $L_m(\xi)$ is well-known Legendre polynomials of order m defined on the interval $[-1, 1]$. In order to obtain orthogonal wavelets, $L_m(\xi)$ is dilated and translated as $L_m\left(2^k\xi - \hat{n}\right)$. It is determined with the recurrence formulae of Legendre polynomials for $\left(2^k\xi - \hat{n}\right)$, that is

$$L_0\left(2^k\xi - \hat{n}\right) = 1, \quad L_1\left(2^k\xi - \hat{n}\right) = 2^k\xi - \hat{n} \tag{13}$$

$$L_{m+1}\left(2^k\xi - \hat{n}\right) = \left(\frac{2m+1}{m+1}\right)\left(2^k\xi - \hat{n}\right)L_m\left(2^k\xi - \hat{n}\right) - \left(\frac{m}{m+1}\right)L_{m-1}\left(2^k\xi - \hat{n}\right), \quad m = 1, 2, 3\ldots \tag{14}$$

If the variable of Legendre polynomial is changed to $\phi = 2\xi - 1$, the shifted Legendre polynomials $Q_m(\phi)$ are obtained, that is $Q_0(\phi) = 1$, $Q_1(\phi) = 2\phi - 1$ so on and in general

$$Q_m(\phi) = \sum_{k=0}^{m}(-1)^{m+k}\frac{(m+k)!}{(m-k)!\,(k!)^2}\phi^k \tag{15}$$

These polynomials are defined on the interval $[0, 1]$ and satisfy the orthogonality relation as:

$$\int_0^1 Q_m(\phi)\,Q_n(\phi)\,d\xi = \begin{cases}\frac{1}{2m+1} & \text{for } m = n\\ 0 & \text{for } m \neq n\end{cases} \tag{16}$$

Any function $f(\xi) \in \mathbf{L}^2([0, 1))$ can be expanded by series of Legendre wavelets described in Eq. (12) as:

$$f(\xi) = \sum_{n=1}^{+\infty}\sum_{m=0}^{+\infty} e_{nm}\eta_{nm}(\xi) \tag{17}$$

where $e_{nm} = \langle f(\xi), \eta_{nm}(\xi)\rangle$ and $\langle ., .\rangle$ denotes the inner product. Eq. (17) contains infinite terms if it is truncated then it can be written as

$$f(\xi) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} e_{nm}\eta_{nm}(\xi) = \mathrm{E}^T\eta(\xi) \tag{18}$$

where E^T and $\eta(\xi)$ are $2^{k-1}M \times 1$ matrices defined as

$$\mathrm{E} = [e_{10}, e_{11}, \ldots, e_{1M-1}, e_{20}, \ldots, e_{2M-1}, \ldots, e_{2^{k-1}0}, \ldots, e_{2^{k-1}M-1}]^T \tag{19}$$

$$\eta(\xi) = [\eta_{10}(\xi), \eta_{11}(\xi), \ldots, \eta_{1M-1}(\xi), \eta_{20}(\xi), \ldots, \eta_{2M-1}(\xi), \ldots, \eta_{2^{k-1}0}(\xi), \ldots, \eta_{2^{k-1}M-1}(\xi)]^T \tag{20}$$

The operational matrix of derivative of Legendre wavelets have been derived in [21]. To be precise, the derivative of the vector $\eta(\xi)$ can be expressed by:

$$\frac{d\eta(\xi)}{d\xi} = \mathbf{D}\eta(\xi) \tag{21}$$

where $\mathbf{D}$ is the $2^{k-1}M \times 2^{k-1}M$ operational matrix of derivative elucidated as:

$$\mathbf{D} = \begin{pmatrix}\mathbf{F} & 0 & \cdots & 0\\ 0 & \mathbf{F} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \mathbf{F}\end{pmatrix} \tag{22}$$

in which $\mathbf{F}$ is $M \times M$ matrix and its (p, q) th element is obtained as:

$$\mathbf{F}_{p,q} = \begin{cases}2^k\sqrt{(2p-1)(2q-1)} & \begin{array}{l}p = 2, \ldots, M,\\ q = 1, \ldots, p-1\\ \text{and } (p+q) \text{ odd}\end{array}\\ 0, & \text{otherwise}\end{cases} \tag{23}$$

Similarly, operational matrix of derivative $\mathbf{D}^\upsilon$ of υ-time derivative of $\eta(\xi)$ can be constructed.

4 Legendre Wavelet Scheme for FBVDE

In this section, Legendre wavelet approach is illustrated for FBVDE defined in Section 2.1. This method is based on operational matrix of derivative of Legendre wavelets. Following the discussion mentioned in [21], consider the parametric form of Eq. (1) and approximate lower and upper functions as:

$$\underline{\kappa}(\xi;\rho) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\eta_{nm}(\xi) = C^T\eta(\xi) \text{ and } \overline{\kappa}(\xi;\rho) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} r_{nm}\eta_{nm}(\xi) = R^T\eta(\xi) \tag{24}$$

where C^T and R^T are the discrete form of unknown coefficients of lower and upper functions, respectively, as described in Eq. (19). In the same way, using operational matrix of derivative Eq. (21), $\kappa'(\xi)$ and $\kappa''(\xi)$ can be approximated as:

$$\underline{\kappa'}(\xi;\rho) = C^T\mathbf{D}\eta(\xi), \qquad \overline{\kappa'}(\xi;\rho) = R^T\mathbf{D}\eta(\xi) \tag{25}$$

$$\underline{\kappa''}(\xi;\rho) = C^T\mathbf{D}^2\eta(\xi), \qquad \overline{\kappa''}(\xi;\rho) = R^T\mathbf{D}^2\eta(\xi) \tag{26}$$

Along with the boundary conditions

$$\underline{\kappa}(0;\rho) = C^T\eta(0) = \underline{\mu}(\rho) \text{ and } \overline{\kappa}(0;\rho) = R^T\eta(0) = \overline{\mu}(\rho) \tag{27}$$

$$\underline{\kappa}(b;\rho) = C^T\eta(0) = \underline{\nu}(\rho) \text{ and } \overline{\kappa}(b;\rho) = R^T\eta(0) = \overline{\nu}(\rho) \tag{28}$$

Substitute Eq. (24)–(26) in parametric form of Eq. (1) we get

$$\left[C^T\mathbf{D}^2\eta(\xi), R^T\mathbf{D}^2\eta(\xi)\right] = \left[\mathrm{H}\left(\xi, C^T\eta(\xi), C^T\mathbf{D}\eta(\xi)\right), \mathrm{H}\left(\xi, R^T\eta(\xi), R^T\mathbf{D}\eta(\xi)\right)\right] \tag{29}$$

Simplify Eq. (29) at the first $2^{k-1}M-2$ roots of shifted Legendre polynomials $Q_{2^{k-1}M}(\xi)$. A system of fuzzy algebraic linear equations are obtained. Solve these equations using Newton's iterative method to calculate values of Legendre wavelet coefficients c_{nm} and r_{nm}. On substitution of these values in Eq. (24) the desired approximated values of lower and upper functions of $\kappa(\xi)$ are obtained.

The simplicity of this method is mainly due to the conversion of the derivatives to matrix form, therefore once constructed is advantageous for approximation of all the derivatives of FBVDE considered under strongly generalized H-differentiability.

5 Illustrative Example

In order to elaborate LWM as a simple computational tool to solve FBVDE, the scheme is applied on a second order FBVDE. The outcomes are presented graphically for different values of ξ and ρ.

Example

Consider the following parametric non homogeneous FBVDE for $0 \le \rho \le 1$,

$$\left[\underline{u''}(\xi;\rho), \overline{u''}(\xi;\rho)\right] = \left[\underline{u'}(\xi;\rho) + 3\underline{u}(\xi;\rho) + \underline{h}(\xi;\rho), \overline{u'}(\xi;\rho) + 3\overline{u}(\xi;\rho) + \overline{h}(\xi;\rho)\right] \tag{30}$$

with boundary conditions

$$[\underline{u}(0;\rho), \overline{u}(0;\rho)] = \left[\frac{2}{9}(\rho-1), \frac{2}{9}(1-\rho)\right] \text{ and } [\underline{u}(1;\rho), \overline{u}(1;\rho)] = \left[\frac{2}{9}(\rho-1), \frac{2}{9}(1-\rho)\right] \tag{31}$$

Where

$$\underline{h}(\xi;\rho) = (\rho-1)(3-2\xi) - \frac{1}{3}\left(9\xi^2 - 9\xi + 2\right)(\rho-1) \tag{32}$$

$$\overline{h}(\xi;\rho) = (1-\rho)(3-2\xi) - \frac{1}{3}\left(9\xi^2 - 9\xi + 2\right)(1-\rho) \tag{33}$$

and the exact solutions $[\underline{u}(\xi;\rho), \overline{u}(\xi;\rho)] = \left[\frac{1}{9}\left(9\xi^2 - 9\xi + 2\right)(\rho-1), \frac{1}{9}\left(9\xi^2 - 9\xi + 2\right)(1-\rho)\right]$.

Since, $u(\xi;\rho)$ is strongly generalized H-differentiable, therefore we apply the proposed scheme to obtain the solution for all the four cases defined in Section 2.1.

Following the approximations defined in Section 4, Eq. (30) becomes

$$C^T\mathbf{D}^2\eta(\xi) = C^T\mathbf{D}\eta(\xi) + 3C^T\eta(\xi) + (\rho-1)(3-2\xi) - \frac{1}{3}\left(9\xi^2 - 9\xi + 2\right)(\rho-1) \tag{34}$$

$$R^T\mathbf{D}^2\eta(\xi) = R^T\mathbf{D}\eta(\xi) + 3R^T\eta(\xi) + (1-\rho)(3-2\xi) - \frac{1}{3}\left(9\xi^2 - 9\xi + 2\right)(1-\rho) \tag{35}$$

Together with Eq. (31) we get

$$\left[C^T\eta(0), R^T\eta(0)\right] = \left[\frac{2}{9}(\rho-1), \frac{2}{9}(1-\rho)\right]; \text{ and } \left[C^T\eta(1), R^T\eta(1)\right] = \left[\frac{2}{9}(\rho-1), \frac{2}{9}(1-\rho)\right] \tag{36}$$

From Eqs. (34)–(36) we attain a system of linear fuzzy algebraic equations, which are further solved to calculate the values of unknown coefficients c_{nm} and r_{nm}. The numerical solutions of lower function $\underline{u}(\xi;\rho)$ and upper function $\overline{u}(\xi;\rho)$ for $k = 1$ and $M = 3$ are shown in Figs. 1–4 for Cases I-IV, respectively. From each Figure different intervals are depicted where the solutions are found valid for $0 \le \rho \le 1$

and $\xi \in [0, 1]$. Fig. 1 presents the comparison of the exact solution and approximated solution which are found to be in a perfect agreement. The valid regions of the solution for Case I are $\xi \le \frac{1}{3}$ and $\xi \ge \frac{2}{3}$. For Case II solutions exists for $\xi \le 0.1262$ and $\xi \ge 0.8738$, as shown in Fig. 2. While Figs. 3 and 4 display invalid solution for the Cases III and IV for $\xi \in [0, 1]$.

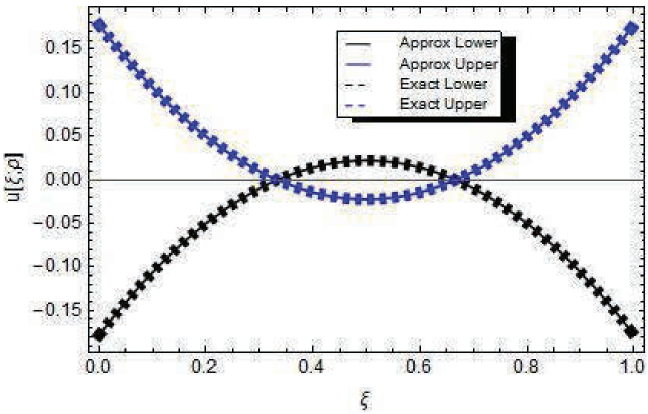

Fig. 1: Comparison of exact solutions versus approximate solutions of Case I obtained by LWM for $\rho = 0.2$.

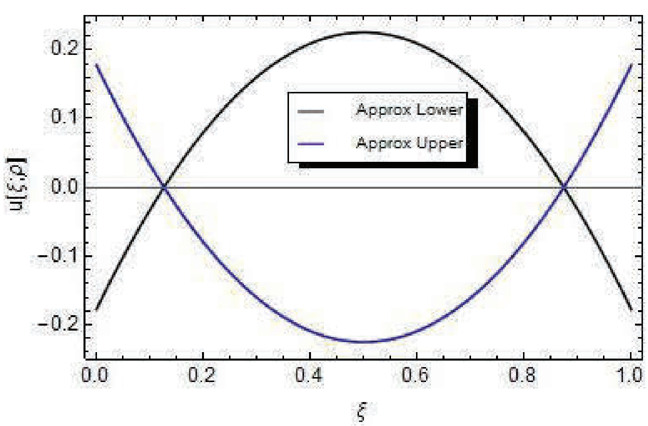

Fig. 2: The approximate solutions of Case II for $\rho = 0.2$.

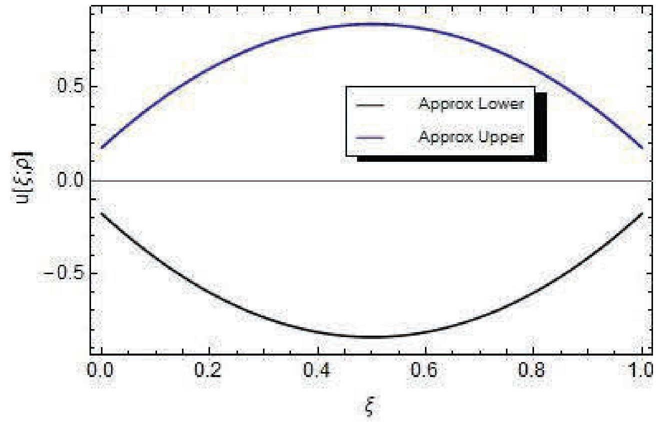

Fig. 3: The invalid region of Case III in the interval $[0, 1]$ for $\rho = 0.2$.

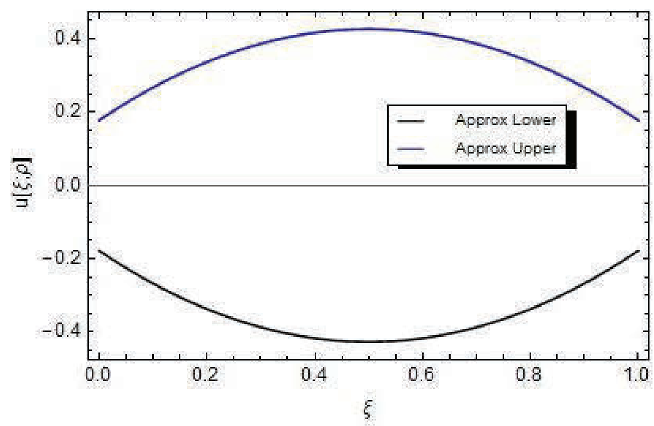

Fig. 4: The invalid region for $\xi \in [0, 1]$ of Case IV for $\rho = 0.2$.

6 Closing Remarks

In this paper, we approximated fuzzy functions and their derivatives using Legendre wavelets series and operational matrix of derivative, respectively. Further, using approximated values solutions of FBVDE was obtained, successfully. Consequently, on basis of the findings demonstrated in Section 5, we picked out the following outcomes:

- Advantageous ability of LWM to reduce the fuzzy differential equations to system of fuzzy algebraic equations made the problem easily computable to attain the solutions more rapidly than the other existing methods in literature.
- Legendre wavelets and operational matrix of derivative once calculated can be utilized for various subsequent problems repeatedly which decreases the working time.
- Small number of M and k is required to generate accurate solutions of FBVDE, which lead to an efficient approximation method to solve fuzzy initial and boundary value problems.
- Due to strongly generalized H-differentiability, four possible cases of the proposed FBVDE was considered, but only first two cases, Case I and Case II, provided valid regions of solution for $\xi \in [0, 1]$. While for Case III and Case IV, valid regions lie outside the

interval $[0, 1]$, which are considered to be invalid for $\xi \in [0, 1]$.

For rapid convergence and simple applicability of LWM, in future, we seek to apply it on other fuzzy differential models with initial and boundary conditions.

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