Amir Younespour* and Hosein Ghaffarzadeh

On using block pulse transform to perform equivalent linearization for a nonlinear Van der Pol oscillator under stochastic excitation

DOI 10.1515/nleng-2015-0022
Received September 5, 2015; accepted January 30, 2016.

Abstract: This paper applied the idea of block pulse (BP) transform in the equivalent linearization of a nonlinear system. The BP transform gives effective tools to approximate complex problems. The main goal of this work is on using BP transform properties in process of linearization. The accuracy of the proposed method compared with the other equivalent linearization including the stochastic equivalent linearization and the regulation linearization methods. Numerical simulations are applied to the nonlinear Van der Pol oscillator system under Gaussian white noise excitation to demonstrate the feasibility of the present method. Different values of nonlinearity are considered to show the effectiveness of the present method. Besides, by comparing the mean-square responses for divers values of nonlinearity and excitation intensity depicted the present method is able to approximate the behavior of nonlinear system and is in agreement with other methods.

Keywords: Equivalent Linearization, block pulse transform, Van der Pol oscillator, Gaussian white noise

1 Introduction

Study of nonlinearity in dynamic systems has been an important area of researches in last decades. In structural dynamics nonlinearities can arise in various forms and usually become progressively more significant as the motion amplitude increases. Nonlinearity may result due to boundary conditions for instant, looseness, clearances, temperature effects [1]. Nonlinear systems may exhibit interesting and complex behavior, such as limit cycles, bifurcations and even chaos which are difficult to predict. Besides in practical applications, due to the high intensity nature and often complex nature of non-stationary environmental loads such as wind and earthquakes, the systems subjected to these loadings may experience large stress or displacement that results inelastic behavior. To describe real observed processes in any field of engineering, researchers often use mathematical models. Its role is to provide a better understanding and characterization of the system. For dynamic processes, these models contain many different types of equations such as ordinary or partial differential equations and algebraic equations. In general case, due to the nature of the considered problems are nonlinear, there is no exact solution for such equations unless for some special cases [2]. Therefore the solutions are approximate. There are diverse approaches which have been developed over the years to treat the nonlinear problems such as harmonic balance, averaging, perturbation, Monte Carlo simulation and normal form transformation [3–7]. Also there are the semi-analytical methods, such as the high order harmonic balance (HOHB) method which has been developed to avoid the tedious algebraic calculations involved in the classical harmonic balance method in processing the nonlinear term in the nonlinear dynamical system [8], the high dimensional harmonic balance (HDHB) method and the time domain collocation (TDC) method [9]. Meanwhile, among methods dealing with nonlinear problems, linearization techniques are the oldest and the most popular methods of approximation. One of the linearization method is the Lyapunov linearization technique used to approximate a nonlinear system with a linear one that is around the equilibrium point, and it is expected the behavior of the linear system will be the same as that of nonlinear one. Feedback linearization is another approach, which is a common approach used in nonlinear control systems. This method is based on the idea of transforming original system model into equivalent linear one which is changed to the state variables and a suitable input instead [10]. The stochastic equivalent linearization method of dynamical system is one of the common approaches to the approximate analysis. The methods can be considered in different fields such

***Corresponding Author: Amir Younespour:** Faculty of Civil Engineering, University of Tabriz, Tabriz, Iran,
E-mail: Younespour@tabrizu.ac.ir
Hosein Ghaffarzadeh: Faculty of Civil Engineering, University of Tabriz, Tabriz, Iran

as state space, frequency domain, distribution space and characteristic functions space. Usually they consist of two main steps, in the first step, deal to find explicit or implicit analytical formulas for linearization coefficients based on the linearization criterion which is depend on unknown response characteristics such as mean value, variance, and higher-order moments. In the second step, deal to replace the unknown characteristics by the corresponding ones determined for linearized systems. It should be considered that accuracy and feasibility of these solutions dependent on the type of nonlinearity and intensity of external excitation forces. The original version of stochastic equivalent linearization present by Caughey the pioneer person in the development of the stochastic equivalent linearization procedure for estimating the mean and variance of the response of a non-linear system to random excitation. He investigated the Van der Pol oscillator and a system with bilinear hysteresis under random excitation in 1956 and 1960 respectively [11, 12]. This method is based on replacing an original nonlinear system under Gaussian random excitation by a linear one under the same excitation for which the coefficients of the equivalent system can be found from a specified optimization criterion, such as the mean-square criterion [12], energy criteria [13], spectral criteria [14, 15], and probability density criteria [16] in some probabilistic sense. The method then has been generalized to random vibration of multi-degree-of-freedom systems by Iwan and Yang and Atalik and Utku [17, 18]. The other similar procedures developed by operating directly on the equations of motion. Usually associated in the literature with the statistical linearization technique introduced by Kazakov [19], but as discussed by Socha and Pawleta the methods proposed by Caughey and Kazakov are not the same approach [20]. Many developments have been present since the fundamental work of Caughey and a comprehensive treatment of the subject can be found in [21]. In 1986, Brückner and Lin [22] generalized the method of equivalent linearization such that the response of a nonlinear oscillator subjected to both parametric and external random white noise excitations can be determined approximately. In 1993, Casciati and Faravelli [23] introduced a novel philosophy for stochastic equivalent linearization and then they studied Duffing and hysteretic oscillators in detail. In 2006, Crandall described some of interesting episodes in the past half century [24]. Some extensions of equivalent linearization were present in [25–28]. In [29] also studied systems under both type of excitations including harmonic and random excitations.

The idea of transformation is a common procedure to approximate various problems in mathematics. After the transformations are defined appropriately, mathematical expressions and solutions of many problems can be simplified. For example, the solutions of certain differential equations are complicated in the time domain, but after they are transformed into the corresponding algebraic equations, their solutions are much easier. The block pulse (BP) transform provides a useful tool to solve difference and integral equations of any order with less computational costs. The BP transform originated from BP functions. The BP functions are a set of orthogonal functions with piecewise constant values and are usually applied as a useful tool in the analysis, identification and other problems of control and systems science. Their extensive application to the area of control and systems was started three decades ago. Some papers discussed the BP transform and BP function in solving certain system and control problems. Ghaffarzadeh and Younespour proposed an equivalent linearization method for deterministic excitation based on BP transform [30]. Optimal control for distributed parameter systems via BP transform discussed in [31]. In active control problem a new method present based on BP functions evolves minimizing computational costs of analytical approaches [32].

The main objective of this study is on using the concept of BP transform in linearization procedure through its simple and easy operation. The effectiveness of the present method is validated by simulation on a nonlinear Van der Pol oscillator system under stationary Gaussian white noise excitation. The linearized system by existing methods have been compared. Numerical simulation results and comparing mean-square response of existing methods depicted the present method have good agreement with stochastic equivalent linearization and regulation linearization methods.

The remaining of this paper is organized as follows. Section 2 presents the BP transform formulation. In section 3 linearization methods including proposed method based on BP transform and stochastic equivalent linearization and regulation linearization are presented. The simulations have been carried out to compare with the other methods in Section 4 and followed by the conclusion in Section 5.

2 Block pulse transform

The aim of transformation in mathematic is to find simple solution for concrete problems. The idea of transformations can also be applied in the block pulse function technique to simplify expressions. Every functions can be expanded into their block pulse series, and operations of functions can be converted to the operations of their block pulse series. The BP transformation can be introduced to express the relations between functions and their block pulse coefficients because block pulse coefficients take information of the corresponding block pulse series of the original functions. The BP transformation provides a technique for transforming a difference equation into an algebraic equation. The BP transformation method is very similar to z-transform method that can greatly facilitate the analysis [33]. The BP transform of function $f(t)$ is defined by:

$$F(z) = \sum_{i=0}^{\infty} f_i z^{-i} = \sum_{i=0}^{\infty} f(ih) z^{-i} \tag{1}$$

where $z \in R$ and f_i are the coefficients of the terms z^{-i} ($i = 0, 1, 2, \ldots$) in the power series which are the values of the $f(t)$ at the corresponding time instants h. h is the sampling period.

By using Tustin integrator (for more details see [33]):

$$s^{-1} = \frac{h}{2}\frac{1+z^{-1}}{1-z^{-1}} \tag{2}$$

The BP transform can be defined in other form, as:

$$F(z) = \frac{1}{1-z^{-1}} F\left(\frac{h}{2}\frac{1+z^{-1}}{1-z^{-1}}\right) \tag{3}$$

where

$$\sum_{i=0}^{\infty} z^{-i} = \frac{1}{1-z^{-1}} \tag{4}$$

The summery of BP transform properties are following:

For addition and subtraction of function $x(t) = f(t) \mp g(t)$, we have:

$$X(z) = F(z) \mp G(z) \tag{5}$$

For multiplied by a scalar $x(t) = kf(t)$, we have:

$$X(z) = kF(z) \tag{6}$$

For multiplication of function $x(t) = f(t)g(t)$, we have:

$$X(z) = \sum_{i=0}^{\infty} f_i g_i z^{-i} \tag{7}$$

For division of function $x(t) = f(t)/g(t)$ with $g(t) \neq 0$, we have:

$$X(z) = \sum_{i=0}^{\infty} (f_i/g_i) z^{-i} \tag{8}$$

For derivation of function $x(t) = \frac{df(t)}{dt}$, we have:

$$X(z) = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} F(z) - f_0^{(0)} \frac{2}{h}\frac{1}{1+z^{-1}} \tag{9}$$

where $f_0^{(0)}$ is the initial value of $f(t)$.

For special case we need in this study ($x(t) = t^n$), we have:

$$X(z) = \frac{h^n}{2^n}\frac{n!(1+z^{-1})^n}{(1-z^{-1})^{n+1}} \tag{10}$$

3 Linearization approach based on BP transform

In this paper a system which is a single degree of freedom with the nonlinear function only depending on two arguments of displacement and velocity considered. We consider a van der pol oscillator of the form

$$\ddot{x} - (2\xi\omega - \alpha\mu x^2)\dot{x} + \omega^2 x = \sigma w(t) \tag{11}$$

where ξ and ω are damping ratio and natural frequency. x is the displacement response of nonlinear Van der Pol oscillator system and the nonlinear function of considered system is $g(x, \dot{x}) = \alpha\mu x^2 \dot{x}$ and α is the nonlinear factor that shows the rate of nonlinearity in nonlinear system. σ is excitation intensity and $w(t)$ is a random excitation and a stationary zero-mean Gaussian white noise which correlation function is

$$E[w(t)\, w(t+\tau)] = \sigma^2 \delta(\tau) \tag{12}$$

where $E[.]$ denotes the expectation of (.). σ^2 is variance and $\delta(\tau)$ is Dirac delta function.

It will be assumed that a stationary solution of Eq. (11) exist. Define the Eq. (13) as a linearized equation with equivalent linear coefficients to be determined to replace the nonlinear term $\alpha\mu x^2\dot{x}$.

$$\ddot{x} + (-2\xi\omega + c_{eq})\dot{x} + (\omega^2 + k_{eq})x = \sigma w(t) \tag{13}$$

The error that yields by replacing equivalent linear system by nonlinear system can be defined as the difference of the two systems. Considering Eqs. (11) and (13) one can be define the error as

$$e = \mu x^2 \dot{x} - c_{eq}\dot{x} - k_{eq}x \tag{14}$$

As the excitation is random, an apparently sensible strategy would be to minimize the average difference between the nonlinear system and the linearized system. In fact this is not sensible as the differences will generally be a mixture of negative and positive and could still average to zero for a wildly inappropriate system. The correct strategy is to minimize the expectation of the squared differences, i.e. find the c_{eq} and k_{eq} which minimize,

$$\frac{\partial E[e^2]}{\partial c_{eq}} = \frac{\partial E[e^2]}{\partial k_{eq}} = 0 \tag{15}$$

Substituting (14) into (15) and performing the partial differentiations in the resulting equation, yield

$$\mu E[x^3\dot{x}] - k_{eq}E[x^2] - c_{eq}E[\dot{x}x] = 0 \tag{16}$$

$$\mu E\left[x^2\dot{x}^2\right] - c_{eq}E[\dot{x}^2] - k_{eq}E[\dot{x}x] = 0 \tag{17}$$

As assumption the excitation is stationary Gaussian white noise and equivalent linear coefficient determined under stationary response, As the stationary displacement and velocity are uncorrelated, i.e., $E[x\dot{x}] = 0$, therefore

$$k_{eq} = 0 \tag{18}$$

and Eq. (17) derived

$$\mu E\left[x^2\dot{x}^2\right] - c_{eq}E[\dot{x}^2] = 0 \tag{19}$$

By performing the BP transform both sides of (19) yields:

$$\mu E\left[\frac{3h^4}{2}\frac{(1+z^{-1})^4}{(1-z^{-1})^5}\right] - c_{eq}E\left[\frac{h^2}{2}\frac{\left(1+z^{-1}\right)^2}{\left(1-z^{-1}\right)^3}\right] = 0 \tag{20}$$

Where $\frac{h}{2}\frac{1+z^{-1}}{1-z^{-1}}$ is Tustin integrator. And equivalent linear coefficient for nonlinear system define as

$$c_{eq} = \mu E\left[\frac{h^2}{2}\frac{\left(1+z^{-1}\right)^2}{\left(1-z^{-1}\right)^3}\right] \tag{21}$$

In practical applications involving numerical algorithms all parameters computing at discrete grid points. In discrete domain the minimization conditions are:

$$\frac{\partial}{\partial k_{eq}}\sum_{i=0}^{T} E\left[e^2\right] = 0 \tag{22}$$

Where T is the final time-instants in simulation. In fact, the equivalent linear coefficient is calculated numerically. The parameters are to be discretized over all the time steps. In order to find the equivalent parameter Eq. (23) can be defined as follow:

$$c_{eq} = \frac{\mu h}{1-z^{-1}}\left(\frac{h}{2}\frac{1+z^{-1}}{1-z^{-1}}\right)^2 \tag{23}$$

In this section we present procedure of the stochastic equivalent linearization (the method proposed by Caughey) and the regulation linearization method (including one-step regulation procedure which proposed by Elishakoff). For stochastic equivalent linearization consider a nonlinear system subject to random excitation with the equation motion:

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x + g(x, \dot{x}) = \sigma w(t) \tag{24}$$

A linearized system can be find as follows:

$$\ddot{x} + (2\xi\omega + c_e)\dot{x} + (\omega^2 + k_e)x = \sigma w(t) \tag{25}$$

where c_e and k_e are the equivalent linear coefficients to be determined by equivalent linearization approach to replace the nonlinear term.

The replacement of a nonlinear system by a linear system is in some probabilistic sense and it will yield the difference or error. The error defined as

$$J = g(x, \dot{x}) - c_e\dot{x} - k_e x \tag{26}$$

Minimizing the mean square value of J is the criterion used here, i.e.

$$J^2 = E[(g(x, \dot{x}) - c_e\dot{x} - k_e x)^2] \tag{27}$$

The coefficients c_e and k_e are determined by the following equation:

$$\frac{\partial J^2}{\partial c_e} = \frac{\partial J^2}{\partial k_e} = 0 \tag{28}$$

Noting $E[x\dot{x}] = 0$ and by considering Eq. (24) for Van der Pol oscillator we have

$$c_e = \frac{E[\dot{x}g(x, \dot{x})]}{E[\dot{x}^2]} \tag{29}$$

and

$$k_e = \frac{E[xg(x, \dot{x})]}{E[x^2]} \tag{30}$$

For more details see [3].

For the one-step regulation linearization procedure, the nonlinear term $\alpha y x^2\dot{x}$ is replaced by a higher-order term, $\alpha y_1 x^4\dot{x}$, and then this nonlinear term is replaced by other nonlinear one, $\alpha y_2 x^2\dot{x}$, and in the final step, the nonlinear term $\alpha y_2 x^2\dot{x}$ is replaced by a linear one, $C_e\dot{x}$. This linearization procedure is illustrated by the following scheme:

$$\alpha y x^2\dot{x} \longrightarrow \alpha y_1 x^4\dot{x} \longrightarrow \alpha y_2 x^2\dot{x} \longrightarrow C_e\dot{x} \tag{31}$$

For more details see [26].

4 Linearization Results

The response of nonlinear Van der Pol system can be investigated numerically by simulation of Eq. (11). The Van der Pol oscillator motion equation is a two dimensional stochastic differential equation. The response of this equation can be find via Milstein scheme [34]. The system parameters are chosen as follows, $\xi = 0.02$, $\omega = 20$, $\mu = 10^3$ and $\sigma^2 = 1$. The initial conditions are set at the equilibrium state, namely, $x(0) = 0$, $\dot{x} = 0$. For $\alpha = 1, 3, 5$ and 10, Figs. 1-4 illustrate the displacement response of original nonlinear system and the displacement response of equivalent linearized system by existing linearization methods.

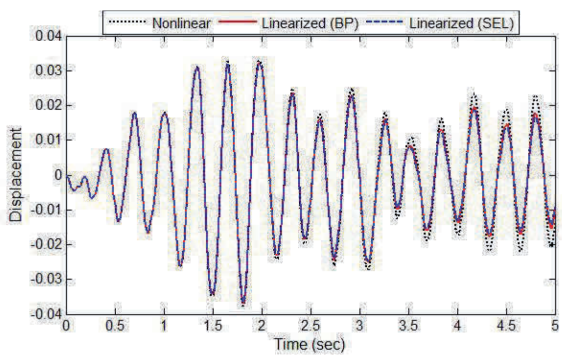

Fig. 1: Displacement response for $\alpha = 1.0$.

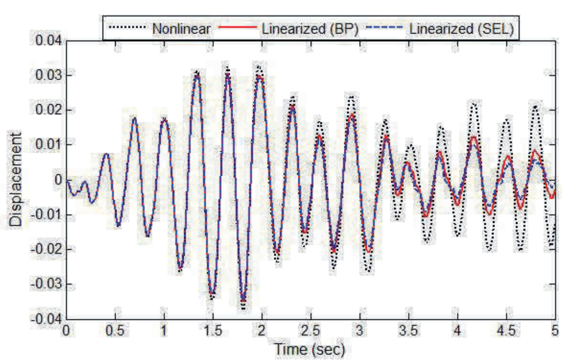

Fig. 2: Displacement response for $\alpha = 3.0$.

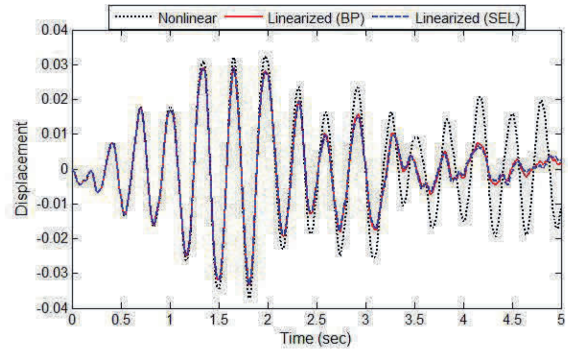

Fig. 3: Displacement response for $\alpha = 5.0$.

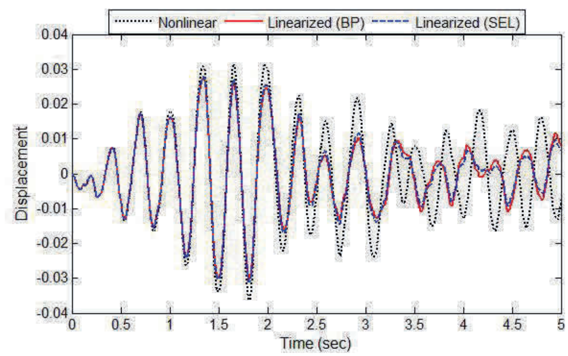

Fig. 4: Displacement response for $\alpha = 10.0$.

The numerical results on considered system reveal that the proposed linearization technique based on BP transform is a promising tool in linearizing nonlinear systems. The agreement is good as it can be seen from Figs. 1-4. But the results demonstrate that the errors increase in high rate of nonlinearity. Also, this fact can be concluded from velocity response of system. Fig. 5 demonstrated the root mean square (RMS) of velocity response of the considered system.

For more investigating and better understanding it is interested in comparing the mean-square displacement response $E[x^2]$ of the present method with the other approximate ones. The values of the approximate solutions including the solutions of the stochastic equivalent linearization method, regulation linearization method and proposed method based on BP transform are compared with a Monte-Carlo simulation for various values of nonlinearity factor α are tabulated in Table 1. Also the results of the approximate mean-square response of the considered system obtained by present methods are compared in Table 2 with $\alpha = 1$ and various values of σ^2.

It is seen that, in general, for different values of α, the errors of the present method give results better than other linearization methods. However, the approximated results get worse for large values of the nonlinearity factor. The obtained comparison shows a good agreement among these methods.

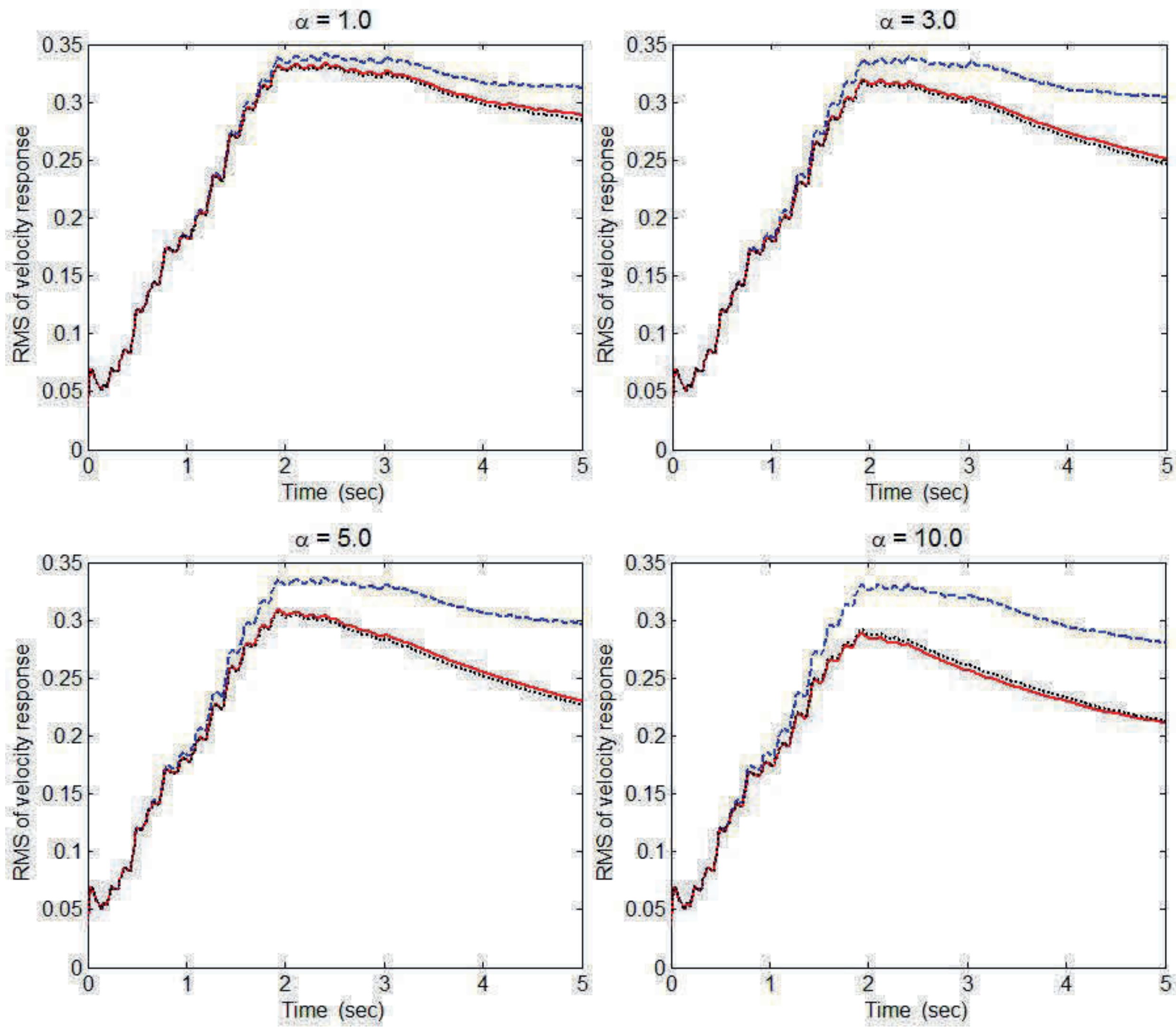

Fig. 5: Root Mean Square of velocity response; Nonlinear system (dashed line), linearized system by proposed method (solid line), linearized system by stochastic equivalent linearization (dotted line).

Table 1: The mean-square responses for different value of nonlinearity.

α	$E[x^2]_{MC}$	$E[x^2]_{BP}$	$Error(\%)$	$E[x^2]_{SEL}$	$Error(\%)$	$E[x^2]_{re}$	$Error(\%)$
1.0	1.69e-03	1.62e-03	4.14	1.56e-03	7.80	1.60e-03	5.32
3.0	5.03e-04	4.74e-04	5.76	4.71e-04	6.32	4.73e-04	5.96
5.0	3.06e-04	2.86e-04	6.54	2.71e-04	11.4	2.76e-04	9.80
8.0	1.90e-04	1.68e-04	11.4	1.69e-04	11.1	1.70e-04	10.5
10.0	1.55e-04	1.38e-04	11	1.31e-04	15.5	1.35e-04	12.9
12.0	1.30e-04	1.18e-04	9.22	1.10e-04	15.4	1.12e-04	13.8
15.0	1.04e-04	9.44e-05	9.58	8.90e-05	14.7	8.94e-05	14.3
20.0	7.89e-05	6.89e-05	12.7	6.62e-05	16.1	6.69e-05	15.2

MC Monte-Carlo simulation, *BP* present linearization method, *SEL* stochastic equivalent linearization method, *re* regulation linearization method

Table 2: The mean-square responses for different excitation intensity.

σ^2	$E[x^2]_{MC}$	$E[x^2]_{BP}$	$Error(\%)$	$E[x^2]_{SEL}$	$Error(\%)$	$E[x^2]_{re}$	$Error(\%)$
0.1	7.893e-04	7.863e-04	0.38	7.804e-04	1.13	7.834e-04	0.75
0.5	1.350e-03	1.321e-03	2.22	1.294e-03	4.15	1.300e-03	3.70
1	1.631e-03	1.648e-03	1.04	1.581e-03	3.07	1.601e-03	1.84
1.5	1.858e-03	1.885e-03	1.45	1.931e-03	3.92	1.911e-03	2.85
2	2.351e-03	2.290e-03	2.55	2.232e-03	5.11	2.250e-03	4.26

MC Monte-Carlo simulation, *BP* present linearization method, *SEL* stochastic equivalent linearization method, *re* regulation linearization method

5 Conclusion

This paper deals with linearization of nonlinear Van der Pol oscillator system. The present method based on BP transform compared with stochastic equivalent linearization and regulation linearization method in different ways. So as to analyze the performance of the present linearization approach, the considered system subjected to Gaussian white noise and the feasibility of the present method is demonstrated. The numerical simulation results and comparisons reveal that for any rate of nonlinearity in nonlinear system, the displacement response of the present method is well-approximated than other existing methods. Lastly it should be pointed out that the errors rise with the nonlinearity, but the present method errors are smaller than the utilization of the other approaches.

References

[1] Kerschen G, Worden K, Vakakisc AF, Golinvala JC.: Past, present and future of nonlinear system identification in structural dynamics. Mech Syst Signal Process 2006; 20: 505–592.

[2] Socha L. Linearization Methods for Stochastic Dynamic Systems. Berlin: Springer; 2008.

[3] Worden K, Tomlinson GR. Nonlinearity in structural dynamics. London: IOP Publishing Ltd; 2001.

[4] Verhulst F. Nonlinear Differential Equations and Dynamical Systems. New York: Springer; 1996.

[5] Glendinning P. Stability, Instability and Chaos. Cambridge, UK: Cambridge University Press; 1994.

[6] Proppe C, Pradlwarter HJ, Schueller GI. Equivalent linearization and Monte Carlo simulation in stochastic dynamics. Probab Eng Mech 2003; 18: 1–15.

[7] Wagg DJ, Nield SA. Nonlinear Vibration with Control. New York: Springer; 2010.

[8] Liu L, Thomas JP, Dowell EH, Attar P, Hall KC. A comparison of classical and high dimensional harmonic balance approaches for a Duffing oscillator. J Computational Phys 2006; 215: 298–320.

[9] Yue XK, Dai HH, Liu CS. Optimal scale polynomial interpolation technique for obtaining periodic solutions to the Duffing oscillator. Nonlinear Dyn 2014; 77: 1455–1468.

[10] Slotine JJ, Li W. Applied nonlinear control. New Jersey: Prentice-Hall; 1991.

[11] Caughey TK. Response of Van der Pol's oscillator to random excitations. Trans. ASME J. Appl. Mech. 1956; 26: 345–348.

[12] Caughey TK.: Random excitation of a system with bilinear hysteresis. Trans. ASME J Appl Mech 1960: 27: 649–652.

[13] Zhang XT, Elishakoff I, Zhang RC. A Stochastic Linearization Technique Based on Minimum Mean-Square Deviation of Potential Energies, Stochastic Structural Dynamics. Y.K. Lin and I. Elishakoff. 327–338. Springer, Berlin (1990)

[14] Crandall SH. Correlation and spectra of nonlinear system response. Nonlinear Vib. Probl. 1973; 14: 39–53.

[15] Apetaur M, Opicka F. Linearization of stochastically excited dynamic systems. J. Sound Vib. 1983; 86: 563–585.

[16] Socha L. Probability density equivalent linearization technique for nonlinear oscillator with stochastic excitations. Z. Angew. Math. Mech. 1998; 78: 1087–1088.

[17] Iwan WD, Yang I. Application of statistical linearization techniques to nonlinear multi-degree-of-freedom systems. Trans. ASME J. Appl. Mech. 1972; 39: 545–550.

[18] Atalik T, Utku S. Stochastic linearization of multi-degree-of-freedom nonlinear system. J. Earth. Eng. Struct. Dyn. 1976; 4: 411–420.

[19] Kazakov IE.: Approximate probabilistic analysis of the accuracy of operation of essentially nonlinear systems, Automatika i Telemekhanika 1956; 17: 423–450.

[20] Socha L, Pawleta M. Are statistical linearization and standard equivalent linearization the same methods in the analysis of stochastic dynamic systems. J Sound Vib 2001; 248: 387–394.

[21] Roberts JB, Spanos PD. Random Vibration and Statistical Linearization. New York: Wiley; 1990.

[22] Brückner A, Lin YK. Generalization of the equivalent linearization method for the nonlinear random vibration problems. Int J Nonlinear Mech 1987; 22: 227–235.

[23] Casciati F, Faravelli L. A new philosophy for stochastic equivalent linearization. Probab Eng Mech 1993; 8: 179–185.

[24] Crandall SH. A half-century of stochastic equivalent linearization. Struct Control Health Monit 2006; 13: 27–40.

[25] Lin YK, Elishakoff I. Stochastic Structural Dynamics–New Theoretical Developments. Berlin: Springer; 1990.

[26] Elishakoff I, Andrimasy L, Dolley M. Application and extension of the stochastic linearization by Anh and Di Paola. Acta Mech 2009; 204: 89–98.

[27] Anh ND, Hieu NN, Linh NN. A dual criterion of equivalent linearization method for nonlinear systems subjected to random excitation. Acta Mech 2012; 223: 645–654.

[28] Jiang WA, Chen LQ.: An equivalent linearization technique for nonlinear piezoelectric energy harvesters under Gaussian white noise. Commun. Nonlinear Sci Numer Simulat 2014; 19: 2897–2904.
[29] Anh ND, Zakovorotny VL, Hao DN. Response analysis of Van der Pol oscillator subjected to harmonic and random excitations. Probab Eng Mech 2014; 34:51–59.
[30] Ghaffarzadeh H, Younespour A. Block pulse transform method for linearization of nonlinear SDOF systems. Nonlinear Engineering 2015; 4(2): 77–82.
[31] Zhu JM, Lu YZ.: Hierarchical optimal control for distributed parameter systems via block pulse operator. Int J Control 1988; 48: 685–703.
[32] Ghaffarzadeh H, Younespour A. Active tendons control of structures using block pulse functions. Struct Control Health Monit 2014; 21: 1453–1464.
[33] Jiang ZH, Schaufelberger W. Block Pulse Functions and Their Applications in Control Systems. Berlin: Springer; 1992.
[34] Milstein GN. Approximate Integration of Stochastic Differential Equations, Theory of Probability and Its Applications, 1957; 19: 557–562.