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Frequency analysis of nonlinear oscillations via the global error minimization

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Abstract: The capacity and effectiveness of a modified variational approach, namely global error minimization (GEM) is illustrated in this study. For this purpose, the free oscillations of a rod rocking on a cylindrical surface and the Duffing-harmonic oscillator are treated. In order to validate and exhibit the merit of the method, the obtained result is compared with both of the exact frequency and the outcome of other well-known analytical methods. The corollary reveals that the first order approximation leads to an acceptable relative error, specially for large initial conditions. The procedure can be promisingly exerted to the conservative nonlinear problems.

Keywords: Analytical approximate solution; nonlinear oscillation; frequency–amplitude relation; global error minimization

1 Introduction

The accurate prediction of nonlinear oscillations in many areas of physics, applied mathematics and structural dynamics has been a significant subject. Surveys of the literature expose that there are different approximate analytical techniques for dealing with the nonlinear problems. Among them, one may allude to the variational iteration method [1, 2], the energy balance method [3–5], the harmonic balance method [6, 7], the parameter expansion method [8, 9], the multiple scales method [10], the homotopy analysis method [11, 12], Max–Min approach [13, 14], Hamiltonian approach [15–17], the iteration perturbation method [18-21], the variational approach [22, 23], the homotopy perturbation method [23, 24], the frequency–amplitude formulation [25–28] and so on [29, 30].

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This study intends to extend the reliability and applicability of the global error minimization [31–33] by considering the governing equation of a uniform rod rocking on the cylindrical surface without slipping [34] and the Duffing-harmonic oscillator [35–37]. The algorithm transforms the nonlinear differential equation into an equivalent optimization problem. After substitution of the trial function into the functional, unknown parameters of it is acquired using a Ritz–like method. It should be mentioned that the construction of the functional in this method is similar to the least squares approach. More details about the technique can be found in the literature [31].

The rest of the manuscript is organized as follows. The outline of the method is presented in section 2. The approach is applied to the governing equation of the uniform rod rocking on the cylindrical surface in section 3. The relationship between the frequency and the initial amplitude of the Duffing-harmonic oscillator is provided for either first- and second-order approximations in section 4. Section 5 ends this study with a brief conclusion.

2 The global error minimization

This section gives the basic idea of the global error minimization. Consider a general nonlinear oscillator as follows:

$$\ddot{u} + F(u) = 0, \qquad u(0) = A, \quad \dot{u}(0) = 0. \tag{1}$$

By defining a functional as follows:

$$E(u) = \int_0^T (\ddot{u} + F(u))^2 dt, \quad T = 2\pi\omega^{-1}, \tag{2}$$

and assuming $F(u)$ is an odd function. One may utilize an approximate trial function in the form of

$$\widehat{u}(t) = \sum_{n=0}^{\infty} \left(a_{(2n+1)} \cos\left((2n+1)\omega t\right)\right) . \tag{3}$$

The unknown parameters (i.e., $a_{(2n+1)}$ & ω) can be find through the following conditions:

$$\frac{\partial E\left(\widehat{u}\right)}{\partial \omega} = 0, \quad \& \quad \frac{\partial E\left(\widehat{u}\right)}{\partial a_{(2n+1)}} = 0 \quad for \quad n \geq 1. \tag{4}$$

To demonstrate the practicality and effectiveness of the aforementioned method, the governing equation of a uniform rod rocking on the cylindrical surface and the Duffing-harmonic oscillator are taken into consideration in the present study. The results are illustrated in next sections.

3 Case 1

Fig. 1 depicts the schematic of the uniform bar rocking on a cylindrical surface. The general equation of the motion is [34]:

$$\left(\frac{l^2}{12}+(r\theta)^2\right)\ddot{\theta}+\left(\left(r\dot{\theta}\right)^2+gr\cos(\theta)\right)\theta=0,$$
$$\theta(0)=\beta,\quad \dot{\theta}(0)=0, \tag{5}$$

where parameters l, r and g are the rod's length, the radius of cylindrical surface and the acceleration of gravity, respectively. Eq. (5) can be rewritten as:

$$\left(1+(\lambda\theta)^2\right)\ddot{\theta}+\left(\left(\lambda\dot{\theta}\right)^2+\Omega^2\cos(\theta)\right)\theta=0, \tag{6}$$

where $\lambda=\sqrt{12}\left(\frac{r}{l}\right)$ and $\Omega=\sqrt{12}\left(\frac{gr}{l^2}\right)^{\frac{1}{2}}$.

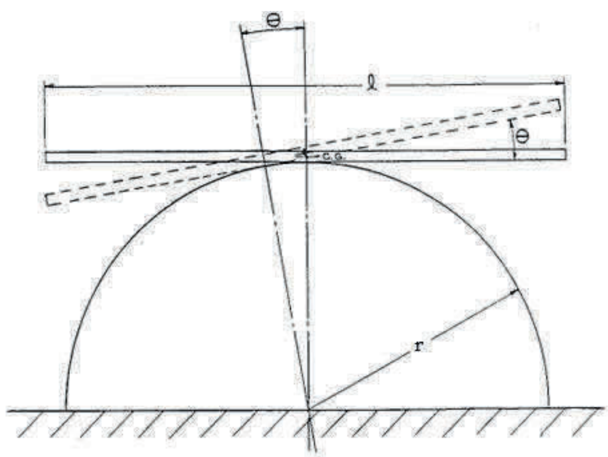

Fig. 1: Thin uniform bar rocking on a cylindrical surface [34].

In the following, the global error minimization is applied to the Eq. (6). As can be seen, a good agreement with exact ones is achieved for the first-order approximation.

Based on the the basic idea of the algorithm, the minimization problem of Eq. (6) is:

$$E(\theta)=\int_0^T\left(\left(1+(\lambda\theta)^2\right)\ddot{\theta}+\left(\left(\lambda\dot{\theta}\right)^2+\Omega^2\cos(\theta)\right)\theta\right)^2dt,$$

$$T=2\pi\omega^{-1}. \tag{7}$$

To determine the approximate frequency, the following trial function is employed:

$$\hat{\theta}(t)=\beta\cos(\omega t) \tag{8}$$

Substituting Eq. (8) into Eq. (7) yields:

$$\begin{aligned}E\left(\hat{\theta}\right)=&\frac{\pi\beta}{2\omega}(\beta((2+2\beta^2\lambda^2+\beta^4\lambda^4)\omega^4\\&+\Omega^4)+\Omega^2(8(-1+\beta^2\lambda^2)\omega^2J_1(\beta)\\&+8\beta(1+(-6+\beta^2)\lambda^2)\omega^2J_2(\beta)+\Omega^2(J_1(2\beta)\\&-2\beta J_2(2\beta)))),\end{aligned} \tag{9}$$

where $J_\alpha(z)$ is the Bessel function of the first kind. by applying $\frac{\partial E\left(\hat{\theta}\right)}{\partial\omega}=0$, the frequency is obtained as:

$$\begin{aligned}\omega=&\frac{1}{\sqrt{3}}\left(\left(\frac{1}{\beta\left(2+2\beta^2\lambda^2+\beta^4\lambda^4\right)}\right.\right.\\&\left(4\Omega^2J_1(\beta)-4\beta^2\lambda^2\Omega^2J_1(\beta)-4\beta\Omega^2J_2(\beta)\right.\\&+24\beta\lambda^2\Omega^2J_2(\beta)-4\beta^3\lambda^2\Omega^2J_2(\beta)\\&+\left(\Omega^4\left(16\left(\left(-1+\beta^2\lambda^2\right)J_1(\beta)\right.\right.\right.\\&\left.+\beta\left(1+\left(-6+\beta^2\right)\lambda^2\right)J_2(\beta)\right)^2\\&+3\beta\left(2+2\beta^2\lambda^2+\beta^4\lambda^4\right)(\beta+J_1(2\beta)\\&\left.\left.\left.\left.-2\beta J_2(2\beta))))^{1/2}\right)\right)^{1/2}\right)\end{aligned} \tag{10}$$

To illustrate the validity and accuracy of the global error minimization, with assumption $r=l/4$ the Eq. (10) is reduced to:

$$\begin{aligned}\omega_{GEM}=&2\left(\frac{1}{\beta(32+24\beta^2+9\beta^4)}(4(4-3\beta^2)yJ_1(\beta)\right.\\&+4\beta(14-3\beta^2)yJ_2(\beta)+(y^2(16((-4+3\beta^2)J_1(\beta)\\&+\beta(-14+3\beta^2)J_2(\beta))^2+3\beta(32+24\beta^2+9\beta^4)(\beta+J_1(2\beta)\\&-2\beta J_2(2\beta))))^{1/2}))^{1/2},\end{aligned} \tag{11}$$

where $y=g/l$. For the first order approximation, Wu et al. [6] acquired:

$$\omega_{Wu}=\left(\frac{24y(J_0(\beta)-J_2(\beta))}{8+3\beta^2}\right)^{1/2}, \tag{12}$$

where the exact frequency for this condition is given as:

$$\omega_e=\pi(6y)^{\frac{1}{2}}\left(\int_0^{\frac{\pi}{2}}\left(\beta^2\cos(\tau)^2\left(4+3\beta^2\sin(\tau)^2\right)/(\cos(\beta)\right.\right.$$
$$\left.\left.-\cos(\beta\sin(\tau))+\beta(\sin(\beta)-\sin(\tau)\sin(\beta\sin(\tau))))\right)^{\frac{1}{2}}d\tau\right)^{-1}. \tag{13}$$

Table 1 compares the approximate frequencies with respect to the exact ones for different initial conditions when $y = 1$. For more convenience, the result is presented in Fig. 2. Moreover, the relative error of both methods is demonstrated in Fig. 3. In contrast to the linearized harmonic balance method (the combination of the linearization of the governing equation with the method of harmonic balance), the relative error of the global error minimization does not increase continuously, and its rate is variable in the domain. As can be seen, the accuracy of the global error minimization is better in some sections, and Eq. (12) gives a lower relative error for small amplitudes.

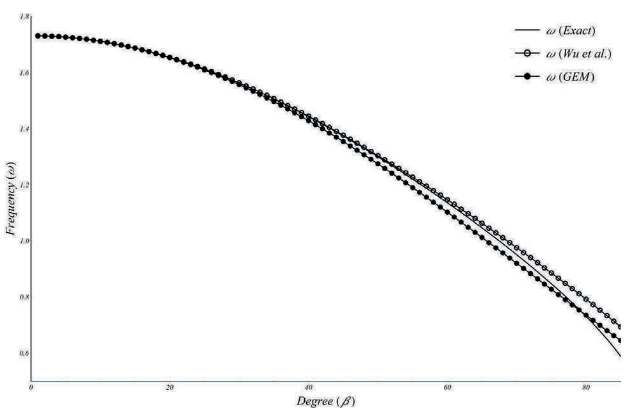

Fig. 2: The value of frequency in the domain ($y = 1$).

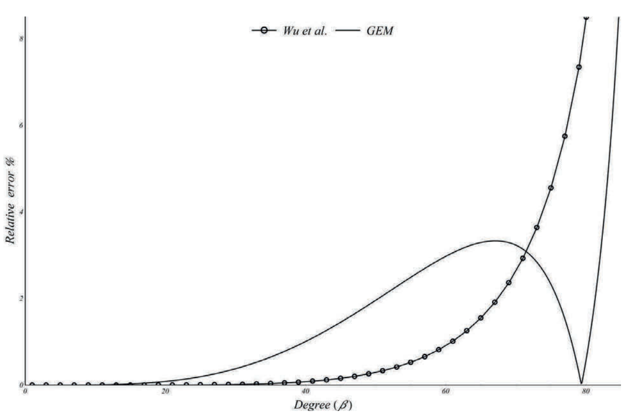

Fig. 3: Comparison between relative errors of the first order approximation for Case I.

4 Case 2

This section investigates accuracy of a the approach by the Duffing-harmonic oscillator. This nonlinear model has a rational form for the restoring force. The governing equation of motion for this type oscillator is:

$$\ddot{u} + u^3\left(1 + u^2\right)^{-1} = 0, \qquad u(0) = A, \quad \dot{u}(0) = 0. \tag{14}$$

Equation (14) is a mathematical model of a conservative system. for small and large values of u, it is a Duffing oscillator (i.e., $\ddot{u} + u^3 \approx 0$) and a linear harmonic oscillator (i.e., $\ddot{u} + u \approx 0$), respectively. The exact frequency of this oscillator is given as:

$$\omega_e = \frac{\pi}{2A} \left(\int_0^1 \left(A^2\left(1 - x^2\right) + \ln\left(\left(1 + A^2x^2\right)\Big/\left(1 + A^2\right)\right)\right)^{-1/2} dx\right)^{-1}. \tag{15}$$

The first- and second-order approximations for this nonlinear model are given in the following context.

4.1 First-order approximation

Based on the section 2, the modified variational approach is exerted. The equation (14) can be rewritten as:

$$\left(1 + u^2\right)\ddot{u} + u^3 = 0. \tag{16}$$

The minimization problem is:

$$E(u) = \int_0^T \left(\left(1 + u^2\right)\ddot{u} + u^3\right)^2 dt, \quad T = 2\pi\omega^{-1}, \tag{17}$$

for the first order approximation, the trial function is:

$$\widehat{u}_1(t) = a_1 \cos(\omega t) \ . \tag{18}$$

Where $a_1 = A$, substituting Eq. (18) into Eq. (17) yields:

$$E\left(\widehat{u}_1\right) = \frac{A^2\pi}{8\omega}\left(8\omega^4 + 12A^2\omega^2(\omega^2 - 1) + 5A^4(\omega^2 - 1)^2\right), \tag{19}$$

by applying $\frac{\partial E\left(\widehat{u}_1\right)}{\partial \omega} = 0$, the approximate frequency is obtained as:

$$\omega_1 = \frac{A}{\sqrt{3}}\sqrt{\frac{6 + 5A^2 + 2\sqrt{39 + 60A^2 + 25A^4}}{8 + 12A^2 + 5A^4}}. \tag{20}$$

Table 1: Comparison of approximate frequencies with exact ones (Case I).

β (Degree)	ω_{GEM} (RE %)	ω_{Wu} [6] (RE %)	ω_e (Eq. 13)
5	1.7271065591268644 (0.0004)	1.7271127875434222 (0.0000003)	1.7271127818492038
10	1.7123006522375672 (0.01)	1.7123979618568987 (0.00002)	1.7123976031210428
15	1.687723383986749 (0.03)	1.6881969036559472 (0.0002)	1.6881929232337736
20	1.6535547330682832 (0.08)	1.6549710318127808 (0.001)	1.6549494461907537
25	1.6100984795905333 (0.19)	1.6133212470119473 (0.005)	1.613242407046901
30	1.557812599993792 (0.38)	1.5639492479882797 (0.01)	1.5637253179110349
35	1.4973246338641835 (0.65)	1.5076154205171215 (0.04)	1.5070808958109516
40	1.4294245459508197 (1.01)	1.4450968213435502 (0.08)	1.4439726688902643
45	1.3550338337819872 (1.45)	1.3771478484815676 (0.16)	1.3749984571501832
50	1.2751562381380954 (1.96)	1.3044649090836247 (0.29)	1.3006439386772928
55	1.190820400643789 (2.49)	1.227654936034564 (0.53)	1.2212304561528435
60	1.1030273049757937 (2.97)	1.147205997567743 (0.91)	1.1368444002767581
65	1.0127162975457542 (3.29)	1.0634562779649626 (1.55)	1.047220857582407
70	0.9207659649820763 (3.23)	0.9765547140194191 (2.63)	0.9515157322021774
75	0.8280556886595868 (2.33)	0.8864008582021395 (4.56)	0.8477781234901954
80	0.7356390479943411 (0.58)	0.7925386096868637 (8.35)	0.7314320558826457

4.2 Second-order approximation

To illustrate the capacity of the approach, the second-order approximation of the algorithm is applied to the Duffing-harmonic oscillator. Substituting (16) into (2), by using the following trial function:

$$\widehat{u}_2(t) = a_1 \cos(\omega t) + a_3 \cos(3\omega t)\ , \tag{21}$$

where $a_1 + a_3 = A$, gives:

$$\begin{aligned} E\left(\widehat{u}_2\right) = & \frac{1}{8\omega}\pi(8A^2\omega^4 + 12A^4\omega^2(-1+\omega^2) \\ & + 5A^6(-1+\omega^2)^2 + (-16A\omega^4 + 32A^3\omega^4 \\ & + 5A^5(-3 - 2\omega^2 + 5\omega^4))a_3 + (656\omega^4 + 8A^2\omega^2(-21 + 97\omega^2) \\ & + A^4(45 - 130\omega^2 + 277\omega^4))a_3^2 - 4A(8\omega^2(-12 + 53\omega^2) \\ & + A^2(25 - 130\omega^2 + 201\omega^4))a_3^3 + 2(12\omega^2(-13 + 77\omega^2) \\ & + A^2(75 - 550\omega^2 + 1051\omega^4))a_3^4 \\ & - 5A(27 - 254\omega^2 + 611\omega^4)a_3^5 + 5(11 - 126\omega^2 + 371\omega^4)a_3^6). \end{aligned} \tag{22}$$

by employing $\frac{\partial E\left(\widehat{u}_2\right)}{\partial\omega} = 0$, yields to:

$$\begin{aligned} \omega_2^{(1)} = & \frac{1}{\sqrt{3y}}\Big(-8A^2 - 6A^4 + y - 4A(-4 + 8A^2 + 5A^4)a_3 \\ & - 4(164 + 173A^2 + 53A^4)a_3^2 + 32A(47 + 17A^2)a_3^3 \\ & - 4(423 + 388A^2)a_3^4 + 2420Aa_3^5 - 1540a_3^6 \\ & + 2\Big(A^8(39 + 60A^2 + 25A^4) + 50A^7(-3 + A^4)a_3 \\ & + 3A^6(1054 + 1120A^2 + 365A^4)a_3^2 \\ & - 6A^5(1516 + 2660A^2 + 1025A^4)a_3^3 \\ & + 3A^4(8824 + 18480A^2 + 8915A^4)a_3^4 \\ & - 6A^3(9979 + 25420A^2 + 14895A^4)a_3^5 \\ & + 6A^2(15253 + 54700A^2 + 39230A^4)a_3^6 \\ & - 24A(3419 + 21100A^2 + 20435A^4)a_3^7 \\ & + 6(5524 + 88120A^2 + 131575A^4)a_3^8 \\ & - 20A(16842 + 47219A^2)a_3^9 + 5(20160 + 158429A^2)a_3^{10} \\ & - 413850Aa_3^{11} + 101325a_3^{12}\Big)^{\frac{1}{2}}\Big)^{\frac{1}{2}} \end{aligned} \tag{23}$$

and the condition $\frac{\partial E\left(\widehat{u}_2\right)}{\partial a_3} = 0$ gives:

$$\begin{aligned} \omega_2^{(2)} = & \frac{1}{\sqrt{\eta}}\Big(5A^5 + 2A^2(84 + 65A^2)a_3 - 12A(48 + 65A^2)a_3^2 \\ & + 8(78 + 275A^2)a_3^3 - 3175Aa_3^4 + 1890a_3^5 \\ & + 4(5A^6(-3 + 6A^2 + 5A^4) + 20A^5(66 + 69A^2 + 23A^4)a_3 \\ & - 2A^4(2958 + 5265A^2 + 2170A^4)a_3^2 \\ & + 4A^3(3276 + 9525A^2 + 4900A^4)a_3^3 \\ & - 5A^2(3207 + 16266A^2 + 10954A^4)a_3^4 \\ & + 4A(2688 + 27075A^2 + 25600A^4)a_3^5 \\ & - 2(1362 + 41985A^2 + 67360A^4)a_3^6 \\ & + 20A(1653 + 6428A^2)a_3^7 - 5(1008 + 17083A^2)a_3^8 \\ & + 34500Aa_3^9 - 6300a_3^{10})^{\frac{1}{2}}\Big)^{\frac{1}{2}} \end{aligned} \tag{24}$$

where $y = (A^2(8 + 12A^2 + 5A^4) + A(-16 + 32A^2 + 25A^4)a_3 + (656 + 776A^2 + 277A^4)a_3^2 - 4A(424 + 201A^2)a_3^3 + 2(924 + 1051A^2)a_3^4 - 3055Aa_3^5 + 1855a_3^6)$ and $\eta = (A\left(-16 + 32A^2 + 25A^4\right) + 2\left(656 + 776A^2 + 277A^4\right)a_3 - 12A\left(424 + 201A^2\right)a_3^2 + 8\left(924 + 1051A^2\right)a_3^3 - 15275Aa_3^4 + 11130a_3^5)$, in the other words, ($\eta = \frac{\partial y}{\partial a_3}$).

By setting $\omega_2^{(1)} = \omega_2^{(2)}$ and with assumption $|a_3| \ll A$, the parameter of a_3 can be determined for a known amplitude. Consequently the value of frequency is achieved. Table 2 presents the ratio of the approximate frequency for different amplitudes. Moreover, Fig. 4 compares the relative error of the first- and second-order approximations with other well-known methods such as harmonic balance and energy balance procedures. Fig. 5 exhibits the phase space trajectory of the Duffing-harmonic oscillator when the initial amplitude is one. As can be seen, the accuracy of the approach is excellent for whole of domain in the second-order approximation.

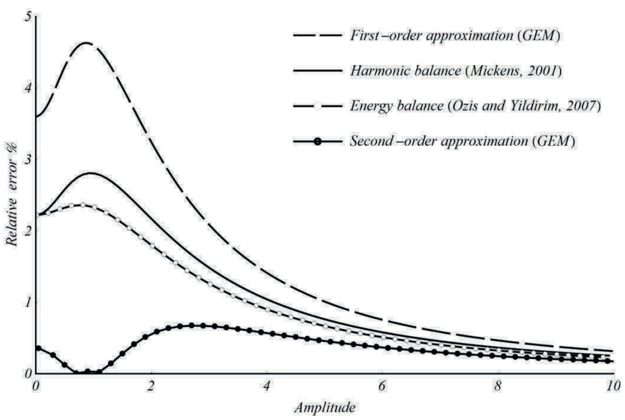

Fig. 4: Comparison between relative errors for Case II.

Table 2: Comparison of approximate frequencies with exact ones for the second-order approximation (Case II).

A	a_3	ω_2	$\frac{\omega_2}{\omega_e}$
0.05	0.002104	0.042470	1.003571
0.1	0.004200	0.084682	1.003482
0.5	0.019571	0.387844	1.001235
1	0.030982	0.636696	0.999868
5	0.013949	0.971407	1.004583
10	0.007104	0.992643	1.001742
50	0.001428	0.999703	1.000095
100	0.000714	0.999926	1.000025
500	0.000143	0.999997	1.000001

The achieved results using the second-order approximation of this straightforward approach show that the technique is easy, convenient and accurate for conservative nonlinear oscillators that the restoring force has a rational form.

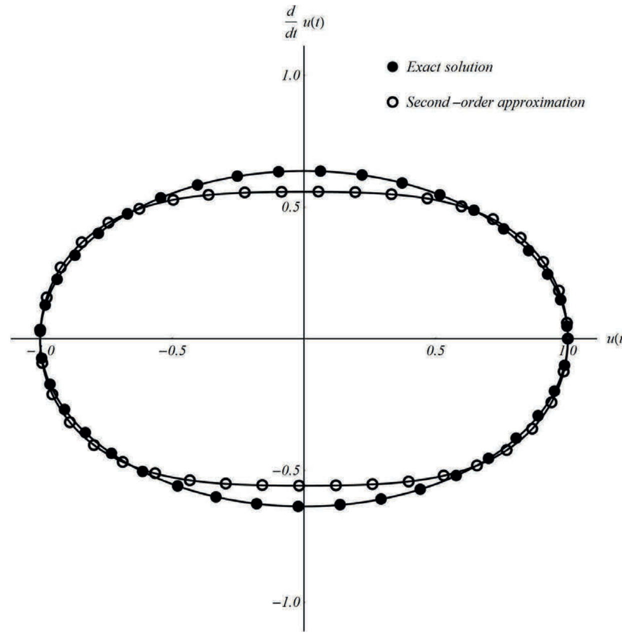

Fig. 5: Phase space diagram of the second-order approximation for Case II (A=1).

5 Conclusions

This study scrutinizes the accuracy of the global error minimization (GEM) by examining two nonlinear equations which arise from the free oscillations of a rigid rod rocking on the cylindrical surface without slipping and the Duffing-harmonic oscillator. The reliable results are validated by the exact solutions. This applicable technique provides a satisfactory approximate frequency for the first order approximation. Higher order estimations using this method should be more accurate for other resembling nonlinear problems with odd and rational restoring forces. At last, The GEM method is a simple and powerful algorithm that can be easily implemented to similar nonlinear systems.

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