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The generalized fractional order of the Chebyshev functions on nonlinear boundary value problems in the semi-infinite domain

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Abstract: A new collocation method, namely the generalized fractional order of the Chebyshev orthogonal functions (GFCFs) collocation method, is given for solving some nonlinear boundary value problems in the semi-infinite domain, such as equations of the unsteady isothermal flow of a gas, the third grade fluid, the Blasius, and the field equation determining the vortex profile. The method reduces the solution of the problem to the solution of a nonlinear system of algebraic equations. To illustrate the reliability of the method, the numerical results of the present method are compared with several numerical results.

Keywords: Generalized fractional order of the Chebyshev functions; Unsteady isothermal flow of a gas equation; Third grade fluid equation; Blasius equation; Field equation; Semi-Infinite domain

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MSC: 34B15; 34B40; 74S25; 34L30

1 Introduction

In this section, some necessary preliminary which are useful for our method have been introduced.

1.1 The Chebyshev functions

The Chebyshev polynomials have frequently been used in numerical analysis including polynomial approximation, Gauss-quadrature integration, integral and differential equations and spectral methods. Chebyshev polynomials have many properties, for example orthogonal, recursive, simple real roots, complete in the space of polynomials. For these reasons, many researchers have employed these polynomials in their research [1, 2].

Using some transformations, the number of researchers extended Chebyshev polynomials to semi-infinite or infinite domains, for example by using $x = \frac{t-L}{t+L}$, $L > 0$ the rational Chebyshev functions are introduced [3, 4].

In the proposed work, by transformation $x = 1-2(\frac{t}{\eta})^{\alpha}$; $\alpha, \eta > 0$ on the Chebyshev polynomials of the first kind, the generalized fractional order of the Chebyshev orthogonal functions (GFCF) in the interval $[0, \eta]$ have been introduced, that we can use them to solve differential equations.

1.2 Basic definitions

In this section, some basic definitions and theorems have been expressed [5–7].

Definition 1. For any real function $f(t)$, $t > 0$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$, is said to be in space C_μ, $\mu \in \Re$, and it is in the space C_μ^n if and only if $f^{(n)} \in C_\mu$, $n \in N$.

Definition 2. Suppose that $f(t)$, $g(t) \in C(0, \eta)$ and $w(t)$ is a positive weight function in $(0, \eta)$, then we define

$$\| f(t) \|_w^2 = \int_0^\eta f^2(t)\, w(t)\, dt,$$

$$\langle f(t), g(t)\rangle_w = \int_0^\eta f(t)g(t)\, w(t)\, dt.$$

Theorem 1. (*Taylor's formula*) Suppose that $f^{(k)}(t) \in C(0, \eta)$ where $k = 0, 1, \ldots, m$ and $\eta > 0$. Then we have

$$f(t) = \sum_{i=0}^{m-1} \frac{t^i}{i!} f^{(i)}(0^+) + \frac{t^m}{m!} f^{(m)}(\xi), \tag{1}$$

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with $0 < \xi \leqslant t,\ \forall t \in [0,\eta]$. And thus for $M \geqslant |f^{(m)}(\xi)|$:

$$|f(t) - \sum_{i=0}^{m-1} \frac{t^i}{i!} f^{(i)}(0^+)| \leqslant M\frac{t^m}{m!}. \tag{2}$$

Proof. See Ref. [13]. □

We know that, the solution of some equations is generated by fractional powers or the structure of the solution of some equations is not exactly known. For example, one of the famous equations that its solution is generated by fractional powers is Thomas-Fermi equation [14, 15]. Baker [15] has proved that the solution of Thomas-Fermi equation is generated by the powers of $t^{\frac{1}{2}}$. For these reasons, in this paper, we decided to solve equations using the fractional basis, namely the generalized fractional order of the Chebyshev function (GFCF), in order to obtain acceptable results.

The GFCFs are introduced as a new basis for Spectral methods and this basis can be used to develop a framework or theory in Spectral methods. In this research, the fractional basis was used for solving nonlinear ordinary differential equations and it provided insight into an important issue.

Recently, some researchers have introduced the fractional basis of various basic functions and have used them in their research, such as the fractional-order Euler functions [4], the fractional-order Legendre functions [6, 8], the fractional-order Bessel functions [9], the fractional-order Jacobi functions [10, 11], and the fractional-order Bernoulli functions [12].

The organization of the paper is expressed as follows: In section 2, the GFCFs and their properties are obtained. In section 3, the proposed method is applied for solving some nonlinear boundary value problems in the semi-infinite domain. Finally, a brief conclusion is given in the last section.

2 Generalized Fractional order of the Chebyshev Functions

In this section, first, the generalized fractional order of the Chebyshev functions (GFCFs) of the first kind have been defined and then some properties and convergence of them for our method have been provided.

2.1 The GFCFs definition

The efficient methods have been used by many researchers to solve the differential equations (DE) is based on series expansion of the form $\sum_{i=0}^{n} c_i t^i$, such as Adomian's decomposition method and Homotopy perturbation method. But the exact solution of some DEs can not be estimated by polynomials basis, for a simple example: the ODE of $4yy'' = 3t,\ y(0) = y'(0) = 0$, that the exact solution is $y(t) = t^{\frac{3}{2}}$, therefore we have decided to define a new basis for Spectral methods to solve them as follows:

$$\Phi_n(t) = \sum_{i=0}^{n} c_i t^{i\alpha}$$

Now by transformation $z = 1 - 2(\frac{t}{\eta})^\alpha;\ \alpha,\ \eta > 0$ on classical Chebyshev polynomials of the first kind, we is defined the GFCFs in the interval $[0,\eta]$, which is denoted by ${}_\eta FT_n^\alpha(t) = T_n(1 - 2(\frac{t}{\eta})^\alpha)$.

The ${}_\eta FT_n^\alpha(t)$ can be obtained using the recursive relation as follows:

$$\begin{aligned} &{}_\eta FT_0^\alpha(t) = 1 \quad , \quad {}_\eta FT_1^\alpha(t) = 1 - 2(\frac{t}{\eta})^\alpha, \\ &{}_\eta FT_{n+1}^\alpha(t) = (2 - 4(\frac{t}{\eta})^\alpha)\ {}_\eta FT_n^\alpha(t) - {}_\eta FT_{n-1}^\alpha(t), \\ &\qquad n = 1, 2, \cdots., \end{aligned}$$

The analytical form of ${}_\eta FT_n^\alpha(t)$ of degree $n\alpha$ is given by

$${}_\eta FT_n^\alpha(t) = \sum_{k=0}^{n} \beta_{n,k,\eta,\alpha}\, t^{\alpha k}, \quad t \in [0,\eta], \tag{3}$$

where

$$\beta_{n,k,\eta,\alpha} = (-1)^k \frac{n 2^{2k}(n+k-1)!}{(n-k)!(2k)!\eta^{\alpha k}} \quad and \quad \beta_{0,k,\eta,\alpha} = 1.$$

Note that ${}_\eta FT_n^\alpha(0) = 1$ and ${}_\eta FT_n^\alpha(\eta) = (-1)^n$.

The GFCFs are orthogonal with respect to the weight function $w(t) = \frac{t^{\frac{\alpha}{2}-1}}{\sqrt{\eta^\alpha - t^\alpha}}$ in the interval $(0,\eta)$:

$$\int_0^\eta {}_\eta FT_n^\alpha(t)\, {}_\eta FT_m^\alpha(t) w(t) dt = \frac{\pi}{2\alpha} c_n \delta_{mn}. \tag{4}$$

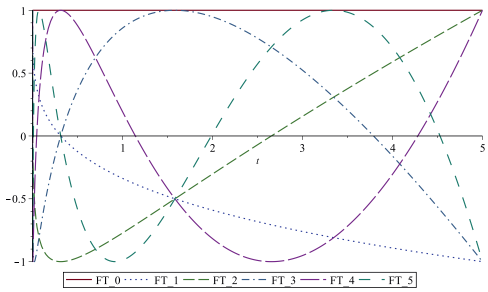
where δ_{mn} is Kronecker delta, $c_0 = 2$, and $c_n = 1$ for $n \geqslant 1$. The Eq. (4) is provable using the properties of orthogonality in the Chebyshev polynomials.

Figs. 1 show graphs of GFCFs for various values of n and α and $\eta = 5$.

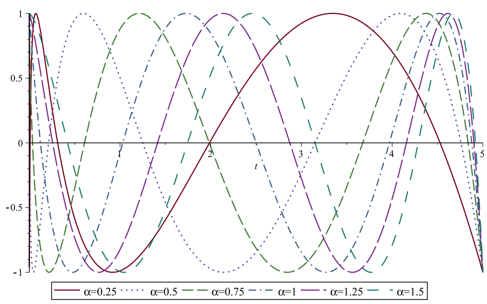
2.2 Approximation of functions

Any function $y(t) \in C[0,\eta]$ can be expanded as the follows:

$$y(t) = \sum_{n=0}^{\infty} a_n\, {}_\eta FT_n^\alpha(t),$$



(a) Graph of the GFCFs with $\alpha = 0.25$ and various values of n



(b) Graph of the GFCFs with $n = 5$ and various values of α

Fig. 1: Graphs of the GFCFs for various values of n and α.

where the coefficients a_n are obtained by inner product:

$$\langle y(t),\ {}_\eta FT_n^\alpha(t)\rangle_w = \langle \sum_{n=0}^{\infty} a_n\, {}_\eta FT_n^\alpha(t),\ {}_\eta FT_n^\alpha(t)\rangle_w$$

and using the property of orthogonality in the GFCFs:

$$a_n = \frac{2\alpha}{\pi c_n} \int_0^\eta {}_\eta FT_n^\alpha(t) y(t) w(t) dt, \quad n = 0, 1, 2, \cdots .$$

In practice, we have to use first m-terms GFCFs and approximate $y(t)$:

$$y(t) \approx y_m(t) = \sum_{n=0}^{m-1} a_n\, {}_\eta FT_n^\alpha(t) = A^T \Phi(t), \tag{5}$$

where

$$A = [a_0, a_1, \ldots, a_{m-1}]^T, \tag{6}$$
$$\Phi(t) = [{}_\eta FT_0^\alpha(t),\ {}_\eta FT_1^\alpha(t), \ldots,\ {}_\eta FT_{m-1}^\alpha(t)]^T. \tag{7}$$

2.3 Convergence of the method

The following theorem shows that by increasing m, the approximation solution $f_m(t)$ is convergent to $f(t)$ exponentially.

Theorem 2. Suppose that $f^{(k)}(t) \in C[0, \eta]$ for $k = 0, 1, \ldots, m$, and ${}_\eta F_m^\alpha$ is the subspace generated by $\{{}_\eta FT_0^\alpha(t), {}_\eta FT_1^\alpha(t), \ldots, {}_\eta FT_{m-1}^\alpha(t)\}$. If $f_m(t) = A^T \Phi(t)$ (in Eq. (5)) is the best approximation to $f(t)$ from ${}_\eta F_m^\alpha$, then the error bound is presented as follows

$$\| f(t) - f_m(t) \|_w \le \frac{\eta^m M}{m!} \sqrt{\frac{\sqrt{\pi}\Gamma(\frac{2m}{\alpha} + \frac{1}{2})}{\alpha \Gamma(\frac{2m}{\alpha} + 1)}}, \tag{8}$$

where $M \ge |f^{(m)}(t)|, \quad t \in [0, \eta]$.

Proof. By theorem 1, $y(t) = \sum_{i=0}^{m-1} \frac{t^i}{i!} f^{(i)}(0^+)$ and

$$|f(t) - y(t)| \le M \frac{t^m}{m!}.$$

Since $A^T \Phi(t)$ is the best approximation to $f(t)$ from ${}_\eta F_m^\alpha$, and $y(t) \in {}_\eta F_m^\alpha$, one has

$$\begin{aligned} \| f(t) - f_m(t) \|_w^2 &\le \| f(t) - y(t) \|_w^2 \\ &\le \frac{M^2}{m!^2} \int_0^\eta \frac{t^{\frac{\alpha}{2} + 2m\alpha - 1}}{\sqrt{\eta^\alpha - t^\alpha}} dt \\ &= \frac{M^2}{m!^2} \frac{\eta^{2m} \sqrt{\pi} \Gamma(\frac{2m}{\alpha} + \frac{1}{2})}{\alpha \Gamma(\frac{2m}{\alpha} + 1)}. \end{aligned}$$

Now, by taking the square roots, the theorem can be proved. Eq. (8) shows that if $m \to \infty$ then $\| f(t) - f_m(t) \|_w \to 0$. □

Theorem 3. The generalized fractional order of the Chebyshev function, ${}_\eta FT_n^\alpha(t)$, has precisely n real zeros on interval $(0, \eta)$ in the form

$$t_k = \eta \left(\frac{1 - \cos(\frac{(2k-1)\pi}{2n})}{2} \right)^{\frac{1}{\alpha}}, \quad k = 1, 2, \ldots, n.$$

Moreover, $\frac{d}{dt}\, {}_\eta FT_n^\alpha(t)$ has precisely $n - 1$ real zeros on interval $(0, \eta)$ in the following points:

$$t_k' = \eta \left(\frac{1 - \cos(\frac{k\pi}{n})}{2} \right)^{\frac{1}{\alpha}}, \quad k = 1, 2, \ldots, n - 1.$$

Proof. The Chebyshev polynomial $T_n(x)$ has n real zeros [16]:

$$x_k = \cos(\frac{(2k-1)\pi}{2n}), \quad k = 1, 2, \ldots, n.$$

Therefore $T_n(x)$ can be written as

$$T_n(x) = (x - x_1)(x - x_2) \ldots (x - x_n).$$

Using transformation $x = 1 - 2(\frac{t}{\eta})^\alpha$ yields to

$$ {}_\eta FT_n^\alpha(t) = \left(\left(1 - 2\left(\frac{t}{\eta}\right)^\alpha\right) - x_1\right)\left(\left(1 - 2(\frac{t}{\eta})^\alpha\right) - x_2\right) \dots \left(\left(1 - 2\left(\frac{t}{\eta}\right)^\alpha\right) - x_n\right), $$

so, the real zeros of ${}_\eta FT_n^\alpha(t)$ are $t_k = \eta(\frac{1-x_k}{2})^{\frac{1}{\alpha}}$.
Also, we know that, the real zeros of $\frac{d}{dt}T_n(t)$ occurs in the following points:

$$ x_k^{'} = \cos(\frac{k\pi}{n}), \qquad k = 1, 2, \dots, n-1 $$

Same as in the previous, the absolute extremes of ${}_\eta FT_n^\alpha(t)$ are $t_k^{'} = \eta(\frac{1-x_k^{'}}{2})^{\frac{1}{\alpha}}$. □

3 Examples

Many problems arising in fluid dynamics, quantum mechanics, astrophysics, and other fields are defined on infinite or semi-infinite domains. There are different approaches for solving this type of equation, such as numerical, analytical, and semi-analytical methods. In this section, we attempt to introduce a numerical method, based on the generalized fractional order of the Chebyshev orthogonal functions for solving this type of equations.

3.1 The unsteady isothermal flow of a gas

One of the important nonlinear ordinary differential equations that occurs on semi-infinite domain is the unsteady gas equation:

$$ \frac{d^2y}{dt^2} + \frac{2t}{\sqrt{1-\beta y(t)}}\frac{dy}{dt} = 0, \quad t \in [0, \infty), \tag{9} $$

where $0 \leqslant \beta \leqslant 1$ is a real constant and the boundary conditions are:

$$ y(0) = 1, \qquad \lim_{t\to\infty} y(t) = 0. \tag{10} $$

A substantial amount of numerical and analytical work has been invested so far in this model [17]. The main reason of this interest is that the approximation can be used for many engineering purposes. As stated before, the problem of Eq. (9) was handled by Kidder [17] where a perturbation technique is carried out to include terms of the second order. Wazwaz [18] has solved this equation nonlinearly by modifying the decomposition method and Pade approximation. Parand et al. [19, 20], and Taghavi et al. [21] have also applied the rational Jacobi functions, Bessel function collocation method, and modified generalized Laguerre polynomials for solving this equation. Rezaei et al. [22] have applied two numerical methods based on Sinc and rational Legendre functions to solve gas flow through a micro-nano-porous media. Rad et al. [23] have solved this equation by two numerical and analytical solutions based on Homotopy analysis method and Hermite functions collocation method. Recently, Parand et al. [19] have used the combination of the quasilinearization method and the rational Jacobi function collocation method, and have obtained an accurate solution to the equation, the value of initial slope is calculated as $-1.191790649719421734122828$4 for $\beta = 0.50$.

Now, we solve this equation by using GFCF collocation method.

For satisfying the boundary conditions, we satisfy the Eq. (10) as follows:

$$ \widehat{y_m}(t) = \frac{\lambda}{t^2+\lambda} + t\, e^{-2t}\, y_m(t), \tag{11} $$

where λ is an arbitrary real constant and $y_m(t)$ is defined in Eq. (5). So, $\widehat{y_m}(0) = 1$ and $\widehat{y_m}(t) = 0$ when t tends to ∞, and the boundary conditions (10) are satisfied for all $\lambda > 0$, and is defined in the semi-infinite domain.

To apply the collocation method, we construct the residual function by substituting $\widehat{y_m}(t)$ in Eq. (11) for $y(t)$ in the unsteady gas equation (9):

$$ Res(t) = \frac{d^2}{dt^2}\widehat{y_m}(t) + \frac{2t}{\sqrt{1-\beta\widehat{y_m}(t)}}\frac{d}{dt}\widehat{y_m}(t). \tag{12} $$

The equations for obtaining the coefficient $\{a_i\}_{i=0}^{m-1}$ arise from equalizing $Res(t)$ to zero on m collocation points:

$$ Res(t_i) = 0, \quad i = 0, 1, \dots, m-1. $$

In this study, the roots of the GFCFs in the interval $[0, \eta]$ (Theorem 3) are used as collocation points. By solving the obtained set of equations, we have the approximating function $\widehat{y_m}(t)$.

Table 1 shows the value of $y^{'}(0)$ by the present method and comparison it with the values of Bessel function collocation (BFC) [20], Shooting method [24], and RBF [24] for $\beta = 0.25,\ 0.50,\ 0.75$ with $m = 35$ and $\alpha = 0.50$.

Table 2 shows the obtained values of $y^{'}(0)$ and $\| Res \|_w^2$ by the present method for various values of m, $\beta = 0.50$, and $\alpha = 0.50$.

Table 3 shows the values of $y(t)$ for various values of t by the present method and comparison it with the values of perturbation (PB) [17], Bessel function collocation (BFC) [20], modified generalized Laguerre (MGL) [21],

Table 1: Comparison of the obtained values of $y'(0)$ for $\beta = 0.25,\ 0.50,\ 0.75$ by the present method, Shooting method[24], BFC [20], and RBF [24], with $m = 35$ and $\alpha = 0.50$

β	λ	Present	BFC	RBF	Shooting	Abs. Err.
0.25	0.1754	−1.156572055	−1.156275	−1.156557	−1.156572	5.514e−8
0.50	0.1755	−1.191790893	−1.191718	−1.191498	−1.191791	5.078e−7
0.75	0.0167	−1.239760055	−1.239760	−1.239671	−1.239760	5.565e−8

Runge-Kutta method (RK) [24], RBF [24], and finite difference (FD) [25], with $m = 35$ and $\alpha = 0.50$.

Figure 2 shows the graphs of $y(t)$ and $y'(t)$ from the solution of unsteady gas equation for $\beta = 0.25,\ 0.50$, and 0.75 with $m = 35,\ \alpha = 0.5$ and $\eta = 10$.

Figure 3 shows the graphs of residual errors of Eq. (12) and the logarithm of coefficients $|a_i|$ for $\beta = 0.50,\ 0.75$, with $m = 35$ and $\alpha = 0.50$, to show the convergence of the method.

Table 2: Obtained values of $y'(0)$ by the present method for various values of m, $\beta = 0.50$, and $\alpha = 0.50$

m	λ	$y'(0)$	$\Vert Res \Vert_w^2$
15	0.3005	-1.191794097	2.2591e-02
25	0.1788	-1.191773155	9.2586e-05
35	0.1755	-1.191790893	1.0379e-05

3.2 The third grade fluid in a porous half space

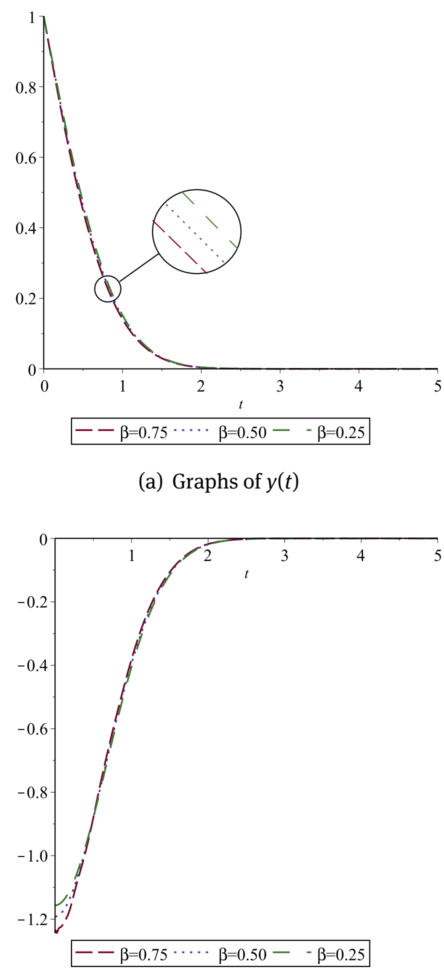
The boundary value problem modelling the steady state flow of a third grade fluid in a porous half space on the semi-infinite domain is as follows:

$$\frac{d^2y}{dt^2}\left(1 + b_1(\frac{dy}{dt})^2\right) - \frac{b_1 b_2}{3} y \left(\frac{dy}{dt}\right)^2 - b_2 y = 0, \quad t \in [0, \infty), \tag{13}$$

where b_1 and b_2 are real constants and the boundary conditions for this equation:

$$y(0) = 1, \quad \lim_{t \to \infty} y(t) = 0. \tag{14}$$

Some researchers approximate the third grade fluid equations in a porous half space; for example, Ahmad [26] by applying the Homotopy analysis method, Kazem et al. [27] by applying the radial basis functions collocation method, Parand and Hajizadeh [28] by applying the modified rational Christov functions collocation method, Baharifard et al. [29] by applying the rational and exponential Legendre Tau method.



(a) Graphs of $y(t)$

(b) Graphs of $y'(t)$

Fig. 2: Obtained graphs of unsteady gas equation for $\beta = 0.25,\ 0.50,\ 0.75$

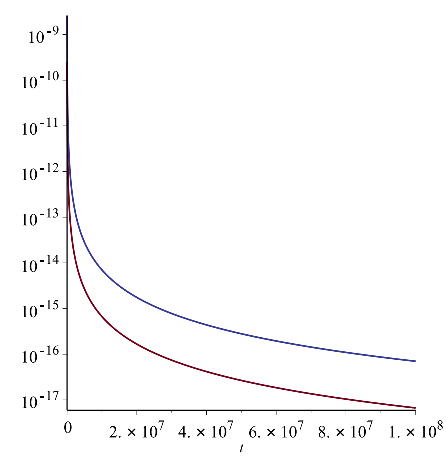
For satisfying the boundary conditions, we satisfy the Eq. (14) as follows:

$$\widehat{y_m}(t) = \frac{\lambda}{t^2 + \lambda} + t\, e^{-2t}\, y_m(t), \tag{15}$$

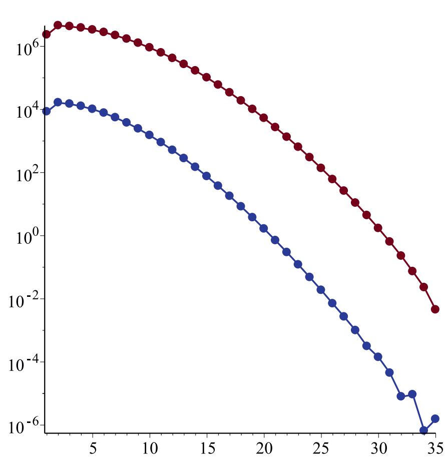
where λ is an arbitrary real constant and $y_m(t)$ is defined in Eq. (5). So, $\widehat{y_m}(0) = 1$, and $\widehat{y_m}(t) = 0$ when t tends to ∞, and the boundary conditions (14) are satisfied for all $\lambda > 0$, and is defined in the semi-infinite domain. We construct

Table 3: Obtained values of $y(t)$ for $\beta = 0.5$, $\alpha = 0.5$, $m = 35$, and $\lambda = 0.1755$, comparison between the present method, BFC [20], RBF [24], FD [25], PB [17], RK [24], and MGL [21]

t	Present	BFC [20]	RBF [24]	FD [25]	PB [17]	RK [24]	MGL [21]
0.1	0.881364979	0.88137178	0.88139802	0.88136465	0.88165882	0.88136461	0.90931873
0.2	0.765828998	0.76582874	0.76588029	0.76582882	0.76630767	0.76582874	0.81748763
0.3	0.656000141	0.65602143	0.65606928	0.65600069	0.65653799	0.65600058	0.71522344
0.4	0.553897686	0.55392593	0.55399431	0.55389894	0.55440240	0.55389881	0.60982075
0.5	0.460943164	0.46097551	0.46107202	0.46094276	0.46136502	0.46094260	0.51632348
0.6	0.377981056	0.37801957	0.37814380	0.37798158	0.37831093	0.37798140	0.41932385
0.7	0.305350982	0.30539503	0.30554174	0.30535232	0.30559765	0.30535212	0.40982377
0.8	0.242953430	0.24313254	0.24316554	0.24295437	0.24313254	0.24295416	0.31999068
0.9	0.190333678	0.19038471	0.19056419	0.19033421	0.19046236	0.19033399	0.20820285
1.0	0.146772618	0.14682691	0.14702048	0.14677328	0.15876898	0.14677305	0.21991074



(a) Graphs of residual errors



(b) Graphs of $log(|a_i|)$

Fig. 3: Graphs of residual errors and $\log(|a_i|)$ for $\beta = 0.50$(Blue) and $\beta = 0.75$(Red)

the residual function as follows:

$$Res(t) = \frac{d^2\widehat{y_m}}{dt^2}\left(1 + b_1(\frac{d\widehat{y_m}}{dt})^2\right) - \frac{b_1 b_2}{3}\widehat{y_m}\left(\frac{d\widehat{y_m}}{dt}\right)^2 - b_2\widehat{y_m}. \tag{16}$$

As before, by solving the obtained set of equations, we have the approximating function $\widehat{y_m}(t)$.

Table 4 shows the obtained values of $y'(0)$ by the present method and comparison it with the values of Gaussian RBF (G-RBF) [27], rational Christov functions (RCF) [28], Shooting method [29], rational Legendre Tau method (RLT) [29], and exponential Legendre Tau method (ELT) [29] for various values of b_1 and b_2, with $m = 20$ and $\alpha = 0.50$.

Table 5 shows the obtained values of $y'(0)$ and $\| Res \|_w^2$ by the present method for various values of m, $b_1 = 0.60$, $b_2 = 0.50$, and $\alpha = 0.50$.

Table 6 shows the obtained values of $y'(0)$ and $\| Res \|_w^2$ by the present method for various values of α, $b_1 = 0.60$, $b_2 = 0.50$, and $m = 20$.

Table 7 shows the obtained values of $y(t)$ for various values of t by the present method and comparison it with the values of Homotopy analysis method (HAM) [26], Gaussian RBF (G-RBF) [27], rational Christov functions (RCF) [28], rational Legendre Tau method (RLT) [29], and exponential Legendre Tau method (ELT) [29], with $m = 20$ and $\alpha = 0.50$.

Figure 4 shows the graphs of residual error of Eq. (16) and the logarithm of coefficients $|a_i|$ for $b_1 = 0.60$, $b_2 = 0.50$, with $m = 20$ and $\alpha = 0.50$, to show the convergence of the method.

3.3 The Blasius equation

Another of important third-order nonlinear ordinary differential equations that occurs in the semi-infinite domain is the Blasius equation:

$$2\frac{d^3y}{dt^3} + y\frac{d^2y}{dt^2} = 0, \qquad t \in [0, \infty), \tag{17}$$

Table 4: Comparison of the obtained values of $y'(0)$ for various values of b_1 and b_2 by the present method, shooting method, RLT, ELT, RCF, and G-RBF

b_1	b_2	λ	Present	Shooting	RLT [29]	ELT [29]	RCF [28]	G-RBF[27]
0.3	0.5	0.80	−0.69128036	−0.691280	−0.691493	−0.691279		
0.6		0.80	−0.67830298	−0.678301	−0.678511	−0.678302	−0.6783017	−0.6783013
0.9		0.80	−0.66732796	−0.667327	−0.667528	−0.667327		
1.2		0.80	−0.65783825	−0.657838	−0.658029	−0.657837		
0.6	0.3	0.81	−0.53331017	−0.533303	−0.533545	−0.533302		
	0.6	0.81	−0.73800824	−0.738008	−0.738116	−0.738007		
	0.9	0.81	−0.88746913	−0.887467	−0.887350	−0.887467		
	1.2	0.81	−1.00865474	−1.008653	−1.008516	−1.008653		

Table 5: Obtained values of $y'(0)$ by the present method for various values of m and $\alpha = 0.50$

m	λ	$y'(0)$	$\|Res\|_w^2$
10	0.4	−0.6757266461	6.098e-03
15	0.7	−0.6782935857	5.147e-05
20	0.8	−0.6783029829	1.198e-05

Table 6: Obtained values of $y'(0)$ by the present method for various values of α and $m = 20$

α	λ	$y'(0)$	$\|Res\|_w^2$
0.25	0.4	−0.67827627	9.6011e-04
0.50	0.8	−0.67830298	1.1986e-05
0.75	1.0	−0.67844196	1.1074e-06
1.00	1.8	−0.67830171	2.7556e-08

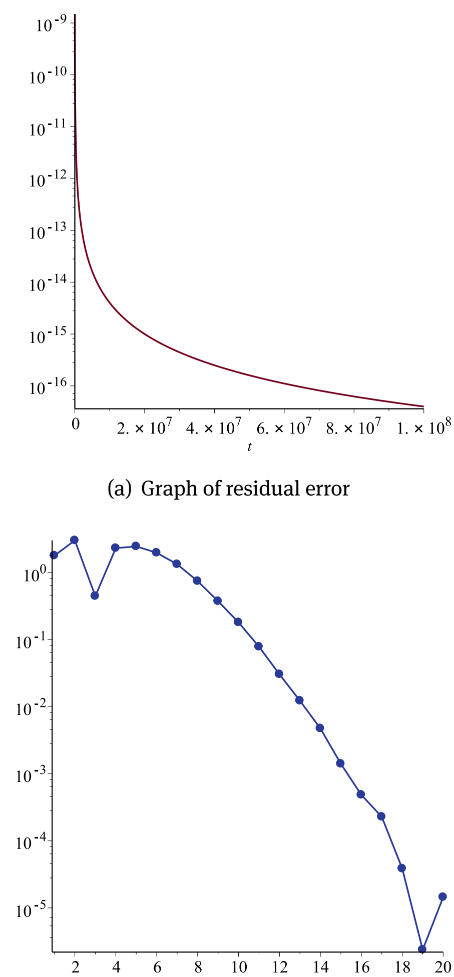
where the boundary conditions for this equation are as follows:

$$y(0) = 0, \quad y'(0) = 0, \quad \lim_{t\to\infty} y'(t) = 1. \qquad (18)$$

In recent years, different methods have been used to solve the Blasius equation. For example, Liao [30] by applying the Homotopy analysis method (HAM), Yu and Chen [31] by applying the differential transformation method, Wang [32] by applying the Adomian decomposition method (ADM), Hashim [33] by applying the ADM Pade approach, Cortell [34] by applying the Runge-Kutta algorithm, Wazwaz [35] by applying the modified Adomian decomposition method, Parand et al. [36, 37] by applying Bessel functions of the first kind and the rational Chebyshev functions. In 2008, Boyd [38] has solved the Blasius equation and has reported the accurate solution of 0.33205733621519630 for $f''(0)$.

For satisfying the boundary conditions, we satisfy the Eq. (18) as follows:

$$\widehat{y_m}(t) = \frac{t^3}{(t+\lambda)^2} + t^2\, e^{-2t-2}\, y_m(t), \qquad (19)$$



(a) Graph of residual error

(b) Graph of $log(|a_i|)$

Fig. 4: Obtained graphs of residual error and the logarithm of coefficients $|a_i|$ for $b_1 = 0.60$ and $b_2 = 0.50$, to show the convergence of the method.

where λ is an arbitrary real constant and $y_m(t)$ is defined in Eq. (5). So, $\widehat{y_m}(0) = 0$, $\widehat{y_m}'(0) = 0$, and $\widehat{y_m}'(t) = 1$ when t tends to ∞, and the boundary conditions (18) are satisfied for all $\lambda > 0$, and is defined in the semi-infinite domain. We

Table 7: Comparison of the obtained values of $y(t)$ for various values of t by the present method, RLT, ELT, RCF, G-RBF, and HAM

t	Present	RLT [29]	ELT [29]	RCF [28]	G-RBF[27]	HAM [26]
0.2	0.87261020	0.87261	0.87261	0.872608596	0.87265264	0.87220
0.4	0.76062562	0.76063	0.76063	0.760626805	0.76074843	0.76010
0.6	0.66243220	0.66243	0.66243	0.662431176	0.66261488	0.66190
0.8	0.57650340	0.57650	0.57650	0.576502298	0.57671495	0.57600
1.0	0.50143723	0.50143	0.50144	0.501436165	0.50164542	0.50100
1.2	0.43595139	0.43595	0.43595	0.435950397	0.43613322	0.43560
1.6	0.32920399	0.32920	0.32920	0.329202052	0.32930679	0.32890
2.0	0.24838691	0.24837	0.24838	0.248384348	0.24842702	0.24820
2.5	0.17455143	0.17455	0.17455	0.174547634	0.17456033	0.17440
3.0	0.12261790	0.12264	0.12261	0.122612386	0.12262652	0.12250
3.5	0.08612124	0.08617	0.08611	—	—	—
4.0	0.06048471	0.06054	0.06047	0.060473418	0.06049038	0.06042
4.5	0.04248174	0.04252	0.04247	—	—	—
5.0	0.02984241	0.02984	0.02982	0.029819128	0.02982446	0.02979

Table 8: Comparison of the obtained values of $y''(0)$ between the present method, Boyd [38], Parand et al. [3, 36], and Liao [30]

Present	Boyd [38]	Parand [36]	Parand [3]	Liao [30]
0.3320573049	0.33205733621519630	0.33205733621519542	0.33205733	0.33206

construct the residual function as follows:

$$Res(t) = 2\frac{d^3\widehat{y_m}}{dt^3} + \widehat{y_m}\frac{d^2\widehat{y_m}}{dt^2}. \quad (20)$$

Table 8 shows the value of $y''(0)$ by the present method with $m = 20,\ \alpha = 0.50$ and $\lambda = 0.9859$ and comparison it with the values of Liao [30], Parand et al. [3, 36], and Boyd [38].

Table 9 shows the obtained values of $y''(0)$ and $\| Res \|_w^2$ by the present method for various values of m and $\alpha = 0.50$.

Table 10 shows the obtained values of $y''(0)$ and $\| Res \|_w^2$ by the present method for various values of α and $m = 20$.

Tables 11 - 13 show the values of $y(t),\ y'(t)$ and $y''(t)$ for various values of t by the present method and comparison it with the values of Parand et al. [3, 36], Cortell [34], and Howarth [39], with $m = 20,\ \alpha = 0.50$ and $\lambda = 0.9859$.

Figure 5 shows the graph of residual error of Eq. (20) and the graph of obtained solution for the Blasius equation by the present method with $m = 20$ and $\alpha = 0.50$.

3.4 The field equation determining the vortex profile

Finally, we consider the field equation determining the vortex profile:

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} + \left(1 - \frac{n^2}{t^2}\right) y(t) - y^3(t) = 0, \quad t \in [0, \infty), \quad (21)$$

Table 9: Obtained values of $y''(0)$ by the present method for various values of m and $\alpha = 0.50$

m	λ	$y'(0)$	$\| Res \|_w^2$
10	0.77	0.33238931	2.0196e-01
15	1.09	0.33213594	1.0365e-03
20	0.98	0.33205367	2.8918e-08

Table 10: Obtained values of $y''(0)$ by the present method for various values of α and $m = 20$

α	λ	$y''(0)$	$\| Res \|_w^2$
0.25	1.0164	0.3320600170	1.1698e-02
0.50	0.9859	0.3320573049	2.8930e-08
0.75	1.7191	0.3320718355	3.4739e-08
1.00	0.9168	0.3320573541	1.5661e-06

where the boundary conditions for this equation are as follows:

$$\lim_{t\to 0} y(t) = k_n t^n + O(t^{n+2}), \qquad \lim_{t\to\infty} y(t) = 1. \quad (22)$$

where $y(t)$ is a real function and $n \in \mathbf{Z}$.

A major problem of the this boundary conditions amounts to find the value(s) of the free parameter, k_n, to ensure the boundary conditions (22) of $y(t)$ at $t = \infty$.

This equation is examined in the static, rotationally symmetric global vortex in a Ginzburg-Landau effective theory [40], which has numerous applications ranging from condensed matter to cosmic strings [41]. Some re-

Table 11: Comparison of the obtained values of $y(t)$ for various values of t by the present method, Parand et al. [3, 36], Howarth [39], and Cortell [34]

t	Present	Parand [36]	Howarth [39]	Cortell [34]	Parand [3]
1.0	0.1655717346	0.1655717	0.16557	0.16557	0.1655724
2.0	0.6500244189	0.6500243	0.65003	0:65003	0.6500351
3.0	1.3968083309	1.3968082	1.39682	1.39682	1.3968223
4.0	2.3057465742	2.3057464	2.30576	2.30576	2.3057618
5.0	3.2832738023	3.2832736	3.28329	3.28330	3.2832910

Table 12: Comparison of the obtained values of $y'(t)$ for various values of t by the present method, Parand et al. [3, 36], Howarth [39], and Cortell [34]

t	Present	Parand [36]	Howarth [39]	Cortell [34]	Parand [3]
1.0	0.3297800807	0.32978003	0.32979	0.32978	0.3297963
2.0	0.6297657868	0.62976573	0.62977	0.62977	0.6297763
3.0	0.8460444987	0.84604444	0.84605	0.84605	0.8460595
4.0	0.9555182849	0.95551823	0.95552	0.95552	0.9555236
5.0	0.9915413606	0.99154190	0.99155	0.99155	0.9915546

Table 13: Comparison of the obtained values of $y''(t)$ for various values of t by the present method, Parand et al. [3, 36], Howarth [39], and Cortell [34]

t	Present	Parand [36]	Howarth [39]	Cortell [34]	Parand [3]
0.0	0.3320573049	0.3320573362	0.33206	0.33206	0.3320571
1.0	0.3230074062	0.3230071168	0.32301	0.32301	0.3230136
2.0	0.2667515144	0.2667515456	0.26675	0.26675	0.2667557
3.0	0.1613603185	0.1613603194	0.16136	0.16136	0.1613637
4.0	0.0642341187	0.0642341209	0.06424	0.06423	0.0642411
5.0	0.0159026175	0.0159067985	0.01591	0.01591	0.0159134

searchers approximate the field equation determining the vortex profile; for example, Boisseau et al. [42] by applying the analytical method based on replacing the original ODEs by a sequence of auxiliary first-order polynomial ODEs with constant coefficients, and Amore & Fernandez [43] by applying the Pade-Hankel method.

For satisfying the boundary conditions, we satisfy the Eq. (22) as follows:

$$\widehat{y_m}(t) = \frac{t^n}{t^n + \lambda} + t^n\, e^{-2t-2}\, y_m(t), \tag{23}$$

where λ is an arbitrary real constant and $y_m(t)$ is defined in Eq. (5). So the boundary conditions (22) are satisfied for all $\lambda > 0$, and is defined in the semi-infinite domain. We construct the residual function as follows:

$$Res(t) = \frac{d^2\widehat{y_m}}{dt^2} + \frac{1}{t}\frac{d\widehat{y_m}}{dt} + \left(1 - \frac{n^2}{t^2}\right)\widehat{y_m}(t) - \widehat{y_m}^3(t). \tag{24}$$

Table 14 shows the values of k_n by the present method with $m = 40$ and $\alpha = 0.50$, and comparison it with the values of Shooting method [42], Boisseau et al. [42], and Amore & Fernandez [43].

Table 15 shows the obtained values of k_2 and $\| Res \|_w^2$ by the present method for various values of m and $\alpha = 0.50$.

Table 16 shows the obtained values of k_2 and $\| Res \|_w^2$ by the present method for various values of α and $m = 40$. As can be seen, the results for $\alpha = 0.50$ and $\alpha = 1.00$ are almost identical.

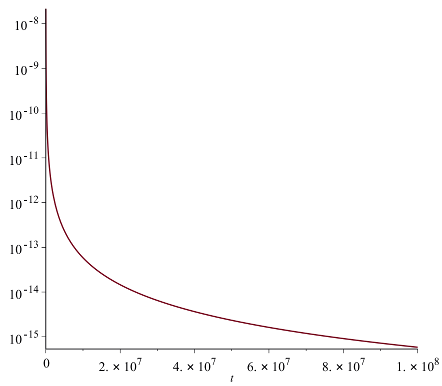
Table 17 shows the obtained values of $y(t)$ by the present method for various values of n and t with $m = 40$ and $\alpha = 0.50$.

Figure 6 shows the graphs of residual errors of Eq. (24) and the graphs of the obtained solutions for the field equation by the present method with $m = 40$ and $\alpha = 0.50$.

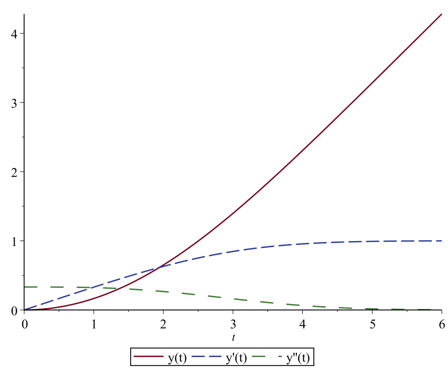
Figure 7 shows the graphs of $\frac{1}{n!}y^{(n)}(t)$ for $n = 1,\ 2,\ 3,$ and 4, and the graphs of $\log(|a_i|)$ for $n = 2,\ 3,$ and 4 to show the convergence of the method.

Table 14: Comparison of the obtained values of k_n between the present method, Shooting method [42], Boisseau et al. [42], and Amore & Fernandez [43].

k_n	λ	Present	Shooting [42]	Boisseau [42]	Amore [43]
k_1	0.1	0.583189717489	0.5831894959	0.5831894936	0.5831894958601
k_2	2	0.153099102829	0.1530991029	0.15309	0.15309910286
k_3	23	0.026183420209	0.02618342072	0.026185	0.0261834207
k_4	45	0.003327173339	0.00332717340	0.0033	0.0033271734



(a) Graph of residual error



(b) Graphs of solution

Fig. 5: Obtained graphs of residual error and the Blasius solution.

Table 15: Obtained values of k_2 by the present method for various values of m and $\alpha = 0.50$

m	λ	k_2	$\Vert Res \Vert_w^2$
10	2	0.1529258473004.3939e-05	
20	2	0.1530990508703.5332e-10	
30	2	0.1530991055357.3141e-13	
40	2	0.1530991028291.5483e-16	

Table 16: Obtained values of k_2 by the present method for various values of α and $m = 40$

α	λ	k_2	$\Vert Res \Vert_w^2$
0.25	2	0.153092105254	2.7708e-04
0.50	2	0.153099102829	1.5483e-16
0.75	2	0.153068774773	6.7629e-08
1.00	2	0.153099103672	8.0842e-16
1.25	1	0.151696304817	8.8242e-05
1.50	1	0.137087413881	7.2232e-04

Table 17: Obtained values of $y(t)$ by the present method for various values of n and t.

t	$n = 1$	$n = 2$	$n = 3$	$n = 4$
0.5	0.28282173	0.03748381	0.00322210	0.00020535
1.0	0.52005163	0.14078358	0.02458744	0.00316426
1.5	0.69208735	0.28531419	0.07662991	0.01503527
2.0	0.80495682	0.44008751	0.16224506	0.04343598
2.5	0.87497019	0.57972330	0.27383162	0.09430307
3.0	0.91748107	0.69154754	0.39661914	0.16902827
3.5	0.94337703	0.77435581	0.51495766	0.26313174
4.0	0.95946829	0.83300459	0.61788070	0.36737260
6.0	0.98473111	0.93670460	0.84845320	0.71026581
8.0	0.99184533	0.96681821	0.92294713	0.85614939
10.	0.99487378	0.97930359	0.95265849	0.91367234

4 Conclusion

The main goal of this paper is introducing a new orthogonal fractional basis, namely the generalized fractional order of the Chebyshev orthogonal functions (GFCF), for solving nonlinear boundary value problems in the semi-infinite domain. Solving these problems is difficult because they have a boundary condition in the infinite. But we used these new basis to solve them and obtained the good results. The present results show that new basis for the collocation Spectral method is efficient and applicable. A comparison was made of the numerical solution of other researchers and the present method. It has been shown that the present method has provided an acceptable approach for solving these types of equations.

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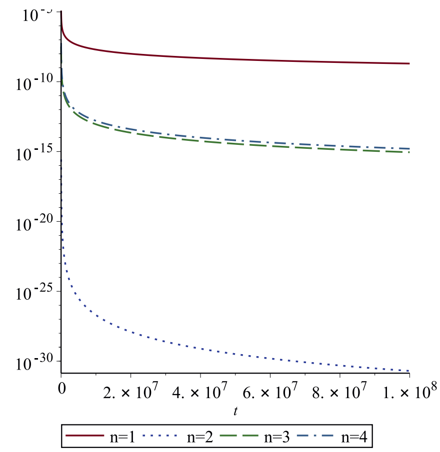
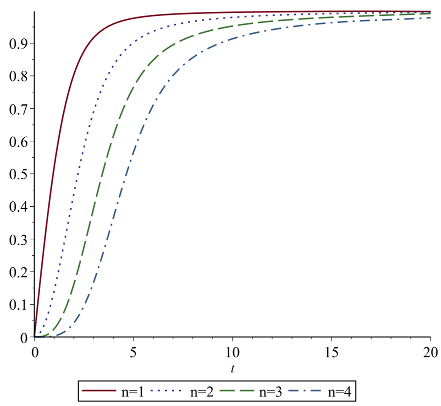
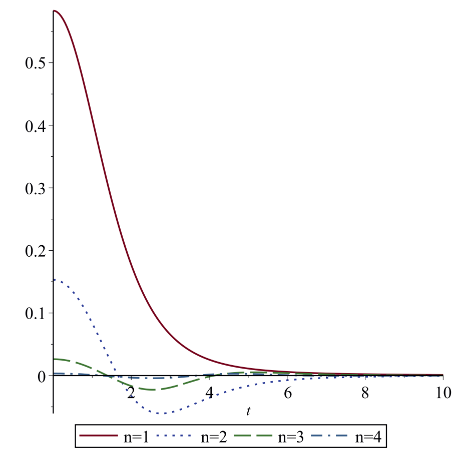


(a) Graph of residual errors

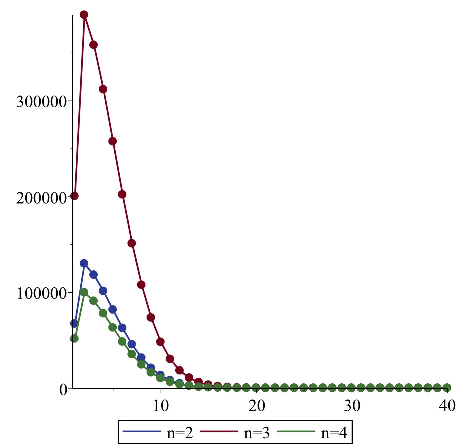


(b) Graph of solutions

Fig. 6: Obtained graphs of the residual errors and the obtained solutions for various values of n.



(a) Graphs of $\frac{1}{n!}y^{(n)}(t)$



(b) Graphs of $\log(|a_i|)$

Fig. 7: (a) Graphs of $\frac{1}{n!}y^{(n)}(t)$ for $n = 1, 2, 3, 4$ (b) Graphs of $\log(|a_i|)$ for $n = 2, 3, 4$ to show the convergence.

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