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A note on Riccati-Bernoulli Sub-ODE method combined with complex transform method applied to fractional differential equations

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Abstract: In this paper, the fractional derivatives in the sense of modified Riemann–Liouville and the Riccati-Bernoulli Sub-ODE method are used to construct exact solutions for some nonlinear partial fractional differential equations via the nonlinear fractional Zoomeron equation and the (3 + 1) dimensional space-time fractional mKDV-ZK equation. These nonlinear fractional equations can be turned into another nonlinear ordinary differential equation by complex transform method. This method is efficient and powerful in solving wide classes of nonlinear fractional order equations. The Riccati-Bernoulli Sub-ODE method appears to be easier and more convenient by means of a symbolic computation system.

Keywords: Modified Riemann-Liouville derivative, Riccati-Bernoulli Sub-ODE method, exact solution, fractional Zoomeron equation, (3 + 1) dimensional space-time fractional mKDV-ZK equation

MSC: 26A33, 34A08, 35A99, 35R11, 83C15, 65Z05

1 Introduction

Many physical phenomena such as mathematical biology, signal processing, optics, fluid mechanics, electromagnetic theory, etc., can be modeled using the fractional derivatives. Consequently, the investigation of exact solutions for FDEs turns out to be very useful in the study of scientific research. Moreover generalized forms of differential equations are described as fractional differential equations FDEs.

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Recent past, an strong attention has been purposed by the researchers concerning the fractional partial differential equations FDEs. For an interesting overview and more applications of nonlinear FDEs, we refer to [1–4].

However, even in most useful studies, there is no an efficient and general methods to solve them. Actually, various analytical and numerical methods to construct approximate and exact solutions of nonlinear FDEs have been put forward, such as the fractional sub-equation method [5, 6], the tanh-sech method [7], the $(\frac{G'}{G})$– expansion method [8, 19], the first integral method [10], the modified Kudryashov method [11], the exponential function method [12, 13] and others [14, 15].

The novelties of this paper are mainly exhibited in two aspects: First, we introduce a new method, which is not familiar, the so called Riccati-Bernoulli Sub-ODE method. We use this method to solve the nonlinear fractional Zoomeron equation and the (3 + 1) dimensional space-time fractional mKDV-ZK equation. Moreover, we show that the proposed method gives infinite sequence of solutions. Second, we obtain new types of exact analytical solutions. Moreover comparing our results with other results, one can see that our results are new and most extensive.

Actually, the proposed two fractional equations have many applications in various fields of theoretical physics,applied mathematics and engineering such as control theory viscoelasticity, modelling heat transfer, control, diffusion, signal and image processing, and many other physical and engineering processes. In more details, the (3 + 1) dimensional space-time fractional mKDV-ZK equation is derived for a plasma comprised of cool and hot electrons and a species of fluid ions, which have so many direct and indirect in engineering models. Furthermore the nonlinear fractional Zoomeron equation is a convenient model to display the novel phenomena associated with boomerons and trappons and further interesting engineering applications.

The Riccati-Bernoulli Sub-ODE technique has been used to solve some partial and fractional differential equations, see for example [16–21]. These works show that this method is efficacious, robust and adequate for solving further equations.

The rest of the paper is arranged as follows: In Section 2, we recall some basic definitions and notions dealing with fractional calculus theory, which are used in the sequel in this article. In Sections 4 and 3, two examples, namely the nonlinear fractional Zoomeron equation and the (3 + 1) dimensional space-time fractional mKDV-ZK equation, are solved by the Riccati-Bernoulli Sub-ODE method. Conclusion will appear in Section 5.

2 Preliminaries and notation

We introduce some fundamental definitions and properties of fractional calculus theory, which turn to be very useful in order to complete this paper in a completely unified way. These are the Riemann–Liouville, the *Grünwald*-Letnikov, the Caputo and the modified Riemann-Caputo, Liouville derivative. The most commonly used definitions are the modified Riemann–Liouville and Caputo derivatives [22, 23]. Jumarie proposed a modified Riemann–Liouville derivative [24]. Firstly, we present some properties and definitions of the modified Riemann–Liouville derivative. Secondly, we give the description of the Riccati-Bernoulli Sub-ODE method.

Assume that $f(t)$ denotes a continuous $\mathbb{R} \to \mathbb{R}$ function (but not necessarily first-order differentiable). The Jumarie's modified Riemann–Liouville derivative is defined as

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha-1}(f(\xi)-f(0))d\xi, & \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, & 0 < \alpha < 1 \\ (f^{(n)}(t))^{(\alpha-n)}, & n \leq \alpha < n+1,\ n \geq 1\,. \end{cases}, \tag{2.1}$$

Important property of the fractional modified Riemann-Liouville derivative is [25]

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}\,. \tag{2.2}$$

Step 1. Any nonlinear fractional differential equation in two independent variables x and t can be expressed in following form:

$$G(u, D_t^\alpha u, D_x^\alpha u, D_t^\alpha D_x^\alpha u, D_x^\alpha D_x^\alpha u, \ldots) = 0, \tag{2.3}$$

where $0 < \alpha \leq 1$, $D_t^\alpha u, D_x^\alpha u$ are modified Riemann-Liouville derivative of u and G is a polynomial in $u(x,t)$ and its partial fractional derivatives.

Step 2. Using the traveling wave transformation

$$u(x,t) = U(\xi), \quad \xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}, \tag{2.4}$$

where k, λ are non zero constants and $1 < \alpha \leq 1$. By using the chain rule,

$$\begin{aligned} D_t^\alpha u &= \sigma_t' \frac{dU}{d\xi} D_t^\alpha \xi\,, \\ D_x^\alpha u &= \sigma_x' \frac{dU}{d\xi} D_x^\alpha \xi\,, \end{aligned} \tag{2.5}$$

where σ_t' and σ_x' are called the sigma indexes, [26], without loss of generality, we can take $\sigma_t' = \sigma_x' = L$, where L is a constant.

Superseding (2.4) with (2.2) and (2.5) into (2.3), the equation (2.3) transformed into the following ODE:

$$H(U, U', U'', U''', \ldots\ldots) = 0, \tag{2.6}$$

where prime denotes the derivation with respect to ξ.

Step 3. Based on the Riccati-Bernoulli Sub-ODE method [16–18], we assume that equation (2.6) has the following solution:

$$U' = aU^{2-n} + bU + cU^n, \tag{2.7}$$

where a, b, c and n are constants calculated later. From equation (2.7), we have

$$\begin{aligned} U'' &= ab(3-n)U^{2-n} + a^2(2-n)U^{3-2n} + nc^2U^{2n-1} \\ &\quad + bc(n+1)U^n + (2ac+b^2)U, \end{aligned} \tag{2.8}$$

$$\begin{aligned} U''' &= (ab(3-n)(2-n)U^{1-n} + a^2(2-n)(3-2n)U^{2-2n} \\ &\quad + n(2n-1)c^2U^{2n-2} + bcn(n+1)U^{n-1} + (2ac+b^2))U'\,. \end{aligned} \tag{2.9}$$

The exact solutions of equation (2.7), for an arbitrary constant μ are given as follow:

1. For $n = 1$, the solution is

$$U(\xi) = \mu e^{(a+b+c)\xi}\,. \tag{2.10}$$

2. For $n \neq 1$, $b = 0$ and $c = 0$, the solution is

$$U(\xi) = \left(a(n-1)(\xi+\mu)\right)^{\frac{1}{n-1}}\,. \tag{2.11}$$

3. For $n \neq 1$, $b \neq 0$ and $c = 0$, the solution is

$$U(\xi) = \left(\frac{-a}{b} + \mu e^{b(n-1)\xi}\right)^{\frac{1}{n-1}}\,. \tag{2.12}$$

4. For $n \neq 1, a \neq 0$ and $b^2 - 4ac < 0$, the solution is

$$U(\xi) = \left(\frac{-b}{2a} + \frac{\sqrt{4ac-b^2}}{2a}\tan\left(\frac{(1-n)\sqrt{4ac-b^2}}{2}(\xi+\mu)\right)\right)^{\frac{1}{1-n}} \quad (2.13)$$

and

$$U(\xi) = \left(\frac{-b}{2a} - \frac{\sqrt{4ac-b^2}}{2a}\cot\left(\frac{(1-n)\sqrt{4ac-b^2}}{2}(\xi+\mu)\right)\right)^{\frac{1}{1-n}}. \quad (2.14)$$

5. For $n \neq 1, a \neq 0$ and $b^2 - 4ac > 0$, the solution is

$$U(\xi) = \left(\frac{-b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\coth\left(\frac{(1-n)\sqrt{b^2-4ac}}{2}(\xi+\mu)\right)\right)^{\frac{1}{1-n}} \quad (2.15)$$

and

$$U(\xi) = \left(\frac{-b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\tanh\left(\frac{(1-n)\sqrt{b^2-4ac}}{2}(\xi+\mu)\right)\right)^{\frac{1}{1-n}}. \quad (2.16)$$

6. For $n \neq 1$, $a \neq 0$ and $b^2 - 4ac = 0$, the solution is

$$U(\xi) = \left(\frac{1}{a(n-1)(\xi+\mu)} - \frac{b}{2a}\right)^{\frac{1}{1-n}}. \quad (2.17)$$

2.1 Bäcklund transformation

When $U_{m-1}(\xi)$ and $U_m(\xi)(U_m(\xi) = U_m(U_{m-1}(\xi)))$ are the solutions of equation (2.7), we have

$$\frac{dU_m(\xi)}{d\xi} = \frac{dU_m(\xi)}{dU_{m-1}(\xi)}\frac{dU_{m-1}(\xi)}{d\xi} = \frac{dU_m(\xi)}{dU_{m-1}(\xi)}(aU_{m-1}^{2-n} + bU_{m-1} + cU_{m-1}^n),$$

namely

$$\frac{dU_m(\xi)}{aU_m^{2-n} + bU_m + cU_m^n} = \frac{dU_{m-1}(\xi)}{aU_{m-1}^{2-n} + bU_{m-1} + cU_{m-1}^n}. \quad (2.18)$$

Integrating equation (2.18) once with respect to ξ, we get a Bäcklund transformation of equation (2.7) as follows:

$$U_m(\xi) = \left(\frac{-cK_1 + aK_2\left(U_{m-1}(\xi)\right)^{1-n}}{bK_1 + aK_2 + aK_1\left(U_{m-1}(\xi)\right)^{1-n}}\right)^{\frac{1}{1-n}}, \quad (2.19)$$

where K_1 and K_2 are arbitrary constants. We use equation (2.19) to obtain infinite sequence of solutions for equation (2.7), as well for equation (2.3).

3 The nonlinear fractional Zoomeron equation

We are concerned with the nonlinear fractional Zoomeron equation([27]),

$$D_{tt}^{2\alpha}\left[\frac{u_{xy}}{u}\right] - \left[\frac{u_{xy}}{u}\right]_{xx} + 2D_t^{\alpha}\left[u^2\right]_x = 0, \quad 0 < \alpha \leq 1, \quad (3.1)$$

where $u(x, y, t)$ is the amplitude of the relevant wave mode.

Using the transformation

$$u(x, y, t) = U(\xi), \quad (3.2)$$

$$\xi = lx + \gamma y - \frac{wt^{\alpha}}{\Gamma(1+\alpha)}, \quad (3.3)$$

where l, γ and w are non zero constants and $0 < \alpha \leq 1$.

Substituting (3.3) with (2.2) and (2.5) into (3.1), we have the ODE

$$l\gamma w^2\left(\frac{U''}{U}\right)'' - \gamma l^3\left(\frac{U''}{U}\right)'' - 2lw(U^2)'' = 0. \quad (3.4)$$

Integrating this equation twice, with the second constant of integration is vanishing, we obtain

$$l\gamma(w^2 - l^2)U'' - 2lwU^3 - kU = 0, \quad (3.5)$$

where k is a nonzero constant of integration.

Substituting equations (2.8) into equation (3.5), we obtain

$$l\gamma(w^2-l^2)\Big(ab(3-m)U^{2-m} + a^2(2-m)U^{3-2m} + mc^2U^{2m-1} + bc(m+1)U^m + (2ac+b^2)U\Big) - 2lwU^3 - kU = 0. \quad (3.6)$$

Setting $m = 0$, equation (3.6) is reduced to

$$l\gamma(w^2-l^2)(3abU^2 + 2a^2U^3 + bc + (2ac+b^2)U) - 2lwU^3 - kU = 0. \quad (3.7)$$

Equating each coefficient of $U^i (i = 0, 1, 2, 3)$ to zero, we have

$$l\gamma(w^2 - l^2)bc = 0, \quad (3.8)$$

$$l\gamma(w^2 - l^2)(2ac + b^2) - k = 0, \quad (3.9)$$

$$3l\gamma(w^2 - l^2)ab = 0, \quad (3.10)$$

$$2l\gamma(w^2 - l^2)a^2 - 2lw = 0. \quad (3.11)$$

Solving equations (3.8)-(3.11), we get

$$b = 0\,, \tag{3.12}$$

$$ac = \frac{k}{2l\gamma(w^2 - l^2)}\,, \tag{3.13}$$

$$a = \pm\sqrt{\frac{w}{\gamma(w^2 - l^2)}}\,, \tag{3.14}$$

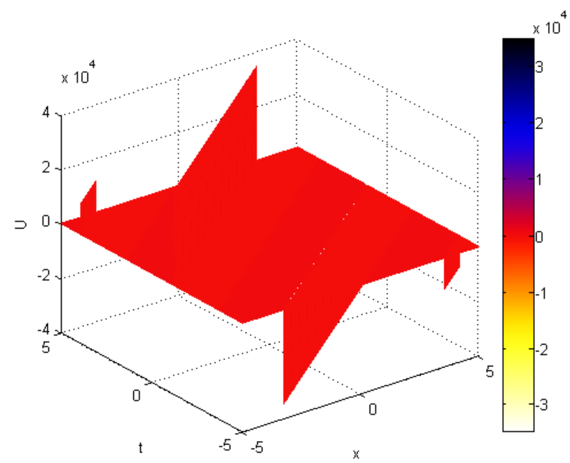

Fig. 1: The solution $U_1(x, 0, t)$ in (3.15) for l=1.5, k=1, a=1, μ=0, w = 2 and $-5 \le t, x \le 5$.

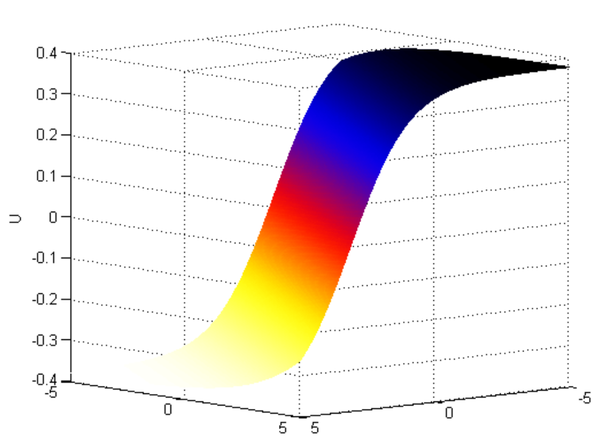

Fig. 2: The solution $U_5(x, 0, t)$ in (3.17) for l=2, k=2, a=1, μ=1, w = 3.5 and $-5 \le t, x \le 5$.

Trigonometric function solutions:

When $\frac{k}{l\gamma(w^2-l^2)} < 0$, substituting equations (3.12)-(3.14) and (3.3) into equations (2.13) and (2.14), we get the exact solutions for equation (3.1),

$$U_{1,2}(x, y, t) = \pm\sqrt{\frac{k}{2wl}}\tan\left(\sqrt{\frac{k}{2lw(w^2 - l^2)}}(lx + \gamma y - \frac{wt^\alpha}{\Gamma(1+\alpha)} + \mu)\right) \tag{3.15}$$

and

$$U_{3,4}(x, y, t) = \pm\sqrt{\frac{k}{2wl}}\cot\left(\sqrt{\frac{k}{2lw(w^2 - l^2)}}(lx + \gamma y - \frac{wt^\alpha}{\Gamma(1+\alpha)} + \mu)\right)\,, \tag{3.16}$$

where l, k, γ, w, μ are arbitrary constants and $0 < \alpha \le 1$. Figure 1 illustrated the solution U_2 .

Hyperbolic function solutions:

When $\frac{k}{l\gamma(w^2-l^2)} > 0$, substituting equations (3.12)-(3.14) and (3.3) into equations (2.15) and (2.16), we obtain exact solutions for equation (3.1),

$$U_{5,6}(x, y, t) = \pm\sqrt{\frac{-k}{2wl}}\tanh\left(\sqrt{\frac{k}{2lw(l^2 - w^2)}}(lx + \gamma y - \frac{wt^\alpha}{\Gamma(1+\alpha)} + \mu)\right) \tag{3.17}$$

and

$$U_{7,8}(x, y, t) = \pm\sqrt{\frac{-k}{2wl}}\tanh\left(\sqrt{\frac{k}{2lw(l^2 - w^2)}}(lx + \gamma y - \frac{wt^\alpha}{\Gamma(1+\alpha)} + \mu)\right)\,, \tag{3.18}$$

where l, k, γ, w, μ are arbitrary constants and $0 < \alpha \le 1$. Figure 2 illustrated the solution U_5.

Remark 3.1. *Applying equation* (2.19) *to* $u_i(x, t)$, *i* =1,2,...,8, *we obtain an infinite sequence of solutions of equation* (3.1). *For illustration, by applying equation* (2.19) *to* $u_i(x, t)$, *i* =1,2,...,8, *once, we have new solutions of equation* (3.1)

$$u^*_{1,2}(x, t) = \frac{-\frac{k}{2lw} \pm A_3\sqrt{\frac{k}{2wl}}\,tan\left(\sqrt{\frac{k}{2lw(w^2-l^2)}}(lx + \gamma y - \frac{wt^\alpha}{\Gamma(1+\alpha)} + \mu)\right)}{A_3 \pm \sqrt{\frac{k}{2wl}}\,tan\left(\sqrt{\frac{k}{2lw(w^2-l^2)}}(lx + \gamma y - \frac{wt^\alpha}{\Gamma(1+\alpha)} + \mu)\right)}\,, \tag{3.19}$$

$$u^{\star}_{3,4}(x,t) = \frac{-\frac{k}{2lw} \pm A_3\sqrt{\frac{k}{2wl}}\, cot\left(\sqrt{\frac{k}{2lw(w^2-l^2)}}(lx+\gamma y-\frac{wt^\alpha}{\Gamma(1+\alpha)}+\mu)\right)}{A_3 \pm \sqrt{\frac{k}{2wl}}\, cot\left(\sqrt{\frac{k}{2lw(w^2-l^2)}}(lx+\gamma y-\frac{wt^\alpha}{\Gamma(1+\alpha)}+\mu)\right)}, \tag{3.20}$$

$$u^{\star}_{5,6}(x,t) = \frac{-\frac{k}{2lw} \pm A_3\sqrt{-\frac{k}{2wl}}\, tanh\left(\sqrt{\frac{k}{2lw(l^2-w^2)}}(lx+\gamma y-\frac{wt^\alpha}{\Gamma(1+\alpha)}+\mu)\right)}{A_3 \pm \sqrt{\frac{k}{2wl}}\, tanh\left(\sqrt{\frac{k}{2lw(l^2-w^2)}}(lx+\gamma y-\frac{wt^\alpha}{\Gamma(1+\alpha)}+\mu)\right)}, \tag{3.21}$$

$$u^{\star}_{7,8}(x,t) = \frac{-\frac{k}{2lw} \pm A_3\sqrt{-\frac{k}{2wl}}\, coth\left(\sqrt{\frac{k}{2lw(l^2-w^2)}}(lx+\gamma y-\frac{wt^\alpha}{\Gamma(1+\alpha)}+\mu)\right)}{A_3 \pm \sqrt{\frac{k}{2wl}}\, coth\left(\sqrt{\frac{k}{2lw(l^2-w^2)}}(lx+\gamma y-\frac{wt^\alpha}{\Gamma(1+\alpha)}+\mu)\right)}, \tag{3.22}$$

where $A_3, l, k, \gamma, w, \mu$ are arbitrary constants and $0 < \alpha \leq 1$.

4 The (3 + 1) dimensional space-time fractional mKDV-ZK equation

The second equation is the (3 + 1) dimensional space-time fractional mKDV-ZK equation which has the form ([28])

$$D_t^\alpha u + lu^2 D_x^\alpha u + D_x^{3\alpha} u + D_x^\alpha D_y^{2\alpha} u + D_x^\alpha D_z^{2\alpha} u = 0, \quad , t > 0, \quad 0 < \alpha \leq 1\,, \tag{4.1}$$

where l is an nonzero constant and $0 < \alpha \leq 1$. The mKdV equation is used for representing physical and engineering phenomena such as to describe the ion-acoustic waves in a magnetized plasma, dipole blocking and study of coastal waves in ocean etc., see e.g. [29–31].

Using the transformation

$$u(x,y,z,t) = U(\xi)\,, \tag{4.2}$$

$$\xi = \frac{\beta x^\alpha}{\Gamma(1+\alpha)} + \frac{\gamma y^\alpha}{\Gamma(1+\alpha)} + \frac{\delta z^\alpha}{\Gamma(1+\alpha)} - \frac{wt^\alpha}{\Gamma(1+\alpha)}\,, \tag{4.3}$$

where β, γ, δ and w are non zero constants and $0 < \alpha \leq 1$.

Substituting (4.3) with (2.2) and (2.5) into (4.1), we have the ODE

$$(\beta^3 + \beta\gamma^2 + \beta\delta^2)U''' + l\beta U^2 U' - wU' = 0\,, \tag{4.4}$$

Integrating equation (4.4) once with respect to ξ with the zero constant of integration, we have

$$(\beta^3 + \beta\gamma^2 + \beta\delta^2)U'' + \frac{l\beta}{3}U^3 - wU = 0\,. \tag{4.5}$$

Substituting equations (2.8) into equation (4.5), we obtain

$$(\beta^3+\beta\gamma^2+\beta\delta^2)\left(ab(3-m)U^{2-m} + a^2(2-m)U^{3-2m} + mc^2U^{2m-1} + bc(m+1)U^m + (2ac+b^2)U\right) + \frac{l\beta}{3}U^3 - wU = 0\,. \tag{4.6}$$

Putting $m = 0$, equation (4.6) becomes

$$(\beta^3 + \beta\gamma^2 + \beta\delta^2)(3abU^2 + 2a^2U^3 + bc + (2ac + b^2)U) + \frac{l\beta}{3}U^3 - wU = 0\,. \tag{4.7}$$

Putting each coefficient of $U^i (i = 0, 1, 2, 3)$ to zero, we obtain

$$(\beta^3 + \beta\gamma^2 + \beta\delta^2)bc = 0\,, \tag{4.8}$$

$$(\beta^3 + \beta\gamma^2 + \beta\delta^2)(2ac + b^2) - w = 0\,, \tag{4.9}$$

$$3(\beta^3 + \beta\gamma^2 + \beta\delta^2)ab = 0\,, \tag{4.10}$$

$$2a^2(\beta^3 + \beta\gamma^2 + \beta\delta^2) + \frac{l\beta}{3} = 0\,. \tag{4.11}$$

Solving equations (4.8)-(4.11), we get

$$b = 0\,, \tag{4.12}$$

$$ac = \frac{w}{2(\beta^3 + \beta\gamma^2 + \beta\delta^2)}\,, \tag{4.13}$$

$$a = \pm\sqrt{\frac{-l}{6(\beta^2 + \gamma^2 + \delta^2)}}\,, \tag{4.14}$$

Trigonometric function solutions:

When $\frac{w}{\beta^3+\beta\gamma^2+\beta\delta^2} < 0$, substituting equations (4.12)-(4.14) and (4.3) into equations (2.13) and (2.14), we obtain the exact solutions of equation (4.1),

$$\tilde{U}_{1,2}(x,y,z,t) = \pm\sqrt{\frac{-3w}{\beta l}}\tan\left(\sqrt{\frac{w}{2(\beta^3+\beta\gamma^2+\beta\delta^2)}}\left(\frac{\beta x^\alpha}{\Gamma(1+\alpha)} + \frac{\gamma y^\alpha}{\Gamma(1+\alpha)} + \frac{\delta z^\alpha}{\Gamma(1+\alpha)} - \frac{wt^\alpha}{\Gamma(1+\alpha)} + \mu\right)\right) \tag{4.15}$$

and

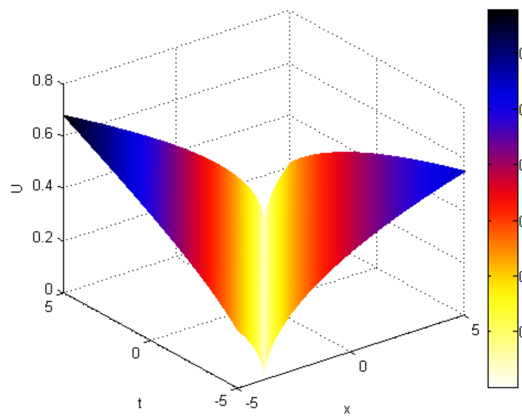

Fig. 3: The solution $\tilde{U}_1(x, 0, 0, t)$ in (4.15) for β=-2.6, γ=-3.5, δ=3,l=-2, α=1, μ=0, w = -2 and $-5 \le t, x \le 5$.

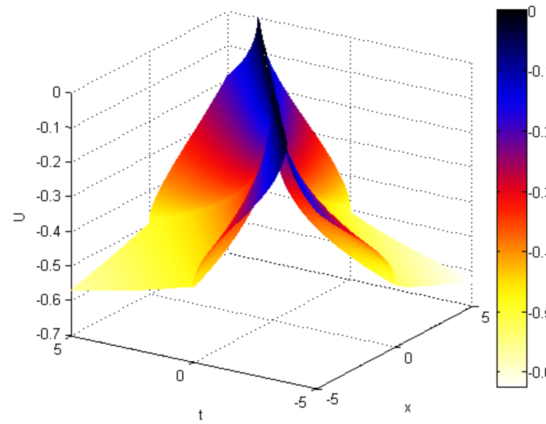

Fig. 4: The solution $\tilde{U}_5(x, 0, 0, t)$ in (4.17) for β=2.6, γ=-3.5, δ=3,l=-2, α=0.5, μ=0, w = 3 and $-5 \le t, x \le 5$.

$$\tilde{U}_{3,4}(x, y, z, t) = \pm\sqrt{\frac{-3w}{\beta l}}\cot\left(\sqrt{\frac{w}{2(\beta^3+\beta\gamma^2+\beta\delta^2)}}\left(\frac{\beta x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma y^\alpha}{\Gamma(1+\alpha)}+\frac{\delta z^\alpha}{\Gamma(1+\alpha)}-\frac{w t^\alpha}{\Gamma(1+\alpha)}+\mu\right)\right), \tag{4.16}$$

where β, γ, δ, w and μ are non zero constants and $0 < \alpha \le 1$. Figure 3 illustrated the solution $\tilde{U}_1$.

Hyperbolic function solutions:

When $\frac{w}{\beta^3+\beta\gamma^2+\beta\delta^2} > 0$, substituting equations (3.12)-(3.14) and (4.3) into equations (2.15) and (2.16), we obtain exact solutions of equation (4.1),

$$\tilde{U}_{5,6}(x, y, z, t) = \pm\sqrt{\frac{3w}{\beta l}}\tanh\left(\sqrt{\frac{-w}{2(\beta^3+\beta\gamma^2+\beta\delta^2)}}\left(\frac{\beta x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma y^\alpha}{\Gamma(1+\alpha)}+\frac{\delta z^\alpha}{\Gamma(1+\alpha)}-\frac{w t^\alpha}{\Gamma(1+\alpha)}+\mu\right)\right) \tag{4.17}$$

and

$$\tilde{U}_{7,8}(x, y, z, t) = \pm\sqrt{\frac{3w}{\beta l}}\coth\left(\sqrt{\frac{-w}{2(\beta^3+\beta\gamma^2+\beta\delta^2)}}\left(\frac{\beta x^\alpha}{\Gamma(1+\alpha)}+\frac{\gamma y^\alpha}{\Gamma(1+\alpha)}+\frac{\delta z^\alpha}{\Gamma(1+\alpha)}-\frac{w t^\alpha}{\Gamma(1+\alpha)}+\mu\right)\right), \tag{4.18}$$

where β, γ, δ, w and μ are non zero constants and $0 < \alpha \le 1$. Figure 4 illustrated the solution $\tilde{U}_5$.

Remark 4.1. *Similarly as shown in Remark* (3.1), *we can give an infinite solutions of equation* (4.1).

Remark 4.2.

1. *Comparing our results concerning equation* (3.1) *with the results in [27, 32], one can see that our results are new and most extensive.*
2. *Comparing our results concerning equation* (4.1) *with the results in [31, 33], one can see that our results are new and most extensive.*
3. *Comparing our solutions for equations* (3.1) *and* (4.1) *with [27, 31–33], it can be seen that by choosing suitable values for the parameters similar solutions can be verified.*
4. *Actually the Riccati-Bernoulli Sub-ODE technique has a very important feature, that admits infinite sequence of solutions of equation, which is explained clearly in Section 2.1. In fact this feature has never given for any another method.*
5. *Consequently, the method is efficacious, robust and adequate for solving other type of space-time fractional differential equations.*

5 Conclusions

In this work, a Riccati-Bernoulli Sub-ODE technique has successfully been applied to exact solutions for the nonlinear fractional Zoomeron equation and the (3 + 1) dimensional space-time fractional mKDV-ZK equation with modified Riemann–Liouville derivative. Fractional complex transform is also used as the basic ingredient to obtain exact solutions for these nonlinear equations. As a result, some new exact solutions for them have successfully been obtained. The graphs of some solutions are depicted for suitable coefficients. Actually this method can be applied for many other nonlinear FDEs appearing in mathematical physics and natural sciences.

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