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Accelerated HPSTM: An efficient semi-analytical technique for the solution of nonlinear PDE's

https://doi.org/10.1515/nleng-2020-0019
Received Mar 23, 2020; accepted Jun 9, 2020.

Abstract: In this paper a novel technique i.e. accelerated homotopy perturbation Sumudu transformation method (AHPSTM), which is a hybrid of accelerated homotopy perturbation method and Sumudu transformation to obtain an approximate analytic solution of nonlinear partial differential equation (PDE) with proportional delay, is used. This approach is based on the new form of calculating He's polynomial, which accelerates the convergence of the series solution. The series solutions obtained from the proposed method are found to converge rapidly to exact solution. In order to affirm the effectiveness and legitimacy of proposed method, the proposed technique is implemented on nonlinear partial differential equation (PDE) with proportional delay. The condition of convergence of series solution is analyzed. Moreover, statistical analysis has been performed to analyze the outcome acquired by AHPSTM and other semi-analytic techniques.

Keywords: accelerated He's polynomial; nonlinear delay partial differential equation; homotopy perturbation method; Sumudu transforms

1 Introduction

The model including delay differential equations may display physical frameworks for which the advancement rely upon the present and past circumstances. This type of model is found in the area of epidemiology and population dynamics, where the delay is due to the gesture or maturity period, or is in numerical control, where there is a delay in taking care of the controller input circle. The partial differential equation (PDE) with proportionate delay is a particular case of a delay differential equation which emerge uniquely in the field of medicine, population ecology, control frameworks, biology, and climate models [1]. Different authors have adopted different numerical techniques like Sarkar et.al [2] use homotopy perturbation method (HPM) to obtain the solution of time-fractional non-linear partial differential equation (PDE) with proportional delay. Chen and Wang [3] use variational iteration method (VIM) while Biazar and Ghanbari [4] apply HPM for solving neutral-functional differential equation with proportionate delay. Singh and Kumar [5] use fractional variational iteration method (VIM) for the solution of fractional PDE with proportional delay. Abazari and Ganji [6], Abazari and Kilicman [7] use differential transform method (DTM) to solve delay (PDE) partial differential equation. Many researchers use various techniques for the solution of such nonlinear PDE's like HPM [8–11], homotopy perturbation transformation method (HPTM) [12–16], homotopy analysis method (HAM) [17], homotopy perturbation Sumudu transformation method (HPSTM) [18–20], homotopy perturbation Elzaki transformation method HPETM [21, 22], Variational iteration method [23] etc. Khan [12] introduce HPTM which is blend of HPM and Laplace transformation for solving nonlinear equations using He's polynomial. Here, we have used a new form of He's polynomial called accelerated He's polynomial [24] which accelerates the rate of convergence of the method. So, we propose a new form of semi-analytic technique named as AHPSTM (Accelerated homotopy perturbation Sumudu transformation method) to study the following type of PDE with proportional delay.

$$\begin{aligned} w_t(x,\ t) &= F(w(p_1 x,\ q_1 t),\ w_x(p_2 x,\ q_2 t), \\ & w_{xx}(p_3 x,\ q_3 t),\ \ldots),\ w(x,0) = g(x) \end{aligned} \tag{1}$$

$p_i,\ q_j \in (0,1),\ i,j \in \mathbb{N}$, $g(x)$ is the initial condition and F is the partial differential operator.

Definition 1.1: The Sumudu transformation over the set of functions

$$A = \{f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_j}},$$

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$$if\ t \in (-1)^j \times [0, \infty),\ j = 1, 2\},$$

is defined as $S[f(t)] = \frac{1}{u} \int_0^\infty f(t)\ e^{-\frac{t}{u}} dt,\ t > 0.$

Properties of Sumudu Transform

1. $S\{1\} = 1$
2. $S\left\{\frac{t^m}{m!}\right\} = u^m$,
3. $S\{f^n(t)\} = \frac{1}{u^n} S\{f(t)\} - \sum_{k=0}^{n-1} \frac{1}{u^{n-k}} f^k(0).$ (*)

2 Accelerated homotopy perturbation Sumudu transform method (AHPSTM)

To elucidate the proposed technique, consider the following nonlinear equations.

$$\frac{\partial^n \phi}{\partial t^n} + L\phi(x, t) + N\phi(x, t) = G(x, t) \tag{2}$$

with condition $\phi^i(x, 0) = k_i(x), i = 0, 1, 2, \ldots n - 1.$

On applying Sumudu transformation to Eq. (2) , we have

$$S\left\{\frac{\partial^n \phi}{\partial t^n} + L\phi(x, t) + N\phi(x, t)\right\} = S\{G(x, t)\} \tag{3}$$

Applying properties of Sumudu transformation (*) to Eq. (3), we get

$$S\{\phi(x, t)\} = u^n \sum_{k=0}^{n-1} \frac{1}{u^{n-k}} \phi^k(x, 0)$$
$$+ u^n S\{G(x, t) - \{L\phi(x, t) + N\phi(x, t)\}\} \tag{4}$$

Further, operating inverse Sumudu transformation to Eq. (4) gives,

$$\phi(x, t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} \phi^k(x, 0)$$
$$+ S^{-1}\{u^n S\{G(x, t) - \{L\phi(x, t) + N\phi(x, t)\}\}\} \tag{5}$$

Now, appling homotpoy perturbation method to Eq. (5), we have

$$0 = (1 - p)(\phi(x, t) - \phi(x, 0)) + p\left(\phi(x, t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \phi^k(x, 0)\right)$$
$$-p\left(S^{-1}\{u^n S\{G(x, t) - \{L\phi(x, t) + N\phi(x, t)\}\}\}\right)$$

where $p \in [0, 1]$ is a parameter. Let

$$\phi(x, t) = \sum_{n=0}^{\infty} \phi_n p^n \tag{6}$$

and

$$N\phi(x, t) = \sum_{n=0}^{\infty} \tilde{H}_n p^n \tag{7}$$

where $\tilde{H}_n$ represents accelerated He's polynomial with

$$\tilde{H}_n(\phi_0, \phi_1, \phi_2 \ldots, \phi_n) = N(S_n) - \sum_{i=0}^{n-1} \tilde{H}_i \tag{8}$$

with $\tilde{H}_0 = N(\phi(x_0))$, and $S_k = (\phi_0 + \phi_1 + \phi_2 + \ldots + \phi_k)$. Using (6), (7) and (8) the Eq. (5) gives,

$$\sum_{n=0}^{\infty} \phi_n p^n = \phi(x, 0) + p\left\{\sum_{k=1}^{n-1} \frac{t^k}{k!} \phi^k(x, 0)\right.$$
$$\left. S^{-1}\left\{+u^n S\left\{G(x, t) - \left\{L \sum_{n=0}^{\infty} \phi_n p^n + \sum_{n=0}^{\infty} \tilde{H}_n p^n\right\}\right\}\right\}\right\} \tag{9}$$

Now, on comparing coefficients of the like power of p, we have

$$p^0 : \phi_0 = \phi(x, 0)$$

$$p^1 : \phi_1 =$$
$$= \sum_{k=1}^{n-1} \frac{t^k}{k!} \phi^k(x, 0) + S^{-1}\{u^n S\{G(x, t) - \{L\phi_0 + \tilde{H}_0\}\}\}$$

$$p^2 : \phi_2 = -S^{-1}\{u^n S\{L\phi_1 + \tilde{H}_1\}\}$$

$$\vdots$$

Hence, the solution of Eq.(1) is obtained by taking $p \to 1$, i.e.

$$\phi(x, t) = \sum_{n=0}^{\infty} \phi_n \tag{10}$$

3 Condition of convergence of accelerated HPSTM

Here, we emphasize on condition of convergence of the above introduced method.

Theorem 3.1: Let ϕ and ϕ_n be elements of a Banach space, then the series

$$\phi(x, t) = \sum_{n=0}^{\infty} \phi_n p^n$$

converges to the solution of Eq. (1) if $\|\phi_{n+1}\| \le \kappa \|\phi_n\|$, where $0 < \kappa < 1$.This conditions of convergence of the series is proved in [15, 20].

4 Application

Example 4.1: Solution of generalized Burgers' equation with proportional delay.

Consider the following initial value problem [6]

$$\frac{\partial\psi(x,t)}{\partial t} = \psi_{xx}(x,t) + \psi_x\left(x,\frac{t}{2}\right)\psi\left(\frac{x}{2},\frac{t}{2}\right) + \frac{1}{2}\psi(x,t),$$
$$t > 0,\ x \in R \tag{11}$$

with $\psi(x,0) = x$. By applying Sumudu transformation on Eq. (11), we have

$$S\left\{\frac{\partial\psi(x,t)}{\partial t} - \frac{1}{2}\psi(x,t)\right\} = S\left\{\psi_{xx}(x,t) + \psi_x\left(x,\frac{t}{2}\right)\psi\left(\frac{x}{2},\frac{t}{2}\right)\right\} \tag{12}$$

Or

$$S\{\psi(x,t)\} = x\left(\frac{2}{2-u}\right) + \frac{2u}{2-u}\left(S\left\{\psi_{xx}(x,t) + \psi_x\left(x,\frac{t}{2}\right)\psi\left(\frac{x}{2},\frac{t}{2}\right)\right\}\right) \tag{13}$$

By applying inverse Sumudu transformation on Eq.(13), we have

$$\psi(x,t) = xe^{\frac{t}{2}} + S^{-1}\left\{\frac{2u}{2-u}\left(S\left\{\psi_{xx}(x,t) + \psi_x\left(x,\frac{t}{2}\right)\psi\left(\frac{x}{2},\frac{t}{2}\right)\right\}\right)\right\} \tag{14}$$

Now, apply AHPSTM on Eq. (14), we get

$$\sum_{n=0}^{\infty}\psi_n(x,t)p^n = xe^{\frac{t}{2}} + p\,S^{-1}\left\{\frac{2u}{2-u}\left(S\left\{\sum_{n=0}^{\infty}(p^n\psi_n(x,t))_{xx} + \sum_{n=0}^{\infty}p^n\tilde{H}_n\right\}\right)\right\} \tag{15}$$

and the initial couple of terms of $\tilde{H}_n$ are given as

$$\tilde{H}_0 = \psi_{0x}\left(x,\frac{t}{2}\right)\psi_0\left(\frac{x}{2},\frac{t}{2}\right),$$

$$\tilde{H}_1 = \psi_{0x}\left(x,\frac{t}{2}\right)\psi_1\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{1x}\left(x,\frac{t}{2}\right)\psi_0\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{1x}\left(x,\frac{t}{2}\right)\psi_1\left(\frac{x}{2},\frac{t}{2}\right),$$

$$\tilde{H}_2 = \psi_{0x}\left(x,\frac{t}{2}\right)\psi_2\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{1x}\left(x,\frac{t}{2}\right)\psi_2\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{2x}\left(x,\frac{t}{2}\right)\psi_0\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{2x}\left(x,\frac{t}{2}\right)\psi_2\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{2x}\left(x,\frac{t}{2}\right)\psi_1\left(\frac{x}{2},\frac{t}{2}\right),$$

$$\tilde{H}_3 = \psi_{0x}\left(x,\frac{t}{2}\right)\psi_3\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{1x}\left(x,\frac{t}{2}\right)\psi_3\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{2x}\left(x,\frac{t}{2}\right)\psi_3\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{3x}\left(x,\frac{t}{2}\right)\psi_3\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{3x}\left(x,\frac{t}{2}\right)\psi_2\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{3x}\left(x,\frac{t}{2}\right)\psi_1\left(\frac{x}{2},\frac{t}{2}\right),$$

$$\vdots$$

On looking at the like powers of p of Eq.(15), we get

$$p^0 : \psi_0 = xe^{\frac{t}{2}};$$

$$p^1 : \psi_1 = \left(\frac{t}{2}\right)xe^{\frac{t}{2}};$$

$$p^2 : \psi_2 = xe^{\frac{t}{2}}\left(\left(\frac{t^2}{2^2 2!}\right) + \frac{1}{2}\left(\frac{t^3}{2^3 3!}\right)\right);$$

$$p^3 : \psi_3 = xe^{\frac{t}{2}}\left(\frac{1}{2}\left(\frac{t^3}{2^3 3!}\right) + \frac{7}{8}\left(\frac{t^4}{2^4 4!}\right) + \frac{5}{8}\left(\frac{t^5}{2^5 5!}\right) + \frac{5}{16}\left(\frac{t^6}{2^6 6!}\right) + \frac{5}{64}\left(\frac{t^7}{2^7 7!}\right)\right);$$

$$p^4 : \psi_4 = x\,e^{\frac{t}{2}}\left(\frac{1}{8}\left(\frac{t^4}{2^4 4!}\right) + \frac{23}{64}\left(\frac{t^5}{2^5 5!}\right) + \frac{5}{8}\left(\frac{t^6}{2^6 6!}\right) + \frac{395}{512}\left(\frac{t^7}{2^7 7!}\right) + \frac{2455}{4096}\left(\frac{t^8}{2^8 8!}\right) + \ldots\right) \tag{16}$$

$$\vdots$$

As $p \to 1$, we get the series solution of Eq. (11) as

$$\psi(x,t) = \sum_{n=0}^{\infty}\psi_n(x,t)$$

Using Eq. (16), we get

$$\psi(x,t) = xe^{\frac{t}{2}}\left(1 + \left(\frac{t}{2}\right) + \left(\frac{t^2}{2^2 2!}\right) + \left(\frac{t^3}{2^3 3!}\right) + \left(\frac{t^4}{2^4 4!}\right) + \frac{63}{64}\left(\frac{t^5}{2^5 5!}\right) + \frac{15}{16}\left(\frac{t^6}{2^6 6!}\right) + \frac{435}{512}\left(\frac{t^7}{2^7 7!}\right) + \ldots\right) \tag{17}$$

The exact solution of Eq. (11) in closed form is

$$\psi(x,t) = xe^t \tag{18}$$

On comparing (18) and (17), we find that the series solution rapidly converges to the actual solution. So,

Table 1: Approximate solution of Eq. (11) using AHPSTM

x	t	ψ_2	ψ_3	ψ_4
	0.25	0.002259289	0.000048675	0.000000387068
0.25	0.5	0.010449426	0.00046536	0.00000754081
	0.75	0.027174522	0.00187525	0.0000464816
	1	0.055816085	0.00530269	0.000178857
	0.25	0.004518577	0.0000973499	0.000000774136
0.5	0.5	0.020898851	0.000930721	0.0000150816
	0.75	0.054349045	0.003750499	0.0000929631
	1	0.111632169	0.01060538	0.000357713
	0.25	0.006777866	0.000146025	0.0000011612
0.75	0.5	0.031348277	0.001396081	0.0000226224
	0.75	0.081523567	0.005625749	0.000139445
	1	0.167448254	0.015908069	0.00053657

we discover that the accelerated homotopy perturbation Sumudu transformation method provides us the faster rate of convergence which can be seen in table 1 that the value of ψ_n (APSTM) decreases rapidly.
Example 4.2: Solution of non-linear PDE with proportional delay,

Consider the following initial value problem [6]

$$\frac{\partial\psi(x,t)}{\partial t} = \psi_{xx}\left(x,\frac{t}{2}\right)\ \psi_x\left(x,\frac{t}{2}\right) - \psi(x,t),$$
$$t>0,\ x\in R \tag{19}$$

with $\psi(x,0)=x^2$. By applying Sumudu transformation on Eq. (19), we have

$$S\left\{\frac{\partial\psi(x,t)}{\partial t}+\psi(x,t)\right\} = S\left\{\psi_{xx}\left(x,\frac{t}{2}\right)\ \psi_x\left(x,\frac{t}{2}\right)\right\} \tag{20}$$

Or

$$S\{\psi(x,t)\} = x^2\left(\frac{1}{1+u}\right)+\frac{u}{1+u}\left(S\left\{\psi_{xx}\left(x,\frac{t}{2}\right)\ \psi_x\left(x,\frac{t}{2}\right)\right\}\right) \tag{21}$$

By applying inverse Sumudu transformation on Eq. (21), we have

$$\psi(x,t) = x^2e^{-t}+S^{-1}\left\{\frac{u}{1+u}\left(S\left\{\psi_{xx}\left(x,\frac{t}{2}\right)\ \psi_x\left(x,\frac{t}{2}\right)\right\}\right)\right\} \tag{22}$$

Now, apply AHPSTM on Eq.(22), we get

$$\sum_{n=0}^{\infty}\psi_n(x,t)p^n = x^2e^{-t}+p\,S^{-1}\left\{\frac{u}{1+u}\left(S\left\{\sum_{n=0}^{\infty}p^n\tilde{H}_n\right\}\right)\right\} \tag{23}$$

and the initial couple of terms of $\tilde{H}_n$ are given as

$$\tilde{H}_0 = \psi_{0xx}\left(x,\frac{t}{2}\right)\psi_0\left(x,\frac{t}{2}\right),$$

$$\tilde{H}_1 = \psi_{0xx}\left(x,\frac{t}{2}\right)\psi_1\left(x,\frac{t}{2}\right)+\psi_{1xx}\left(x,\frac{t}{2}\right)\psi_0\left(x,\frac{t}{2}\right) + \psi_{1xx}\left(x,\frac{t}{2}\right)\psi_1\left(x,\frac{t}{2}\right),$$

$$\tilde{H}_2 = \psi_{0xx}\left(x,\frac{t}{2}\right)\psi_2\left(x,\frac{t}{2}\right)+\psi_{1xx}\left(x,\frac{t}{2}\right)\psi_2\left(x,\frac{t}{2}\right) + \psi_{2xx}\left(x,\frac{t}{2}\right)\psi_2\left(x,\frac{t}{2}\right)+\psi_{2xx}\left(x,\frac{t}{2}\right)\psi_0\left(x,\frac{t}{2}\right) + \psi_{2xx}\left(x,\frac{t}{2}\right)\psi_1\left(x,\frac{t}{2}\right),$$

$$\tilde{H}_3 = \psi_{0xx}\left(x,\frac{t}{2}\right)\psi_3\left(x,\frac{t}{2}\right)+\psi_{1xx}\left(x,\frac{t}{2}\right)\psi_3\left(x,\frac{t}{2}\right) + \psi_{2xx}\left(x,\frac{t}{2}\right)\psi_3\left(x,\frac{t}{2}\right)+\psi_{3xx}\left(x,\frac{t}{2}\right)\psi_3\left(x,\frac{t}{2}\right) + \psi_{3xx}\left(x,\frac{t}{2}\right)\psi_2\left(x,\frac{t}{2}\right)+\psi_{3xx}\left(x,\frac{t}{2}\right)\psi_1\left(x,\frac{t}{2}\right),$$

$$\vdots$$

On looking at the like powers of p of Eq. (23), we get

$$p^0 : \psi_0 = x^2e^{-t};$$
$$p^1 : \psi_1 = x^2e^{-t}(2t);$$
$$p^2 : \psi_2 = x^2e^{-t}\left(\left(\frac{2^2t^2}{2!}\right)+\frac{1}{2}\left(\frac{2^3t^3}{3!}\right)\right);$$
$$p^3 : \psi_3 = x^2e^{-t}\left(\frac{1}{2}\left(\frac{2^3t^3}{3!}\right)+\frac{7}{8}\left(\frac{2^4t^4}{4!}\right) + \frac{5}{8}\left(\frac{2^5t^5}{5!}\right)+\frac{5}{16}\left(\frac{2^6t^6}{6!}\right)+\frac{5}{64}\left(\frac{2^7t^7}{7!}\right)\right);$$

$$p^4 : \psi_4 = x^2e^{-t}\left(\frac{1}{8}\left(\frac{2^4t^4}{4!}\right)+\frac{23}{64}\left(\frac{2^5t^5}{5!}\right) + \frac{5}{8}\left(\frac{2^6t^6}{6!}\right)+\frac{395}{512}\left(\frac{2^7t^7}{7!}\right)+\frac{2455}{4096}\left(\frac{2^8t^8}{8!}\right)+\dots\right) \tag{24}$$

$$\vdots$$

As $p\to 1$, we get the series solution of Eq. (19) as

$$\psi(x,t)=\sum_{n=0}^{\infty}\psi_n(x,t)$$

Table 2: Approximate solution of Eq. (11) up to 4^{th} order approximation

x	t	Exact solution	AHPSTM	DTM [6]	HPM [2]	Abs. error (AHPSTM)	Abs. error (DTM)	Abs. error (HPM)
	0.25	0.321006	3.21E-01	3.21E-01	0.321004	1.22E-09	2.12E-06	2.12E-06
	0.5	0.412180	0.412180	4.12E-01	0.412109	4.83E-08	7.09E-05	7.09E-05
	0.75	0.529250	0.529249	5.29E-01	0.528687	4.52E-07	5.63E-04	5.63E-04
0.25	1	0.679570	0.679568	0.677083	0.677083	2.35E-06	2.49E-03	2.49E-03
	0.25	0.642012	6.42E-01	6.42E-01	0.642008	2.40E-09	4.24E-06	4.24E-06
	0.5	0.824360	0.824360	8.24E-01	0.824219	9.66E-08	1.42E-04	1.42E-04
	0.75	1.058500	1.058499	1.06E+00	1.057373	9.03E-07	1.13E-03	1.13E-03
0.5	1	1.359140	1.359136	1.354167	1.354167	4.70E-06	4.97E-03	4.97E-03
	0.25	0.963019	0.963019	9.63E-01	0.963013	3.70E-09	6.36E-06	6.36E-06
	0.5	1.236541	1.236541	1.24E+00	1.236328	1.45E-07	2.13E-04	2.13E-04
	0.75	1.587750	1.587749	1.58606	1.58606	1.36E-06	1.69E-03	1.69E-03
0.75	1	2.038711	2.038704	2.03125	2.03125	7.05E-06	7.46E-03	7.46E-03

Table 3: Approximate solution of Eq.(19) using AHPSTM

x	t	ψ_2	ψ_3	ψ_4
	0.25	0.006591413	0.000626203	0.0000211215
0.25	0.5	0.022113097	0.004755562	0.000350289
	0.75	0.041516592	0.015073801	0.001831803
	1	0.06131324	0.033256958	0.005952034
	0.25	0.026365652	0.002504813	0.0000844858
0.5	0.5	0.088452388	0.019022246	0.001401156
	0.75	0.166066366	0.060295204	0.007327211
	1	0.245252961	0.133027834	0.023808135
	0.25	0.059322716	0.00563583	0.000190093
0.75	0.5	0.199017873	0.042800054	0.003152601
	0.75	0.373649324	0.135664209	0.016486225
	1	0.551819162	0.299312626	0.053568305

Using Eq. (16), we get

$$\psi(x,t) = x^2 e^{-t}\left(1 + (2t) + \left(\frac{2^2t^2}{2!}\right) + \left(\frac{2^3t^3}{3!}\right) + \left(\frac{2^4t^4}{4!}\right) + \frac{63}{64}\left(\frac{2^5t^5}{5!}\right) + \frac{15}{16}\left(\frac{2^6t^6}{6!}\right) + \frac{435}{512}\left(\frac{2^7t^7}{7!}\right) + \dots\right) \tag{25}$$

The exact solution of Eq. (11) in closed form is

$$\psi(x,t) = x^2 e^t \tag{26}$$

On comparing (26) and (25), it is clear that the series approaches to the exact solution. Also from the table 3, we find that $\|\psi_4\| < \|\psi_3\| < \|\psi_2\|$, i.e. the series solution satisfy the condition of convergence.

Example 4.3: Consider the following initial value problem [6]

$$\frac{\partial\psi(x,t)}{\partial t} = \psi_{xx}\left(\frac{x}{2},\frac{t}{2}\right)\psi_x\left(\frac{x}{2},\frac{t}{2}\right) - \psi_x(x,t) - \psi(x,t),$$

$$t > 0,\ x \in R \tag{27}$$

with $\psi(x,0) = x^2$.

On operating Sumudu transformation on Eq. (27), we have

$$S\left\{\frac{\partial\psi(x,t)}{\partial t} + \psi(x,t)\right\} = S\left\{\psi_{xx}\left(\frac{x}{2},\frac{t}{2}\right)\psi_x\left(\frac{x}{2},\frac{t}{2}\right) - \psi_x(x,t)\right\} \tag{28}$$

Or

$$S\{\psi(x,t)\} = x^2\left(\frac{1}{1+u}\right) + \frac{u}{1+u}\left(S\left\{\psi_{xx}\left(\frac{x}{2},\frac{t}{2}\right)\psi_x\left(\frac{x}{2},\frac{t}{2}\right) - \psi_x(x,t)\right\}\right) \tag{29}$$

By applying inverse Sumudu transformation on Eq. (29), we have

$$\psi(x,t) = x^2e^{-t} + S^{-1}\left\{\frac{u}{1+u}\left(S\left\{\psi_{xx}\left(\frac{x}{2},\frac{t}{2}\right)\psi_x\left(\frac{x}{2},\frac{t}{2}\right) - \psi_x(x,t)\right\}\right)\right\} \tag{30}$$

Now, apply AHPSTM on Eq. (30), we get

$$\sum_{n=0}^{\infty}\psi_n(x,t)p^n = x^2e^{-t} + p\,S^{-1}\left\{\frac{u}{1+u}\left(S\left\{\sum_{n=0}^{\infty}p^n\tilde{H}_n - \psi_x(x,t)\right\}\right)\right\} \tag{31}$$

and the initial couple of terms of $\tilde{H}_n$ are given as

$$\tilde{H}_0 = \psi_{0x}\left(\frac{x}{2},\frac{t}{2}\right)\psi_{0xx}\left(\frac{x}{2},\frac{t}{2}\right),$$

Table 4: Approximate solution of Eq.(19) up to 4^{th} order approximation

x	t	Exact solution	AHPSTM	DTM[6]	HPM [2]	Abs. Error (AHPSTM)	Abs. Error (DTM)	Abs. Error (HPM)
	0.25	0.080251	8.03E-02	8.03E-02	0.080251	2.77E-07	5.30E-07	5.30E-07
	0.5	0.103045	0.103035	1.03E-01	0.103027	9.80E-06	1.77E-05	1.77E-05
	0.75	0.132312	0.132229	1.32E-01	0.132172	8.30E-05	1.41E-04	1.41E-04
0.25	1	0.169893	0.169499	0.169270	0.169271	3.93E-04	6.22E-04	6.22E-04
	0.25	0.321006	3.21E-01	3.21E-01	0.321004	1.11E-06	2.12E-06	2.12E-06
	0.5	0.412180	0.412141	4.12E-01	0.412109	3.92E-05	7.09E-05	7.09E-05
	0.75	0.529250	0.528917	5.29E-01	0.528687	3.32E-04	5.63E-04	5.63E-04
0.5	1	0.679570	0.677998	0.677083	0.677083	1.57E-03	2.49E-03	2.49E-03
	0.25	0.722264	0.722261	7.22E-01	0.72226	2.50E-06	4.78E-06	4.78E-06
	0.5	0.927405	0.927317	9.27E-01	0.927246	8.82E-05	1.60E-04	1.60E-04
	0.75	1.190812	1.190065	1.189545	1.189545	7.47E-04	1.27E-03	1.27E-03
0.75	1	1.529033	1.525497	1.523437	1.523438	3.54E-03	5.60E-03	5.60E-03

$$\tilde{H}_1 = \psi_{0x}\left(\frac{x}{2},\frac{t}{2}\right)\psi_{1xx}\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{1x}\left(\frac{x}{2},\frac{t}{2}\right)\psi_{0xx}\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{1x}\left(\frac{x}{2},\frac{t}{2}\right)\psi_{1xx}\left(\frac{x}{2},\frac{t}{2}\right),$$

$$\tilde{H}_2 = \psi_{0x}\left(\frac{x}{2},\frac{t}{2}\right)\psi_{2xx}\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{1x}\left(\frac{x}{2},\frac{t}{2}\right)\psi_{2xx}\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{2x}\left(\frac{x}{2},\frac{t}{2}\right)\psi_{2xx}\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{2x}\left(\frac{x}{2},\frac{t}{2}\right)\psi_{0xx}\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{2x}\left(\frac{x}{2},\frac{t}{2}\right)\psi_{1xx}\left(\frac{x}{2},\frac{t}{2}\right),$$

$$\tilde{H}_3 = \psi_{0x}\left(\frac{x}{2},\frac{t}{2}\right)\psi_{3xx}\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{1x}\left(\frac{x}{2},\frac{t}{2}\right)\psi_{3xx}\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{2x}\left(\frac{x}{2},\frac{t}{2}\right)\psi_{3xx}\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{3x}\left(\frac{x}{2},\frac{t}{2}\right)\psi_{3xx}\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{3x}\left(\frac{x}{2},\frac{t}{2}\right)\psi_{2xx}\left(\frac{x}{2},\frac{t}{2}\right) + \psi_{3x}\left(\frac{x}{2},\frac{t}{2}\right)\psi_{1xx}\left(\frac{x}{2},\frac{t}{2}\right),$$

$$\vdots$$

On looking at the like powers of p of Eq. (31), we get

$$p^0 : \psi_0 = x^2 e^{-t};$$
$$p^1 : \psi_1 = 0,$$
$$p^2 : \psi_2 = 0,$$
$$p^3 : \psi_3 = 0,$$
$$\vdots$$

Hence, the solution of Eq. (27) is given by

$$\psi(x,t) = \sum_{n=0}^{\infty} \psi_n(x,t)$$

i.e.

$$\psi(x,t) = x^2 e^{-t} \tag{32}$$

Also, the exact solution of Eq. (27) is in the closed form is

$$\psi(x,t) = x^2 e^{-t} \tag{33}$$

So from Eq.(32) and Eq.(33),we have found this exact solution in only one iteration.

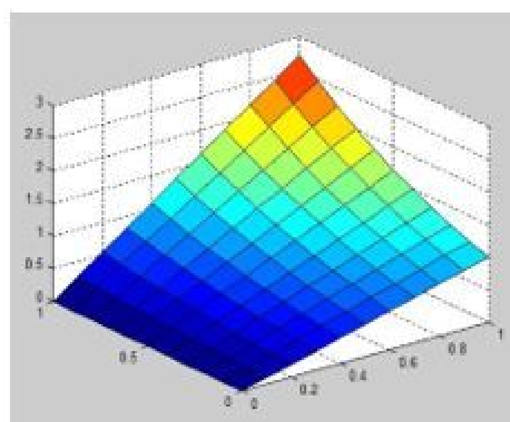

Figure 1: Approximate solution of Eq. (11) using AHPSTM

5 Statistical analysis

In order to validate the solution obtained from the semi-analytic technique AHPSTM, and to investigate the techniques (AHPSTM, HPM and DTM) for their outcome in regard of solution of non-linear problem considered in Eq.

Table 5: Approximate solution of Eq.(19) up to 4^{th} order approximation

x	t	Exact solution	AHPSTM	DTM [6]	HPM [2]	Abs. Error (AHPSTM)	Abs. Error (DTM)	Abs. Error (HPM)
	0.25	0.04868	4.87E-02	4.87E-02	0.0486755	0	4.88E-07	4.88E-07
	0.5	0.03791	0.0379081	3.79E-02	0.0379231	0	1.50E-05	1.50E-05
	0.75	0.02952	0.0295229	2.96E-02	0.0296325	0	1.10E-04	1.10E-04
0.25	1	0.02299	0.0229924	0.023437	0.0234375	0	4.45E-04	4.45E-04
	0.25	0.19470	1.95E-01	1.95E-01	0.1947021	0	1.95E-06	1.95E-06
	0.5	0.15163	0.1516326	1.52E-01	0.1516927	0	6.00E-05	6.00E-05
	0.75	0.11809	0.1180916	1.19E-01	0.1185302	0	4.39E-04	4.39E-04
0.5	1	0.09196	0.0919698	0.09375	0.09375	0	1.78E-03	1.78E-03
	0.25	0.43807	0.4380754	4.38E-01	0.4380798	0	4.39E-06	4.39E-06
	0.5	0.34117	0.3411734	3.41E-01	0.3413085	0	1.35E-04	1.35E-04
	0.75	0.26570	0.2657061	0.266693	0.2666931	0	9.87E-04	9.87E-04
0.75	1	0.20693	0.2069321	0.210937	0.2109375	0	4.01E-03	4.01E-03

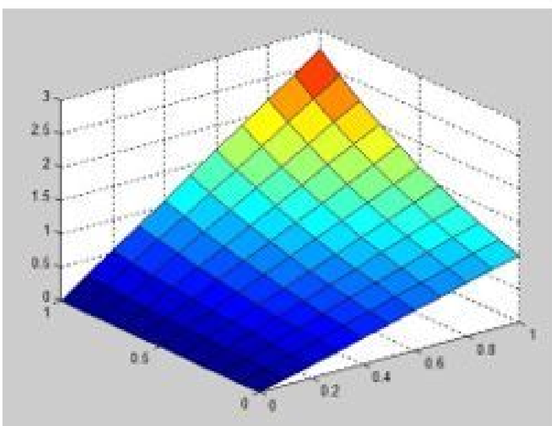

Figure 2: Exact solution

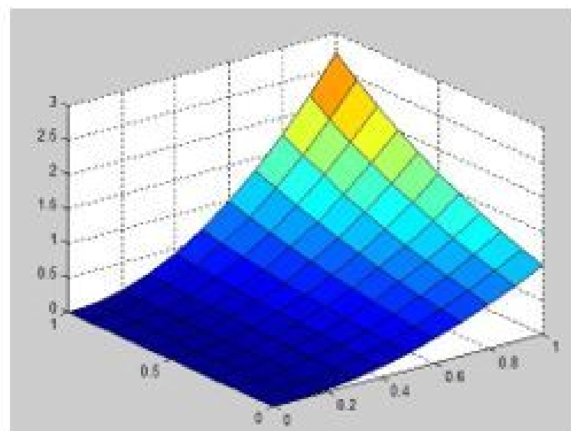

Figure 4: Exact solution

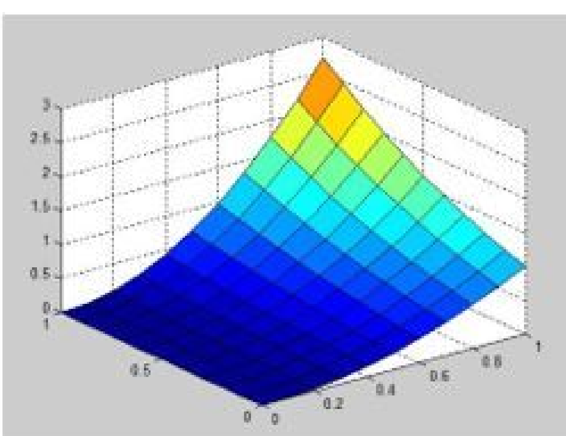

Figure 3: Approximate solution of Eq. (11) using AHPSTM

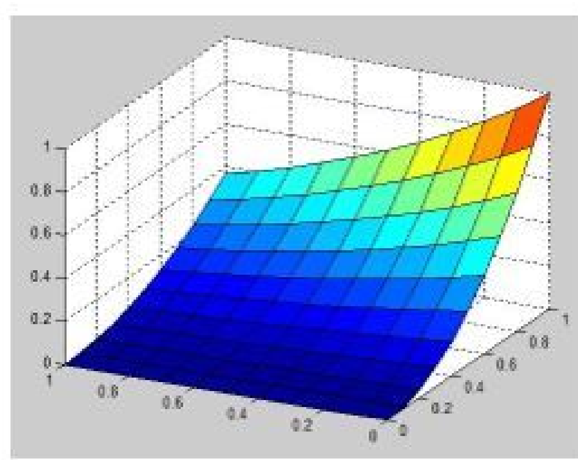

Figure 5: Approximate solution of Eq. (27) using AHPSTM

(11), (19) and (28) we employ a statistical technique i.e. paired student's t-test at 5% level of significance to the data of Tables 2, 4 and 6. The null hypothesis is

Null hypothesis: $H_0^A : \mu_1^A = \mu_{2j}^A$, $H_0^B : \mu_1^B = \mu_{2j}^B$, $H_0^C : \mu_1^C = \mu_{2j}^C$,

where μ_1^k, ; $k = A; B; C$denotes the exact solution of (11), (19) and (28) respectively, while μ_{2j}^k;$k = A; B; C;\ j =$

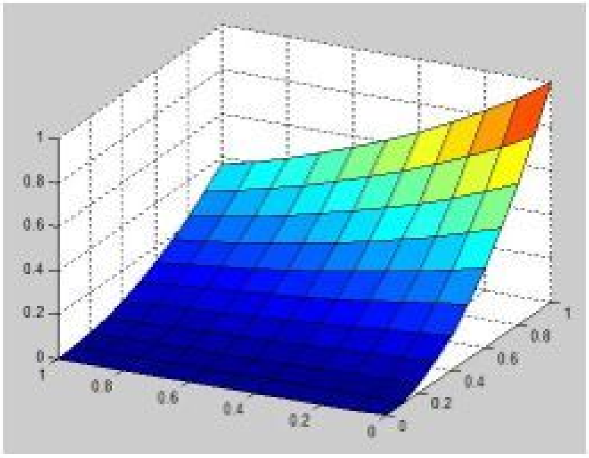

Figure 6: Exact solution

1; 2; 3 denote the approximate solution of Eq. (11), (19) and (27) via AHPSTM, DTM and HPM, respectively. The considered degree of freedom is $n_k - 1 = 12 - 1 = 11$ and the tabulated value of $t$at $\alpha = 5\%$ is $|t_{tab.}| = 2.201$. The calculated values of test statistic of Eq. (11), (19) and (27) for pair AHPSTM with exact solution A_i; DTM with exact solution B_i; and HPM with exact solution C_i, $i = 1, 2, 3$ are given below:

$$|t_{cal.}(A_1)| = 2.192,$$
$$|t_{cal.}(B_1)| = 2.282,$$
$$|t_{cal.}(C_1)| = 2.282,$$
$$|t_{cal.}(A_2)| = 1.884,$$
$$|t_{cal.}(B_2)| = 1.914,$$
$$|t_{cal.}(C_2)| = 1.914,$$
$$|t_{cal.}(B_3)| = 1.954,$$
$$|t_{cal.}(C_3)| = 1.954.$$

From the above analysis, it is clear that null hypothesis H_0 is accepted for Eq. (11) only for pair of AHPSTM solution and exact solution, and is rejected for DTM with pair of exact solution and HPM with exact solution. For Eq. (19) and (27), null Hypothesis is accepted in all the three cases (Note: For Eq. (27), as we get exact solution with AHPSTM , so we do not test statistically). Hence, with this statistical analysis we conclude that AHPSTM gives better solution than other semi analytical technique like DTM and HPM.

6 Conclusion

A new semi analytic technique of accelerated AHPSTM is implemented for the approximate analytical solution of non-linear partial differential equations with proportional delay. It provides the power series solution in the form of a rapidly convergent series. The proposed technique converges faster than other semi- analytic techniques like HPM,VIM and DTM. To validate the efficiency and reliability of the proposed technique, the condition of convergence is verified and statistical analysis is performed. Tables 1 and 3 show that the series solution obtained from the proposed method satisfied condition of convergence, while Tables 2,4,5 and Figures 1, 2, and 3 show that the approximate results are close to the exact solution of the considered models with given initial conditions. Also, the approximate solution obtained from AHPTM gives better result with just four iterations than other methods like HPM, VIM and DTM. The proposed method gives a better result for the solution of non-linear PDEs as no discritizing algorithm and no linearization is required for non-linear problems. Further, it is concluded that with the proposed techniques only few iterations will lead to the solution and hence it reduces the computational cost. Thus, this technique is equally competent for linear and non-linear partial differential equation.

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