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A reliable analytical technique for fractional Caudrey-Dodd-Gibbon equation with Mittag-Leffler kernel

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Abstract: The pivotal aim of the present work is to find the solution for fractional Caudrey-Dodd-Gibbon (CDG) equation using q-homotopy analysis transform method (q-HATM). The considered technique is graceful amalgamations of Laplace transform technique with q-homotopy analysis scheme, and fractional derivative defined with Atangana-Baleanu (AB) operator. The fixed point hypothesis considered in order to demonstrate the existence and uniqueness of the obtained solution for the projected fractional-order model. In order to illustrate and validate the efficiency of the future technique, we analysed the projected model in terms of fractional order. Moreover, the physical behaviour of q-HATM solutions have been captured in terms of plots for diverse fractional order and the numerical simulation is also demonstrated. The obtained results elucidate that, the considered algorithm is easy to implement, highly methodical as well as accurate and very effective to examine the nature of nonlinear differential equations of arbitrary order arisen in the connected areas of science and engineering.

Keywords: Laplace transform; Atangana-Baleanu derivative; Caudrey-Dodd-Gibbon equation; q-homotopy analysis method; fixed point theorem

1 Introduction

Fractional calculus (FC) was originated in Newton's time, but lately, it fascinated the attention of many scholars. From the last thirty years, the most intriguing leaps in scientific and engineering applications have been found within the framework of FC. The concept of the fractional derivative has been industrialized due to the complexities associated with a heterogeneities phenomenon. The fractional differential operators are capable to capture the behaviour of multifaceted media having diffusion process. It has been a very essential tool and many problems can be illustrated more conveniently and more accurately with differential equations having arbitrary order. Due to the swift development of mathematical techniques with computer software's, many researchers started to work on generalised calculus to present their viewpoints while analysing many complex phenomena.

Numerous pioneering directions are prescribed for the diverse definitions of fractional calculus by many senior researchers, and which prearranged the foundation [1–6]. Calculus with fractional order is associated to practical ventures and it extensively employed to nanotechnology [7], human diseases [8, 9], chaos theory [10], and other areas [11–34]. The numerical and analytical solution for these equations illustrating these models have an impartment role in portraying nature of nonlinear problems ascends in connected areas of science.

In order to demonstrate the efficiency of the future scheme, we consider fifth-order nonlinear CGD equation of the form [35, 36]

$$u_t + u_{xxxxx} + 30uu_{xxx} + 30u_xu_{xx} + 180u^2u_x = 0. \qquad (1)$$

The above equation is a class of KdV equation and further, it possesses distinct and diverse properties. The CGD equation is also familiar as Sawada-Kotera equation [37]. Due to the importance of the considered problem, it has been magnetized the attention of many researchers from diverse areas. In 1984, Weiss illustrated the Painleve' property for the Eq. (1) [38]. It has been proved that it has a strong physical background in fluid [39] and also has N-soliton solutions [40].

In the present scenario, many important and nonlinear models are methodically and effectively analysed with the help of fractional calculus. There have been diverse definitions are suggested by many senior research scholars, for instance, Riemann, Liouville, Caputo and Fabrizio.

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However, these definitions have their own limitations. The Riemann–Liouville derivative is unable to explain the importance of the initial conditions; the Caputo derivative has overcome this shortcoming but is impotent to explain the singular kernel of the phenomena. Later, in 2015 Caputo and Fabrizio defeated the above obliges [41], and many researchers are considered this derivative in order to analyse and find the solution for diverse classes of nonlinear complex problems. But some issues were pointed out in CF derivative, like non-singular kernel and nonlocal, these properties are very essential in describing the physical behaviour and nature of the nonlinear problems. In 2016, Atangana and Baleanu introduced and natured the novel fractional derivative, namely AB derivative. This novel derivative defined with the aid of Mittag–Leffler functions [42]. This fractional derivative buried all the above-cited issues and helps us to understand the natural phenomena systematically and effectively.

In the present framework, we consider the fractional Caudrey-Dodd-Gibbon (FCDG) equation of the form

$$
{}_{a}^{ABC}D_t^{\alpha}u\,(x,t) + u_{xxxxx} + 30uu_{xxx} + 30u_xu_{xx} + 180u^2u_x = 0,\quad 0<\alpha\le 0=1, \tag{2}
$$

where α is fractional-order and defined with AB fractional operator. The fractional-order is introduced in order to incorporate the memory effects and hereditary consequence in the phenomenon and these properties aid us to capture essential physical properties of the nonlinear problems.

Recently, many mathematicians and physicists developed very effective and more accurate methods in order to find and analyse the solution for complex and nonlinear problems arisen in science and engineering. In connection with this, the homotopy analysis method (HAM) proposed by Chinese mathematician Liao Shijun [43]. HAM has been profitably and effectively applied to study the behaviour of nonlinear problems without perturbation or linearization. But, for computational work, HAM requires huge time and computer memory. To overcome this, there is an essence of the amalgamation of a considered method with well-known transform techniques.

In the present investigation, we put an effort to find and analysed the behaviour of the solution obtained for the FCDG equation by applying q-HATM. The future algorithm is the combination of q-HAM with LT [44]. Since q-HATM is an improved scheme of HAM; it does not require discretization, perturbation or linearization. Recently, due to its reliability and efficacy, the considered method is exceptionally applied by many researchers to understand physical behaviour diverse classes of complex problems [45–53]. The projected method offers us more freedom to consider the diverse class of initial guess and the equation type complex as well as nonlinear problems; because of this, the complex NDEs can be directly solved. The novelty of the future method is it aids a modest algorithm to evaluate the solution and it natured by the homotopy and axillary parameters, which provides the rapid convergence in the obtained solution for a nonlinear portion of the given problem. Meanwhile, it has prodigious generality because it plausibly contains the results obtained by many algorithms like q-HAM, HPM, ADM and some other traditional techniques. The considered method can preserve great accuracy while decreasing the computational time and work in comparison with other methods. The considered nonlinear problem recently fascinated the attention of researchers from different areas of science. Since FCDG equation plays a significant role in portraying several nonlinear phenomena and also which are the generalizations of diverse complex phenomena, many authors find and analysed the solution using analytical as well as numerical schemes [54–61].

2 Preliminaries

Recently, many authors considered these derivatives to analyse a diverse class of models in comparison with classical order as well as other fractional derivatives, and they prove that AB derivative is more effective while analysing the nature and physical behaviour of the models [62–65]. Here, we define the basic notion of Atangana-Baleanu derivatives and integrals [42].

Definition 1. The fractional Atangana-Baleanu-Caputo derivative for a function $f \in H^1(a,\ b)\,(\,b>a,\ \ \alpha\in[0,\ 1])$is presented as follows

$$
{}_{a}^{ABC}D_t^{\alpha}\,(f\,(t)) = \frac{B\,[\alpha]}{1-\alpha}\int_a^t f'\,(\vartheta)\,E_\alpha\left[\alpha\frac{(t-\vartheta)^\alpha}{\alpha-1}\right]d\vartheta. \tag{3}
$$

where $B\,[\alpha]$ is a normalization function such that $B(0) = B(1) = 1$.

Definition 2. The AB derivative of fractional order for a function $f\in H^1(a,\ b)\,,\ b>a,\ \ \alpha\in[0,\ 1]$ in Riemann-Liouville sense presented as follows

$$
{}_{a}^{ABR}D_t^{\alpha}\,(f\,(t)) = \frac{B\,[\alpha]}{1-\alpha}\frac{d}{dt}\int_a^t f\,(\vartheta)\,E_\alpha\left[\alpha\frac{(t-\vartheta)^\alpha}{\alpha-1}\right]d\vartheta. \tag{4}
$$

Definition 3. The fractional AB integral related to the non-local kernel is defined by

$$ {}_{a}^{AB}I_{t}^{\alpha}\left(f\left(t\right)\right)=\frac{1-\alpha}{B\left[\alpha\right]}f\left(t\right)+\frac{\alpha}{B\left[\alpha\right]\Gamma\left(\alpha\right)}\int_{a}^{t}f\left(\vartheta\right)\left(t-\vartheta\right)^{\alpha-1}d\vartheta \tag{5}$$

Definition 4. The Laplace transform (LT) of AB derivative is defined by

$$L\left[{}_{0}^{ABR}D_{t}^{\alpha}\left(f\left(t\right)\right)\right]=\frac{B\left[\alpha\right]}{1-\alpha}\frac{s^{\alpha}L\left[f\left(t\right)\right]-s^{\alpha-1}f\left(0\right)}{s^{\alpha}+\left(\alpha/(1-\alpha)\right)} \tag{6}$$

Theorem 1. The following Lipschitz conditions respectively hold true for both Riemann-Liouville and AB derivatives defined in Eqs. (3) and (4) [42],

$$\left\|{}_{a}^{ABC}D_{t}^{\alpha}f_{1}\left(t\right)-{}_{a}^{ABC}D_{t}^{\alpha}f_{2}\left(t\right)\right\|<K_{1}\left\|f_{1}\left(x\right)-f_{2}\left(x\right)\right\|, \tag{7}$$

and

$$\left\|{}_{a}^{ABC}D_{t}^{\alpha}f_{1}\left(t\right)-{}_{a}^{ABC}D_{t}^{\alpha}f_{2}\left(t\right)\right\|<K_{2}\left\|f_{1}\left(x\right)-f_{2}\left(x\right)\right\|. \tag{8}$$

Theorem 2. The time-fractional differential equation ${}_{a}^{ABC}D_{t}^{\alpha}f_{1}\left(t\right)=s\left(t\right)$ has a unique solution and which is defined as [42]

$$f\left(t\right)=\frac{1-\alpha}{B\left[\alpha\right]}s\left(t\right)+\frac{\alpha}{B\left[\alpha\right]\Gamma\left(\alpha\right)}\int_{0}^{t}s\left(\varsigma\right)\left(t-\varsigma\right)^{\alpha-1}d\varsigma. \tag{9}$$

3 Fundamental idea of the considered scheme

In this segment, we consider the arbitrary order differential equation in order to demonstrate the fundamental solution procedure of the projected algorithm

$$\begin{aligned}&{}_{a}^{ABC}D_{t}^{\alpha}v\left(x,t\right)+R\,v\left(x,t\right)+N\,v\left(x,t\right)=f\left(x,t\right),\\&n-1<\alpha\leq n\end{aligned} \tag{10}$$

with the initial condition

$$v\left(x,0\right)=g\left(x\right), \tag{11}$$

where ${}_{a}^{ABC}D_{t}^{\alpha}v\left(x,t\right)$ symbolise the AB derivative of $v\left(x,t\right), f\left(x,t\right)$ signifies the source term, R and N respectively denotes the linear and nonlinear differential operator. On using the LT on Eq. (10), we have after simplification

$$\begin{aligned}&L\left[v\left(x,t\right)\right]-\frac{g\left(x\right)}{s}+\frac{1}{B\left[\alpha\right]}\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)\{L\left[Rv\left(x,t\right)\right]\\&+L\left[N\,v\left(x,t\right)\right]-L\left[f\left(x,t\right)\right]\}=0.\end{aligned} \tag{12}$$

The non-linear operator is defined as follows

$$\begin{aligned}&N\left[\varphi\left(x,t;q\right)\right]=L\left[\varphi\left(x,t;q\right)\right]-\frac{g\left(x\right)}{s}\\&+\frac{1}{B\left[\alpha\right]}\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)\{L\left[R\,\varphi\left(x,t;q\right)\right]+L\left[N\varphi\left(x,t;q\right)\right]\\&-L\left[f\left(x,t\right)\right]\}.\end{aligned} \tag{13}$$

Here, $\varphi(x,t;q)$ is the real-valued function with respect to x, t and $\left(q\in\left[0,\frac{1}{n}\right]\right)$. Now, we define a homotopy as follows

$$\left(1-nq\right)L\left[\varphi\left(x,\ t;q\right)-v_{0}\left(x,t\right)\right]=\hbar qN\left[\varphi\left(x,\ t;q\right)\right], \tag{14}$$

where L is signifying LT, $q\in\left[0,\frac{1}{n}\right]$ $(n\geq1)$ is the embedding parameter and $\hbar/=0$ is an auxiliary parameter. For $q=0$ and $q=\frac{1}{n}$, the results are given below hold true

$$\varphi\left(x,t;0\right)=v_{0}\left(x,t\right),\ \ \varphi\left(x,t;\frac{1}{n}\right)=v\left(x,t\right). \tag{15}$$

Thus, by intensifying q from 0 to $\frac{1}{n}$, the solution $\varphi(x,t;q)$ varies from $v_0\left(x,t\right)$ to $v\left(x,t\right)$. By using the Taylor theorem near to q, we defining $\varphi\left(x,t;q\right)$ in series form and then we get

$$\varphi\left(x,t;q\right)=v_{0}\left(x,t\right)+\sum_{m=1}^{\infty}v_{m}\left(x,t\right)q^{m}, \tag{16}$$

where

$$v_{m}\left(x,t\right)=\frac{1}{m!}\frac{\partial^{m}\varphi(x,\ t;q)}{\partial q^{m}}|_{q=0}. \tag{17}$$

The series (14) converges at $q=\frac{1}{n}$ for the proper chaise of $v_0\left(x,t\right)$, n and $\hbar$. Then

$$v\left(x,t\right)=v_{0}\left(x,t\right)+\sum_{m=1}^{\infty}v_{m}\left(x,t\right)\left(\frac{1}{n}\right)^{m}. \tag{18}$$

Now, m-times differentiating Eq. (15) with q and later dividing by $m!$ and then putting $q=0$, we obtain

$$L[v_{m}\left(x,t\right)-k_{m}v_{m-1}\left(x,t\right)]=\hbar R_{m}\left(\vec{v}_{m-1}\right), \tag{19}$$

where the vectors are defined as

$$\vec{v}_{m}=\{v_{0}\left(x,t\right),v_{1}\left(x,t\right),\ldots,v_{m}\left(x,t\right)\}. \tag{20}$$

On applying inverse LT on Eq. (19), one can get

$$v_{m}\left(x,t\right)=k_{m}v_{m-1}\left(x,t\right)+\hbar L^{-1}\left[R_{m}\left(\vec{v}_{m-1}\right)\right], \tag{21}$$

where

$$R_{m}\left(\vec{v}_{m-1}\right)=L\left[v_{m-1}\left(x,t\right)\right]-\left(1-\frac{k_{m}}{n}\right)$$

$$\left(\frac{g(x)}{s}+\frac{1}{B[\alpha]}\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L[f(x,t)]\right)$$
$$+\frac{1}{B[\alpha]}\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L[Rv_{m-1}+H_{m-1}], \tag{22}$$

and

$$k_m=\begin{cases}0, & m\leq 1,\\ n, & m>1.\end{cases} \tag{23}$$

In Eq. (22), H_m signifies homotopy polynomial and presented as follows

$$H_m=\frac{1}{m!}\left[\frac{\partial^m\varphi(x,\ t;q)}{\partial q^m}\right]_{q=0} \tag{24}$$

and $\varphi(x,\ t;q)=\varphi_0+q\varphi_1+q^2\varphi_2+\ldots$.

By the aid of Eqs. (21) and (22), one can get

$$v_m(x,t)=(k_m+\hbar)v_{m-1}(x,t)-\left(1-\frac{k_m}{n}\right)$$
$$L^{-1}\left(\frac{g(x)}{s}+\frac{1}{B[\alpha]}\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L[f(x,t)]\right)$$
$$+\hbar L^{-1}\left\{\frac{1}{B[\alpha]}\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L[Rv_{m-1}+H_{m-1}]\right\}. \tag{25}$$

Using the Eq. (25), one can get the series of $v_m(x,t)$. Lastly, the series q-HATM solution is defined as

$$v(x,t)=v_0(x,t)+\sum_{m=1}^{\infty}v_m(x,t)\left(\frac{1}{n}\right)^m. \tag{26}$$

4 Solution for FCDG equation

In order to present the solution procedure and efficiency of the future scheme, in this segment, we consider FCDG equation of fractional order. Further by the help of obtained results, we made an attempt to capture the behaviour of q-HATM solution for different fractional order. By the help of Eq. (2), we have

$$ {}_{a}^{ABC}D_t^{\alpha}u(x,t)+u_{xxxxx}+30uu_{xxx}+30u_xu_{xx}+180u^2u_x=0,$$
$$0<\alpha\leq 1, \tag{27}$$

with initial condition

$$u(x,0)=\mu^2 sech^2(\mu x). \tag{28}$$

Taking LT on Eq. (27) and then using the Eq. (28), we get

$$L[u(x,t)]=\frac{1}{s}\left(\mu^2 sech^2(\mu x)\right)+\frac{1}{B[\alpha]}\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)$$
$$L\{\frac{\partial^5 u}{\partial x^5}+30u\frac{\partial^3 u}{\partial x^3}+30\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+180u^2\frac{\partial u}{\partial x}\}. \tag{29}$$

The non-linear operator N is presented with the help of future algorithm as below

$$N[\varphi(x,t;q)]=L[\varphi(x,t;q)]-\frac{1}{s}\left(\mu^2 sech^2(\mu x)\right)$$
$$+\frac{1}{B[\alpha]}\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L\{\frac{\partial^5\varphi}{\partial x^5}+30\varphi\frac{\partial^3\varphi}{\partial x^3}+30\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}$$
$$+180\varphi^2\frac{\partial\varphi}{\partial x}\}. \tag{30}$$

The deformation equation of m-th order by the help of q-HATM at $H(x,t)=1$, is given as follows

$$L[u_m(x,t)-k_m u_{m-1}(x,t)]=\hbar R_{1,m}\left[\vec{u}_{m-1},\ \vec{v}_{m-1}\right], \tag{31}$$

where

$$R_m\left[\vec{u}_{m-1}\right]=L[u_{m-1}(x,t)]-\left(1-\frac{k_m}{n}\right)$$
$$\left\{\frac{1}{s}\left(\mu^2 sech^2(\mu x)\right)\right\}+\frac{1}{B[\alpha]}\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)$$
$$L\{\frac{\partial^5 u_{m-1}}{\partial x^5}+30\sum_{i=0}^{m-1}u_i\frac{\partial^3 u_{m-1-i}}{\partial x^3}+30\sum_{i=0}^{m-1}\frac{\partial u_i}{\partial x}\frac{\partial^2 u_{m-1-i}}{\partial x^2}$$
$$+\sum_{j}^{i}\sum_{i=0}^{m-1}u_ju_{i-j}\frac{\partial u_{m-1-i}}{\partial x}\}. \tag{32}$$

On applying inverse LT on Eq. (31), it reduces to

$$u_m(x,t)=k_m u_{m-1}(x,t)+\hbar L^{-1}\left\{R_m\left[\vec{u}_{m-1}\right]\right\}. \tag{33}$$

On simplifying the above equation systematically by using $u_0(x,t)=\frac{1}{s}\left(\mu^2 sech^2(\mu x)\right)$ we can evaluate the terms of the series solution

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{34}$$

5 Existence of solutions for the future model

Here, we considered the fixed-point theorem in order to demonstrate the existence of the solution for the considered model. Since the considered model cited in the system (27) is non-local as well as complex; there are no particular algorithms or methods exist to evaluate the exact solutions. However, under some particular conditions the existence of the solution assurances. Now, the system (27) is considered as follows:

$$ {}_{0}^{ABC}D_t^{\alpha}[u(x,t)]=G(x,t,\ u). \tag{35}$$

The foregoing system is transformed to the Volterra integral equation using the Theorem 2, and which as follows

$$u(x,t)-u(x,0)=\frac{(1-\alpha)}{B(\alpha)}G(x,t,u)$$

$$+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t G(x,\zeta,u)(t-\zeta)^{\alpha-1}d\zeta. \tag{36}$$

Theorem 3. The kernel G satisfies the Lipschitz condition and contraction if the condition $0 \le \left(\delta^5+30\delta\left(2\left(a^2+b^2+ab\right)+\delta\right)\right)<1$ holds.

Proof. In order to prove the required result, we consider the two functions u and u_1, then

$$\begin{aligned}
&\|G(x,t,\ u)-G(x,t,\ u_1)\| = \|(\frac{\partial^5}{\partial x^5}[u(x,t)-u(x,t_1)]\\
&+30\left(u(x,t)\frac{\partial^3 u(x,t)}{\partial x^3}-u(x,t_1)\frac{\partial^3 u(x,t_1)}{\partial x^3}\right)\\
&+30\left(\frac{\partial u(x,t)}{\partial x}\frac{\partial^2 u(x,t)}{\partial x^2}-\frac{\partial u(x,t_1)}{\partial x}\frac{\partial^2 u(x,t_1)}{\partial x^2}\right)\\
&+180\left(u^2(x,t)\frac{\partial u(x,t)}{\partial x}-u^2(x,t_1)\frac{\partial u(x,t_1)}{\partial x}\right)\|\\
&\le \left\|\delta^5+30\delta\left(2\left(a^2+b^2+ab\right)+\delta\right)\right\|\,\|u(x,t)-u(x,t_1)\|\\
&\le \left(\delta^5+30\delta\left(2\left(a^2+b^2+ab\right)+\delta\right)\right)\|u(x,t)-u(x,t_1)\|,
\end{aligned} \tag{37}$$

where δ is the differential operator. Since u and u_1 are bounded, we have $\|u(x,t)\| \le a$ and $\|u(x,t_1)\| \le b$. Putting $\eta = \delta^5+30\delta\left(2\left(a^2+b^2+ab\right)+\delta\right)$ in the above inequality, then we have

$$\|G(x,t,u)-G(x,t,u_1)\| \le \eta\|u(x,t)-u(x,t_1)\|. \tag{38}$$

This gives, the Lipschitz condition is obtained for G_1. Further, we can see that if $0 \le \left(\delta^5+30\delta\left(2\left(a^2+b^2+ab\right)+\delta\right)\right) < 1$, then it implies the contraction. The recursive form of Eq. (36) defined as follows

$$\begin{aligned}
u_n(x,\ t) &= \frac{(1-\alpha)}{B(\alpha)}G(x,t,u_{n-1})\\
&+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t G(x,\zeta,u_{n-1})(t-\zeta)^{\alpha-1}d\zeta.
\end{aligned} \tag{39}$$

The associated initial condition is

$$u(x,0)=u_0(x,t). \tag{40}$$

The successive difference between the terms is presented as follows

$$\begin{aligned}
\phi_n(x,t) &= u_n(x,t)-u_{n-1}(x,t)\\
&=\frac{(1-\alpha)}{B(\alpha)}(G(x,t,u_{n-1})-G(x,t,u_{n-2}))\\
&+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t G(x,\zeta,u_{n-1})(t-\zeta)^{\alpha-1}d\zeta.
\end{aligned} \tag{41}$$

Notice that

$$u_n(x,t)=\sum_{i=1}^n \phi_{1i}(x,t). \tag{42}$$

By using Eq. (38) after applying the norm on the Eq. (41), one can get

$$\begin{aligned}
\|\phi_n(x,t)\| &\le \frac{(1-\alpha)}{B(\alpha)}\eta\|\phi_{(n-1)}(x,t)\|\\
&+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\eta\int_0^t \|\phi_{(n-1)}(x,\zeta)\|d\zeta.
\end{aligned} \tag{43}$$

We prove the following theorem by using the above result.

Theorem 4. The solution for the system (27) will exist and unique if we have specific t_0 then

$$\frac{(1-\alpha)}{B(\alpha)}\eta+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\eta<1.$$

Proof. Let us consider the bounded function $u(x,t)$ satisfying the Lipschitz condition. Then, by Eqs. (42) and (44), we have

$$\|\phi_i(x,t)\| \le \|u_n(x,0)\|\left[\frac{(1-\alpha)}{B(\alpha)}\eta+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\eta\right]^n. \tag{44}$$

Therefore, the continuity as well as existence for the obtained solutions is proved. Subsequently, in order to show the Eq. (44) is a solution for the Eq. (29), we consider

$$u(x,t)-u(x,0)=u_n(x,t)-K_n(x,t). \tag{45}$$

In order to obtain require a result, we consider

$$\begin{aligned}
\|K_n(x,t)\| &= \|\frac{(1-\alpha)}{B(\alpha)}\left(G(x,t,u)-G(x,t,\ u_{n-1})\right)\\
&+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-\zeta)^{\mu-1}(G(x,\zeta,u)-G(x,\zeta,u_{n-1}))d\zeta\|\\
&\le \frac{(1-\alpha)}{B(\alpha)}\|(G(x,t,u)-G(x,t,\ u_{n-1}))\|\\
&+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t \|(G(x,\zeta,u)-G(x,\zeta,u_{n-1}))\|d\zeta\\
&\le \frac{(1-\alpha)}{B(\alpha)}\eta\|u-u_{n-1}\|+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\eta\|u-u_{n-1}\|t.
\end{aligned} \tag{46}$$

Similarly, at t_0 we can obtain

$$\|K_n(x,t)\| \le \left(\frac{(1-\alpha)}{B(\alpha)}+\frac{\alpha\, t_0}{B(\alpha)\Gamma(\alpha)}\right)^{n+1}\eta^{n+1}M. \tag{47}$$

As n approaches to ∞, we can see that form Eq. (50), $\|K_n(x,t)\|$ tends to 0.

Next, it is a necessity to demonstrate uniqueness for the solution of the considered model. Suppose $u^{*}(x,t)$ be the other solution, then we have

$$u(x,t)-u^{*}(x,t)=\frac{(1-\alpha)}{B(\alpha)}\left(G(x,t,u)-G\left(x,t,u^{*}\right)\right)$$
$$+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{0}^{t}\left(G(x,\zeta,u)-G\left(x,\zeta,u^{*}\right)\right)d\zeta. \tag{48}$$

On applying norm, the Eq. (48) simplifies to

$$\left\|u(x,t)-u^{*}(x,t)\right\|=\left\|\frac{(1-\alpha)}{B(\alpha)}\left(G(x,t,u)-G\left(x,t,u^{*}\right)\right)\right.$$
$$\left.+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{0}^{t}\left(G(x,\zeta,u)-G\left(x,\zeta,u^{*}\right)\right)d\zeta\right\|$$
$$\le\frac{(1-\alpha)}{B(\alpha)}\eta\left\|u(x,t)-u^{*}(x,t)\right\|$$
$$+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\eta t\left\|u(x,t)-u^{*}(x,t)\right\|. \tag{49}$$

On simplification

$$\left\|u(x,t)-u^{*}(x,t)\right\|\left(1-\frac{(1-\alpha)}{B(\alpha)}\eta-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\eta t\right)\le 0. \tag{50}$$

From the above condition, it is clear that $u(x,t)-u^{*}(x,t)$, if

$$\left(1-\frac{(1-\alpha)}{B(\alpha)}\eta-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\eta t\right)\ge 0. \tag{51}$$

Hence, Eq. (51) evidences our essential result.

Theorem 5. Suppose $u_n(x,t)$ and $u(x,t)$ define in the Banach space $(B[0,\ T],\ \|\cdot\|)$. Then series solution defined in Eq. (26) converges to the solution of the Eq. (10), if $0<\lambda_1<1$.

Proof: Let consider the sequence $\{S_n\}$ and which is the partial sum of the Eq. (26), we have to prove $\{S_n\}$ is Cauchy sequence in $(B[0,\ T],\ \|\cdot\|)$. Now consider

$$\begin{aligned}\|S_{n+1}(x,t)-S_n(x,t)\| &= \|u_{n+1}(x,t)\|\\ &\le \lambda_1\|u_n(x,t)\|\\ &\le {\lambda_1}^2\|u_{n-1}(x,t)\|\\ &\le \ldots\\ &\le {\lambda_1}^{n+1}\|u_0(x,t)\|.\end{aligned}$$

Now, we have for every $n,\ m\in N\ (m\le n)$

$$\begin{aligned}\|S_n-S_m\| &= \|(S_n-S_{n-1})+(S_{n-1}-S_{n-2})+\ldots\\ &+(S_{m+1}-S_m)\|\le\|S_n-S_{n-1}\|+\|S_{n-1}-S_{n-2}\|+\ldots\\ &+\|S_{m+1}-S_m\|\le\left({\lambda_1}^{n}+{\lambda_1}^{n-1}+\ldots+{\lambda_1}^{m+1}\right)\|u_0\|\\ &\le {\lambda_1}^{m+1}\left({\lambda_1}^{n-m-1}+{\lambda_1}^{n-m-2}+\ldots+\lambda_1+1\right)\|u_0\|\\ &\le {\lambda_1}^{m+1}\left(\frac{1-{\lambda_1}^{n-m}}{1-\lambda_1}\right)\|u_0\|.\end{aligned} \tag{52}$$

But $0<\lambda_1<1$, therefore $\lim_{n,\ m\to\infty}\|S_n-S_m\|=0$. Hence, $\{S_n\}$is the Cauchy sequence. Similarly, we can demonstrate for the second case. This proves the required result.

Theorem 6. For the series solution (26) of the Eq. (10), the maximum absolute error is presented as

$$\left\|u(x,t)-\sum_{n=0}^{M}u_n(x,t)\right\|\le\frac{{\lambda_1}^{M+1}}{1-\lambda_1}\|u_0(x,t)\|.$$

Proof: By the help of Eq. (56), we get

$$\|u(x,t)-S_n\|=\lambda_1^{m+1}\left(\frac{1-{\lambda_1}^{n-m}}{1-\lambda_1}\right)\|u_0(x,t)\|.$$

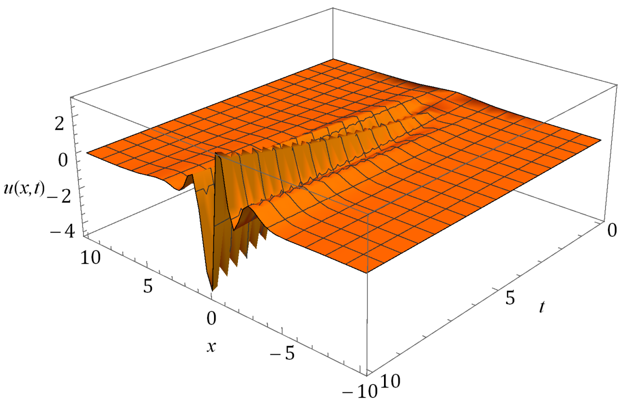
But $0<\lambda_1<0\Rightarrow 1-{\lambda_1}^{n-m}<1$. Hence, we have

$$\left\|u(x,t)-\sum_{n=0}^{M}u_n(x,t)\right\|\le\frac{{\lambda_1}^{M+1}}{1-\lambda_1}\|u_0(x,t)\|.$$

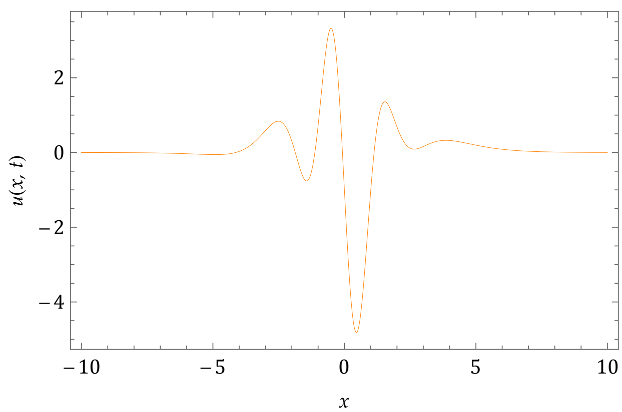
This ends the proof.

6 Results and discussion

In this manuscript, we find the solution for CDG equation having arbitrary order using a novel scheme namely, q-HATM with the help of Mittag-Leffler law. In the present segment, we demonstrate the effect of fractional order in the obtained solution with distinct parameters offered by the future method. In Figures 1 to 3, the nature of q-HATM solution for different arbitrary order is presented in terms of 2D plots. From these plots, we can see that considered problem conspicuously depends on fractional order. In order to analyse the behaviour of obtained solution with respect to homotopy parameter $(\hbar)$, the $\hbar$-curves are drowned for diverse μ and presented in Figure 4. In the plots, the horizontal line represents the convergence region of the q-HATM solution and these curves aid us to adjust and handle the convergence province of the solution. For an appropriate value of $\hbar$, the achieved solution quickly converges to the exact solution. Further, the small variation in the physical behaviour of the complex models stimulates the enormous new results to analyse and understand nature in a better and systematic manner. Moreover, from all the plots we can see that the considered method is more accurate and very effective to analyse the considered complex coupled fractional order equations.

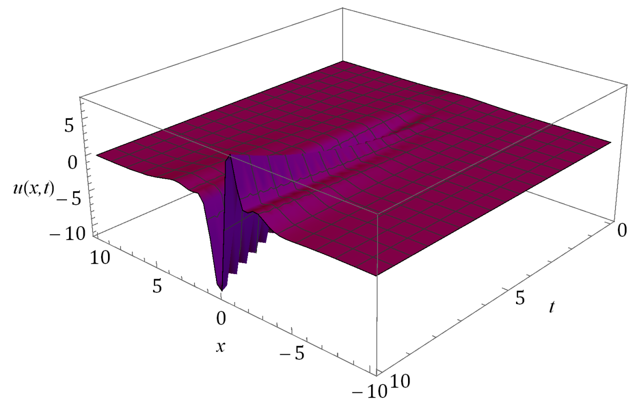


(a)

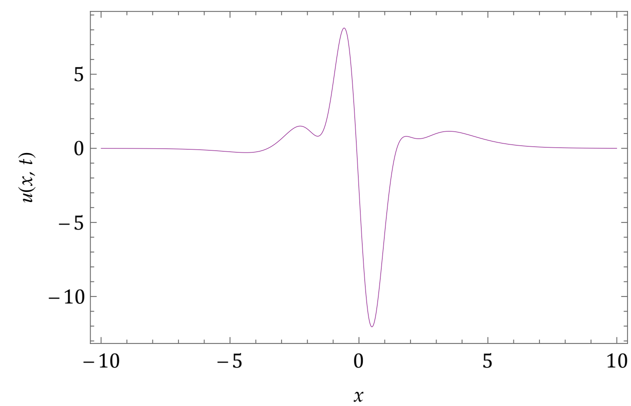


(b)

Figure 1: (a) Surface of $u\,(x,\,t)$, (b) 2D plot of $u\,(x,\,t)$ at $t\;=\;10$ at $\mu = 0.5,\;\hbar = -1,\;n = 1$ and $\alpha = 0.5$.



(a)



(b)

Figure 2: (a) Surface of $u\,(x,\,t)$, (b) 2D plot of $u\,(x,\,t)$ at $t\;=\;10$ at $\mu = 0.5,\;\hbar = -1,\;n = 1$ and $\alpha = 0.75$.

7 Conclusion

In this study, the q-HATM is applied lucratively to find the solution for arbitrary order CDG equations. Since AB derivatives and integrals having fractional order are defined with the help of generalized Mittag-Leffler function as the non-singular and non-local kernel, the present investigation illuminates the effeteness of the considered derivative. The existence and uniqueness of the obtained solution are demonstrated with the fixed point hypothesis. The results obtained by the future scheme are more stimulating as compared to results available in the literature. Further, the projected algorithm finds the solution for the nonlinear problem without considering any discretization, perturbation or transformations. The present investigation illuminates, the considered nonlinear phenomena noticeably depend on the time history and the time instant and which can be proficiently analysed by applying the concept of calculus with fractional order. The present investigation helps the researchers to study the behaviour nonlinear problems gives very interesting and useful consequences. Lastly, we can conclude the projected method is extremely methodical, more effective and very accurate, and which can be applied to analyse the diverse classes of nonlinear problems arising in science and technology.

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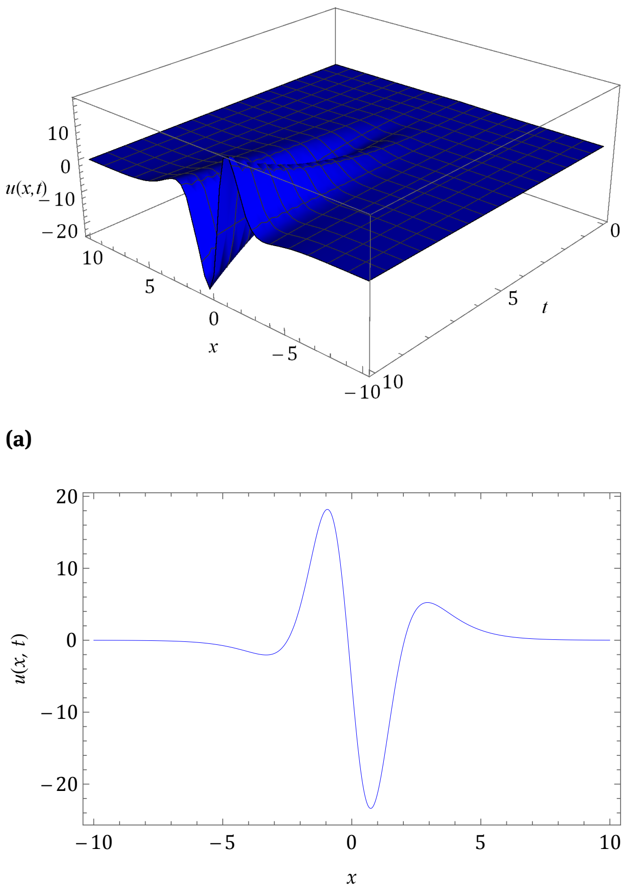
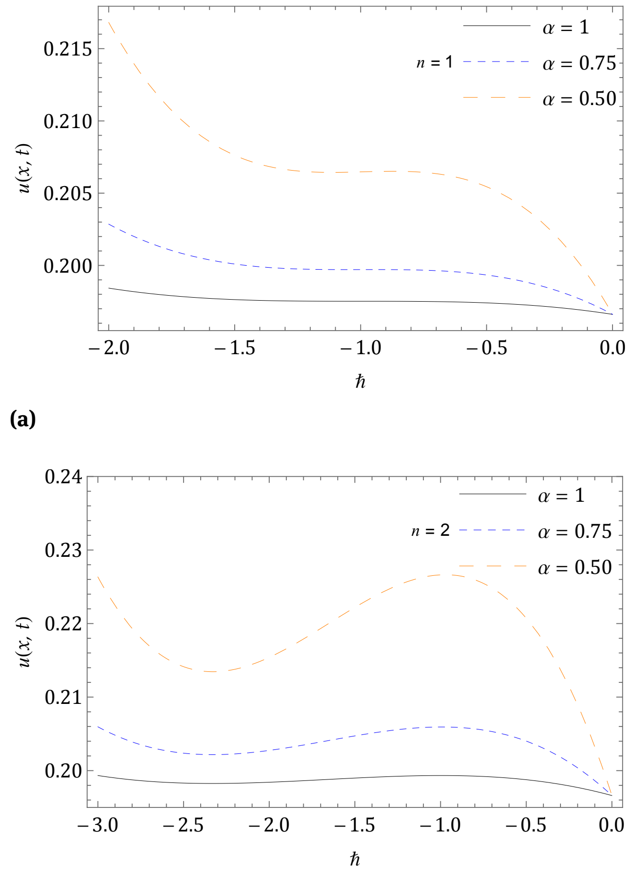


(a)

(b)

Figure 3: (a) Surface of $u\,(x, t)$, (b) 2D plot of $u\,(x, t)$ at $t\ =\ 10$ at $\mu = 0.5,\ \hbar = -1,\ n = 1$ and $\alpha = 1$.



(a)

(b)

Figure 4: $\hbar$-curves for q-HATM solution with distinct α at $x\ =\ 1$ and $t = 0.01$ for $(a)\ n = 1$ and $(b)\ n = 2$.

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