DE GRUYTER Nonlinear Engineering 2020; 9: 265–272

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Qualitative analysis for two fractional difference equations

https://doi.org/10.1515/nleng-2020-0014
Received Sep 4, 2018; accepted Dec 18, 2019.

Abstract: Some difference equations are generally studied by investigating their long behaviours rather than their exact solutions. The proposed equations cannot be solved analytically. Hence, this article discusses the main qualitative behaviours of two rational difference equations. Some appropriate hypotheses are examined and given to show the local and global attractivity. Special cases from the considered equations are solved analytically. The periodicity is also proved in this work. We also illustrate the achieved results in some 2D figures.

Keywords: attractivity, boundedness, difference equations, equilibrium, periodicity, stability

1 Introduction

Difference equations are widely utilized in describing some phenomena occurred in nonlinear sciences such as biology, physics, chemical reactions, economy, probability theory, population growth, genetics, computer science and so on. They are sometimes used to discretise partial and ordinary derivatives appeared in partial and ordinary differential equations, respectively. Such equations have gained their popularity in recent years due to their use in nonlinear problems. It is well known fact that the analytical solutions of most fractional recursive relations cannot be often constructed due to the lack of the relevant mathematical methods. This has prevented some experts from searching the future pattern of most natural situations described by difference equations. Therefore, the majority of researchers have made great efforts to study these phenomena by means of some auxiliary elements such as local stability, global character, periodicity, boundedness and several others. For instance, El-Dessoky et al. [1] investigated the dynamics and periodicity of the fifth order difference equation

$$y_{n+1} = \alpha y_n + \frac{\beta y_n y_{n-3}}{A y_{n-4} + B y_{n-3}}.$$

Almatrafi and Alzubaidi [2] analysed the asymptotic attractivity, periodic nature and boundedness of the eighth order difference equation

$$x_{n+1} = c_1 x_{n-3} + \frac{c_2 x_{n-3}}{c_3 x_{n-3} - c_4 x_{n-7}}.$$

Moreover, Kalabušić et al. [3] studied the global character of the second order equation

$$x_{n+1} = \frac{x_n x_{n-1} + \alpha x_n + \beta x_{n-1}}{a x_n x_{n-1} + b x_{n-1}}.$$

The author in [4] highlighted the semi-cycle behaviour, stability character and solutions for the second order difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

El-Owaidy et al. [5] explored the global stability of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n+1}^p}.$$

In [6], the author utilized Fibonacci sequence to construct the solutions of special type of Riccati difference equations given by

$$x_{n+1} = \frac{1}{1 + x_n}, \quad y_{n+1} = \frac{1}{y_n - 1}.$$

Almatrafi [7] presented forms of exact solutions to the fractional system

$$x_{n+1} = \frac{x_{n-1} y_{n-3}}{y_{n-1}\left(-1 - x_{n-1} y_{n-3}\right)},$$
$$y_{n+1} = \frac{y_{n-1} x_{n-3}}{x_{n-1}\left(\pm 1 \pm y_{n-1} x_{n-3}\right)}, \quad n = 0, 1, \ldots,$$

In order to obtain more qualitative results on difference equations, see refs.[8–28].

The basic task of this work is to point out the local and global stability, periodicity and forms of solutions to the following two recursive equations:

$$u_{m+1} = a u_{m-1} + \frac{b u_{m-1} u_{m-4}}{c u_{m-4} - d u_{m-6}}, \quad m = 0, 1, \ldots,$$

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$$u_{m+1} = au_{m-1} - \frac{bu_{m-1}u_{m-4}}{cu_{m-4} - du_{m-6}}, \quad m = 0, 1, \ldots,$$

where the coefficients a, b, c and d are supposed to be positive real numbers and the initial data u_i *for all* $i = -6, -5, \ldots, 0$, are arbitrary non-zero real numbers.

2 Analysis of first equation

This section is devoted to study the difference equation given by

$$u_{m+1} = au_{m-1} + \frac{bu_{m-1}u_{m-4}}{cu_{m-4} - du_{m-6}}, \quad m = 0, 1, \ldots, \tag{1}$$

where the constants a, b, c and d are assumed to be positive real numbers and the initial conditions u_i *for all* $i = -6, -5, \ldots, 0$, are arbitrary non-zero real numbers.

2.1 Equilibrium and local stability

The discussion in this part concentrates on the local behaviour around the equilibrium point. The equilibrium point is obtained as follows:

$$\overline{u} = a\overline{u} + \frac{b\overline{u}^2}{c\overline{u} - d\overline{u}},$$

which leads to a unique equilibrium point given by $\overline{u} = 0$, if $b \neq (1-a)(c-d)$. In order to linearise about the fixed point, we define a function $g : (0, \infty)^3 \longrightarrow (0, \infty)$ by

$$g(x, y, z) = ax + \frac{bxy}{cy - dz}. \tag{2}$$

Hence,

$$\frac{\partial g(x,y,z)}{\partial x} = a + \frac{by}{(cy - dz)}, \tag{3}$$

$$\frac{\partial g(x,y,z)}{\partial y} = -\frac{bdxz}{(cy - dz)^2}, \tag{4}$$

$$\frac{\partial g(x,y,z)}{\partial z} = \frac{bdxy}{(cy - dz)^2}. \tag{5}$$

Evaluating Eqs. (3), (4) and Eq. (5) at $\bar{u}$ gives us

$$\frac{\partial g(\bar{u},\bar{u},\bar{u})}{\partial x} = a + \frac{b\bar{u}}{(c\bar{u} - d\bar{u})} = a + \frac{b}{c-d} := -q_1,$$

$$\frac{\partial g(\bar{u},\bar{u},\bar{u})}{\partial y} = -\frac{bd\bar{u}^2}{(c\bar{u} - d\bar{u})^2} = -\frac{bd}{(c-d)^2} := -q_2,$$

$$\frac{\partial g(\bar{u},\bar{u},\bar{u})}{\partial z} = \frac{bd\bar{u}^2}{(c\bar{u} - d\bar{u})^2} = \frac{bd}{(c-d)^2} := -q_3.$$

Hence, the linearised equation of Eq. (1) around $\bar{u}$ is shown as follows:

$$U_{n+1} + q_1 U_{n-1} + q_2 U_{n-4} + q_3 U_{n-6} = 0.$$

Theorem 1. *Assume that*

$$\left| a + \frac{b}{c-d} \right| < 1 - \frac{2bd}{(c-d)^2}. \tag{6}$$

Then, the equilibrium point of Eq.(1) is locally asymptotically stable.

Proof. The proof is established according to the hypotheses of Theorem A in [13]. Therefore, the local stability occurs if

$$|q_1| + |q_2| + |q_3| < 1, \tag{7}$$

which implies that

$$\left| -\left(a + \frac{b}{c-d} \right) \right| + \left| \frac{bd}{(c-d)^2} \right| + \left| -\left(\frac{bd}{(c-d)^2} \right) \right| < 1. \tag{8}$$

Inequality (8) can be simply reduced to

$$\left| a + \frac{b}{c-d} \right| < 1 - \frac{2bd}{(c-d)^2},$$

which is the required condition.

2.2 Global attractivity

The global behaviour in the neighbourhood of the equilibrium point is explained in this subsection according to the conditions of Theorem C in [29].

Theorem 2. *Let* $a + \frac{b}{c-d} > 0$. *Then, the equilibrium point of Eq.(1) is a global attractor if* $a \neq 1$.

Proof. Let $r_1, r_2 \in \mathbb{R}$, assume that $g : [r_1, r_2]^3 \longrightarrow [r_1, r_2]$ is a function defined by Eq. (2), and assume that $a + \frac{b}{c-d} > 0$. Then, it can be easily observed from Eqs. (3), (4) and Eq. (5) that g is increasing in x and in z and decreasing in y. Suppose that (ζ, η) is a solution to the following system:

$$\zeta = g(\zeta, \eta, \zeta), \ \eta = g(\eta, \zeta, \eta).$$

Or,

$$\zeta = g(\zeta, \eta, \zeta) = a\zeta + \frac{b\zeta\eta}{c\eta - d\zeta},$$

$$\eta = g(\eta, \zeta, \eta) = a\eta + \frac{b\eta\zeta}{c\zeta - d\eta}.$$

Simplifying this gives

$$c\zeta\eta - d\zeta^2 = ac\zeta\eta - ad\zeta^2 + b\zeta\eta, \tag{9}$$

$$c\zeta\eta - d\eta^2 = ac\zeta\eta - ad\eta^2 + b\zeta\eta. \tag{10}$$

Subtracting Eq.(10) from Eq.(9) yields

$$d(\eta^2 - \zeta^2) = ad(\eta^2 - \zeta^2),$$

Hence, if $a \neq 1$, then $\zeta = \eta$. Consequently, Theorem C [29] concludes that every solution of Eq. (1) converges to $\bar{u}$.

Theorem 3. *Assume that $a + \frac{b}{c-d} < 0$. Then, the equilibrium point of Eq.(1) is a global attractor if $d \neq ac + b$.*

Proof. The proof is similar to the previous proof. Thus, it is omitted.

2.3 Exact solution of $u_{m+1} = u_{m-1} + \frac{u_{m-1}u_{m-4}}{u_{m-4}-u_{m-6}}$

We now begin with presenting the exact solution of the equation

$$u_{m+1} = u_{m-1} + \frac{u_{m-1}u_{m-4}}{u_{m-4} - u_{m-6}}. \tag{11}$$

The coefficients a, b, c and d are positive real numbers and the initial data u_i *for all* $i = -6, -5, \ldots, 0$, are arbitrary non-zero real numbers.

Theorem 4. *Let $\{u_m\}_{m=-6}^{\infty}$ be a solution to Eq. (11) and suppose that $u_{-6} = a,\ u_{-5} = b,\ u_{-4} = c,\ u_{-3} = d,\ u_{-2} = e,\ u_{-1} = r,\ u_0 = k$. Then, for $m = 0, 1, \ldots,$ we have*

$$\begin{aligned}
u_{10m-6} &= \frac{(F_{2m+1}d + F_{2m-1}b)(F_{2m+1}r + F_{2m-1}d)(F_{2m}k + F_{2m-2}e)(F_{2m}c + F_{2m-2}a)(F_{2m}e + F_{2m-2}c)}{ec(d-b)(r-d)},\\
u_{10m-5} &= \frac{(F_{2m+1}k + F_{2m-1}e)(F_{2m+1}e + F_{2m-1}c)(F_{2m+1}c + F_{2m-1}a)(F_{2m}d + F_{2m-2}b)(F_{2m}r + F_{2m-2}d)}{d(c-a)(e-c)(k-e)},\\
u_{10m-4} &= \frac{(F_{2m+1}d + F_{2m-1}b)(F_{2m+1}r + F_{2m-1}d)(F_{2m}k + F_{2m-2}e)(F_{2m}e + F_{2m-2}c)(F_{2m+2}c + F_{2m}a)}{ec(d-b)(r-d)},\\
u_{10m-3} &= \frac{(F_{2m+2}d + F_{2m}b)(F_{2m}r + F_{2m-2}d)(F_{2m+1}k + F_{2m-1}e)(F_{2m+1}e + F_{2m-1}c)(F_{2m+1}c + F_{2m-1}a)}{d(c-a)(e-c)(k-e)},\\
u_{10m-2} &= \frac{(F_{2m+2}e + F_{2m}c)(F_{2m+2}c + F_{2m}a)(F_{2m+1}d + F_{2m-1}b)(F_{2m+1}r + F_{2m-1}d)(F_{2m}k + F_{2m-2}e)}{ec(d-b)(r-d)},\\
u_{10m-1} &= \frac{(F_{2m+2}d + F_{2m}b)(F_{2m+2}r + F_{2m}d)(F_{2m+1}k + F_{2m-1}e)(F_{2m+1}e + F_{2m-1}c)(F_{2m+1}c + F_{2m-1}a)}{d(c-a)(e-c)(k-e)},\\
u_{10m} &= \frac{(F_{2m+2}k + F_{2m}e)(F_{2m+2}e + F_{2m}c)(F_{2m+2}c + F_{2m}a)(F_{2m+1}d + F_{2m-1}b)(F_{2m+1}r + F_{2m-1}d)}{ec(d-b)(r-d)},\\
u_{10m+1} &= \frac{(F_{2m+3}c + F_{2m+1}a)(F_{2m+2}d + F_{2m}b)(F_{2m+2}r + F_{2m}d)(F_{2m+1}k + F_{2m-1}e)(F_{2m+1}e + F_{2m-1}c)}{d(c-a)(e-c)(k-e)},\\
u_{10m+2} &= \frac{(F_{2m+3}d + F_{2m+1}b)(F_{2m+1}r + F_{2m-1}d)(F_{2m+2}k + F_{2m}e)(F_{2m+2}e + F_{2m}c)(F_{2m+2}c + F_{2m}a)}{ec(d-b)(r-d)},\\
u_{10m+3} &= \frac{(F_{2m+2}d + F_{2m}b)(F_{2m+2}r + F_{2m}d)(F_{2m+1}k + F_{2m-1}e)(F_{2m+3}e + F_{2m+1}c)(F_{2m+3}c + F_{2m+1}a)}{d(c-a)(e-c)(k-e)}.
\end{aligned}$$

Here, $\{F_m\}_{m=0}^{\infty} = \{1, 1, 2, 3, 5, 8, \ldots\}$, i.e, $F_m = F_{m-1} + F_{m-2}$, $F_{-1} = 0$ and $F_{-2} = -1$, is called Fibonacci sequence.

Proof. The relations are true for $m = 0$. Now, assume that $m > 0$ and that our assumption holds for $m - 1$. That is,

$$\begin{aligned}
u_{10m-16} &= \frac{(F_{2m-1}d + F_{2m-3}b)(F_{2m-1}r + F_{2m-3}d)(F_{2m-2}k + F_{2m-4}e)(F_{2m-2}c + F_{2m-4}a)(F_{2m-2}e + F_{2m-4}c)}{ec(d-b)(r-d)},\\
u_{10m-15} &= \frac{(F_{2m-1}k + F_{2m-3}e)(F_{2m-1}e + F_{2m-3}c)(F_{2m-1}c + F_{2m-3}a)(F_{2m-2}d + F_{2m-4}b)(F_{2m-2}r + F_{2m-4}d)}{d(c-a)(e-c)(k-e)},\\
u_{10m-14} &= \frac{(F_{2m-1}d + F_{2m-3}b)(F_{2m-1}r + F_{2m-3}d)(F_{2m-2}k + F_{2m-4}e)(F_{2m-2}e + F_{2m-4}c)(F_{2m-2}c + F_{2m-4}a)}{ec(d-b)(r-d)},
\end{aligned}$$

$$\begin{aligned}
u_{10m-13} &= \frac{(F_{2m}d + F_{2m-2}b)(F_{2m-2}r + F_{2m-4}d)(F_{2m-1}k + F_{2m-3}e)(F_{2m-1}e + F_{2m-3}c)(F_{2m-1}c + F_{2m-3}a)}{d(c-a)(e-c)(k-e)},\\
u_{10m-12} &= \frac{(F_{2m}e + F_{2m-2}c)(F_{2m}c + F_{2m-2}a)(F_{2m-1}d + F_{2m-3}b)(F_{2m-1}r + F_{2m-3}d)(F_{2m-2}k + F_{2m-4}e)}{ec(d-b)(r-d)},\\
u_{10m-11} &= \frac{(F_{2m}d + F_{2m-2}b)(F_{2m}r + F_{2m-2}d)(F_{2m-1}k + F_{2m-3}e)(F_{2m-1}e + F_{2m-3}c)(F_{2m-1}c + F_{2m-3}a)}{d(c-a)(e-c)(k-e)},
\end{aligned}$$

$$
\begin{aligned}
u_{10m-10} &= \frac{(F_{2m}k + F_{2m-2}e)(F_{2m}e + F_{2m-2}c)(F_{2m}c + F_{2m-2}a)(F_{2m-1}d + F_{2m-3}b)(F_{2m-1}r + F_{2m-3}d)}{ec(d-b)(r-d)},\\
u_{10m-9} &= \frac{(F_{2m+1}c + F_{2m-1}a)(F_{2m}d + F_{2m-2}b)(F_{2m}r + F_{2m-2}d)(F_{2m-1}k + F_{2m-3}e)(F_{2m-1}e + F_{2m-3}c)}{d(c-a)(e-c)(k-e)},\\
u_{10m-8} &= \frac{(F_{2m+1}d + F_{2m-1}b)(F_{2m-1}r + F_{2m-3}d)(F_{2m}k + F_{2m-2}e)(F_{2m}e + F_{2m-2}c)(F_{2m}c + F_{2m-2}a)}{ec(d-b)(r-d)},\\
u_{10m-7} &= \frac{(F_{2m}d + F_{2m-2}b)(F_{2m}r + F_{2m-2}d)(F_{2m-1}k + F_{2m-3}e)(F_{2m+1}e + F_{2m-1}c)(F_{2m+1}c + F_{2m-1}a)}{d(c-a)(e-c)(k-e)}.
\end{aligned}
$$

Two random relations will be now selected and proved. It can be easily observed from Eq. (11) that

$$
\begin{aligned}
u_{10m-5} &= u_{10m-7} + \frac{y_{10m-7}y_{10m-10}}{y_{10m-10} - y_{10m-12}} = u_{10m-7}\left(1 + \frac{y_{10m-10}}{y_{10m-10} - y_{10m-12}}\right)\\
&= u_{10m-7}\left(1 + \frac{F_{2m}k - F_{2m-2}e}{F_{2m}k - F_{1m-2}e - F_{2m-2}k + F_{2m-4}e}\right)\\
&= u_{10m-7}\left(1 + \frac{F_{2m}k - F_{2m-2}e}{F_{2m-1}k - F_{2m-3}e}\right) = u_{10m-7}\left(\frac{F_{2m+1}k - F_{2m-1}e}{F_{2m-1}k - F_{2m-3}e}\right)\\
&= \tfrac{(F_{2m}d+F_{2m-2}b)(F_{2m}r+F_{2m-2}d)(F_{2m-1}k+F_{2m-3}e)(F_{2m+1}e+F_{2m-1}c)(F_{2m+1}c+F_{2m-1}a)}{d(c-a)(e-c)(k-e)}\left(\frac{F_{2m+1}k - F_{2m-1}e}{F_{2m-1}k - F_{2m-3}e}\right)\\
&= \frac{(F_{2m+1}k + F_{2m-1}e)(F_{2m+1}e + F_{2m-1}c)(F_{2m+1}c + F_{2m-1}a)(F_{2m}d + F_{2m-2}b)(F_{2m}r + F_{2m-2}d)}{d(c-a)(e-c)(k-e)}.
\end{aligned}
$$

We can also see from Eq. (11) that

$$
\begin{aligned}
u_{10m-1} &= u_{10m-3} + \frac{u_{10m-3}u_{10m-6}}{u_{10m-6} - u_{10m-8}} = u_{10m-3}\left(1 + \frac{u_{10m-6}}{u_{10m-6} - u_{10m-8}}\right)\\
&= u_{10m-3}\left(1 + \frac{F_{2m+1}r - F_{2m-1}d}{F_{2m+1}r - F_{1m-1}d - F_{2m-1}r + F_{2m-3}d}\right)\\
&= u_{10m-3}\left(1 + \frac{F_{2m+1}r - F_{2m-1}d}{F_{2m}r - F_{2m-2}d}\right) = u_{10m-3}\left(\frac{F_{2m+2}r - F_{2m}d}{F_{2m}r - F_{2m-2}d}\right)\\
&= \tfrac{(F_{2m+2}d+F_{2m}b)(F_{2m}r+F_{2m-2}d)(F_{2m+1}k+F_{2m-1}e)(F_{2m+1}e+F_{2m-1}c)(F_{2m+1}c+F_{2m-1}a)}{d(c-a)(e-c)(k-e)}\left(\frac{F_{2m+2}r - F_{2m}d}{F_{2m}r - F_{2m-2}d}\right)\\
&= \frac{(F_{2m+2}d + F_{2m}b)(F_{2m+2}r + F_{2m}d)(F_{2m+1}k + F_{2m-1}e)(F_{2m+1}e + F_{2m-1}c)(F_{2m+1}c + F_{2m-1}a)}{d(c-a)(e-c)(k-e)}.
\end{aligned}
$$

Similarly, other relations can be proved.

3 Analysis of second equation

In this section, we are interested in analysing the periodicity of the recursive equation

$$
u_{m+1} = u_{m-1} - \frac{u_{m-1}u_{m-4}}{u_{m-4} - u_{m-6}}, \quad m = 0, 1, \dots, \tag{12}
$$

where the constants a, b, c and d are positive real numbers and the initial data u_i *for all* $i = -6, -5, \dots, 0$, are arbitrary non-zero real numbers.

Theorem 5. *Let $\{u_m\}_{m=-6}^{\infty}$ be a solution to Eq. (12) where $u_k \neq u_{k-2}$ for all $k = -4, -3, -2, \ldots$. Then, every solution of Eq. (12) is periodic with period 60. Furthermore, $\{u_m\}_{m=-6}^{\infty}$ takes the form*

$$\left\{\begin{array}{c}
a,\ b,\ c,\ d,\ e,\ r,\ k,\ \frac{a\,r}{a-c},\ \frac{b\,k}{b-d},\ \frac{a\,c\,r}{(a-c)\,(c-e)},\ \frac{b\,d\,k}{(b-d)\,(d-r)},\ \frac{a\,c\,e\,r}{(a-c)\,(c-e)\,(e-k)},\\
-\frac{b\,d\,k\,(a-c)}{c\,(b-d)\,(d-r)},\ -\frac{a\,c\,e\,r\,(b-d)}{d\,(a-c)\,(c-e)\,(e-k)},\ \frac{b\,d\,k\,(a-c)\,(c-e)}{c\,e\,(b-d)\,(d-r)},\ \frac{a\,c\,e\,(b-d)\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)},\\
-\frac{b\,d\,(a-c)\,(c-e)\,(e-k)}{c\,e\,(b-d)\,(d-r)},\ \frac{c^2\,e\,(b-d)\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)},\ -\frac{d^2\,(a-c)\,(c-e)\,(e-k)}{c\,e\,(b-d)\,(d-r)},\ \frac{c\,e^2\,(b-d)\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)},\\
-\frac{d\,r\,(a-c)\,(c-e)\,(e-k)}{c\,e\,(b-d)\,(d-r)},\ \frac{c\,e\,k\,(b-d)\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)},\ -\frac{a\,d\,r\,(c-e)\,(e-k)}{c\,e\,(b-d)\,(d-r)},\ \frac{b\,c\,e\,k\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)},\\
-\frac{a\,d\,r\,(e-k)}{e\,(b-d)\,(d-r)},\ \frac{b\,c\,e\,k}{(a-c)\,(c-e)\,(e-k)},\ -\frac{a\,d\,r}{(b-d)\,(d-r)},\ -\frac{b\,e\,k}{(c-e)\,(e-k)},\ \frac{a\,r}{d-r},\ \frac{b\,k}{e-k},\ -a,\ -b,\\
-c,\ -d,\ -e,\ -r,\ -k,\ -\frac{a\,r}{a-c},\ -\frac{b\,k}{b-d},\ -\frac{a\,c\,r}{(a-c)\,(c-e)},\ -\frac{b\,d\,k}{(b-d)\,(d-r)},\ -\frac{a\,c\,e\,r}{(a-c)\,(c-e)\,(e-k)},\\
\frac{b\,d\,k\,(a-c)}{c\,(b-d)\,(d-r)},\ \frac{a\,c\,e\,r\,(b-d)}{d\,(a-c)\,(c-e)\,(e-k)},\ -\frac{b\,d\,k\,(a-c)\,(c-e)}{c\,e\,(b-d)\,(d-r)},\ -\frac{a\,c\,e\,(b-d)\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)},\\
\frac{b\,d\,(a-c)\,(c-e)\,(e-k)}{c\,e\,(b-d)\,(d-r)},\ -\frac{c^2\,e\,(b-d)\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)},\ \frac{d^2\,(a-c)\,(c-e)\,(e-k)}{c\,e\,(b-d)\,(d-r)},\ -\frac{c\,e^2\,(b-d)\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)},\\
\frac{d\,r\,(a-c)\,(c-e)\,(e-k)}{c\,e\,(b-d)\,(d-r)},\ -\frac{c\,e\,k\,(b-d)\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)},\ \frac{a\,d\,r\,(c-e)\,(e-k)}{c\,e\,(b-d)\,(d-r)},\ -\frac{b\,c\,e\,k\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)},\\
\frac{a\,d\,r\,(e-k)}{e\,(b-d)\,(d-r)},\ -\frac{b\,c\,e\,k}{(a-c)\,(c-e)\,(e-k)},\ \frac{a\,d\,r}{(b-d)\,(d-r)},\ \frac{b\,e\,k}{(c-e)\,(e-k)},\ -\frac{a\,r}{d-r},\ -\frac{b\,k}{e-k},\ a,\ b,\ c,\\
d,\ e,\ r,\ k,\ \ldots
\end{array}\right\}$$

Proof. The results are true for $m = 0$. Assume that $m > 0$ and the results hold for $m - 1$. That is

$$\begin{aligned}
u_{60m-66} &= a, \quad u_{60m-65} = b, \quad u_{60m-64} = c, \quad u_{60m-63} = d,\\
u_{60m-62} &= e, \quad u_{60m-61} = r, \quad u_{60m-60} = k, \quad u_{60m-59} = \frac{a\,r}{a-c},\\
u_{60m-58} &= \frac{b\,k}{b-d}, \quad u_{60m-57} = \frac{a\,c\,r}{(a-c)\,(c-e)}, \quad u_{60m-56} = \frac{b\,d\,k}{(b-d)\,(d-r)}\\
u_{60m-55} &= \frac{a\,c\,e\,r}{(a-c)\,(c-e)\,(e-k)}, \quad u_{60m-54} = -\frac{b\,d\,k\,(a-c)}{c\,(b-d)\,(d-r)},\\
u_{60m-53} &= -\frac{a\,c\,e\,r\,(b-d)}{d\,(a-c)\,(c-e)\,(e-k)}, \quad u_{60m-52} = \frac{b\,d\,k\,(a-c)\,(c-e)}{c\,e\,(b-d)\,(d-r)},\\
u_{60m-51} &= \frac{a\,c\,e\,(b-d)\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)}, \quad u_{60m-50} = -\frac{b\,d\,(a-c)\,(c-e)\,(e-k)}{c\,e\,(b-d)\,(d-r)},\\
u_{60m-49} &= \frac{c^2\,e\,(b-d)\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)}, \quad u_{60m-48} = -\frac{d^2\,(a-c)\,(c-e)\,(e-k)}{c\,e\,(b-d)\,(d-r)},\\
u_{60m-47} &= \frac{c\,e^2\,(b-d)\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)}, \quad u_{60m-46} = -\frac{d\,r\,(a-c)\,(c-e)\,(e-k)}{c\,e\,(b-d)\,(d-r)},\\
u_{60m-45} &= \frac{c\,e\,k\,(b-d)\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)}, \quad u_{60m-44} = -\frac{a\,d\,r\,(c-e)\,(e-k)}{c\,e\,(b-d)\,(d-r)},\\
u_{60m-43} &= \frac{b\,c\,e\,k\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)}, \quad u_{60m-42} = -\frac{a\,d\,r\,(e-k)}{e\,(b-d)\,(d-r)},\\
u_{60m-41} &= \frac{b\,c\,e\,k}{(a-c)\,(c-e)\,(e-k)}, \quad u_{60m-40} = -\frac{a\,d\,r}{(b-d)\,(d-r)},\\
u_{60m-39} &= -\frac{b\,e\,k}{(c-e)\,(e-k)}, \quad u_{60m-38} = \frac{a\,r}{d-r}, \quad u_{60m-37} = \frac{b\,k}{e-k},\\
u_{60m-36} &= -a, \quad u_{60m-35} = -b, \quad u_{60m-34} = -c, \quad u_{60m-33} = -d, \quad u_{60m-32} = -e,\\
u_{60m-31} &= -r, \quad u_{60m-30} = -k, \quad u_{60m-29} = -\frac{a\,r}{a-c}, \quad u_{60m-28} = -\frac{b\,k}{b-d},\\
u_{60m-27} &= -\frac{a\,c\,r}{(a-c)\,(c-e)}, \quad u_{60m-26} = -\frac{b\,d\,k}{(b-d)\,(d-r)},\\
u_{60m-25} &= -\frac{a\,c\,e\,r}{(a-c)\,(c-e)\,(e-k)}, \quad u_{60m-24} = \frac{b\,d\,k\,(a-c)}{c\,(b-d)\,(d-r)},\\
u_{60m-23} &= \frac{a\,c\,e\,r\,(b-d)}{d\,(a-c)\,(c-e)\,(e-k)}, \quad u_{60m-22} = -\frac{b\,d\,k\,(a-c)\,(c-e)}{c\,e\,(b-d)\,(d-r)},\\
u_{60m-21} &= -\frac{a\,c\,e\,(b-d)\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)}, \quad u_{60m-20} = \frac{b\,d\,(a-c)\,(c-e)\,(e-k)}{c\,e\,(b-d)\,(d-r)},\\
u_{60m-19} &= -\frac{c^2\,e\,(b-d)\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)}, \quad u_{60m-18} = \frac{d^2\,(a-c)\,(c-e)\,(e-k)}{c\,e\,(b-d)\,(d-r)},
\end{aligned}$$

$$
\begin{aligned}
u_{60m-17} &= -\frac{c\,e^2\,(b-d)\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)}, \quad u_{60m-16} = \frac{d\,r\,(a-c)\,(c-e)\,(e-k)}{c\,e\,(b-d)\,(d-r)},\\
u_{60m-15} &= -\frac{c\,e\,k\,(b-d)\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)}, \quad u_{60m-14} = \frac{a\,d\,r\,(c-e)\,(e-k)}{c\,e\,(b-d)\,(d-r)},\\
u_{60m-13} &= -\frac{b\,c\,e\,k\,(d-r)}{d\,(a-c)\,(c-e)\,(e-k)}, \quad u_{60m-12} = \frac{a\,d\,r\,(e-k)}{e\,(b-d)\,(d-r)},\\
u_{60m-11} &= -\frac{b\,c\,e\,k}{(a-c)\,(c-e)\,(e-k)}, \quad u_{60m-10} = \frac{a\,d\,r}{(b-d)\,(d-r)},\\
u_{60m-9} &= \frac{b\,e\,k}{(c-e)\,(e-k)}, \quad u_{60m-8} = -\frac{a\,r}{d-r},\\
u_{60m-7} &= -\frac{b\,k}{e-k}, \quad u_{60m-6} = a, \quad u_{60m-5} = b, \quad u_{60m-4} = c,\\
u_{60m-3} &= d, \quad u_{60m-2} = e, \quad u_{60m-1} = r, \quad u_{60m} = k.
\end{aligned}
$$

Next, some of these relations are verified.

$$
\begin{aligned}
u_{60m-6} &= u_{60m-8} - \frac{u_{60m-8}u_{60m-11}}{u_{60m-11} - u_{60m-13}}\\
&= -\frac{ar}{d-r} - \frac{\frac{ar}{d-r}\frac{bcek}{(a-c)(c-e)(e-k)}}{-\frac{bcek}{(a-c)(c-e)(e-k)} + \frac{bcek(d-r)}{d(a-c)(c-e)(e-k)}}\\
&= -\left\{\frac{ar}{d-r} - \frac{ad}{d-r}\right\} = a.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
u_{60m-5} &= u_{60m-7} - \frac{u_{60m-7}u_{60m-10}}{u_{60m-10} - u_{60m-12}}\\
&= -\frac{bk}{e-k} - \frac{-\frac{bk}{e-k}\frac{adr}{(b-d)(d-r)}}{\frac{adr}{(b-d)(d-r)} - \frac{adr(e-k)}{e(b-d)(d-r)}}\\
&= -\left\{\frac{bk}{e-k} - \frac{be}{e-k}\right\} = b.
\end{aligned}
$$

Thus, the rest of the relations can be similarly proved.

4 Numerical results

In this section, we are mainly interested in confirming the obtained theoretical results by providing some numerical examples.

Example 1. In Figure 1, we confirm that the local stability of the equilibrium occurs if condition (6) is satisfied. Here, the constants and initial values have been taken by $a = 0.1,\ b = 2,\ c = 0.8,\ d = 4,\ u_{-6} = 0.01,\ u_{-5} = -0.02,\ u_{-4} = 0.03,\ u_{-3} = -0.2,\ u_{-2} = 0.3,\ u_{-1} = -0.3,\ u_0 = 0.4$.

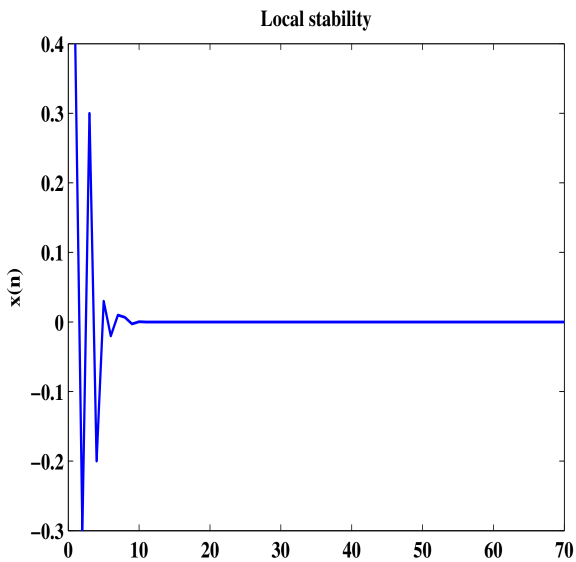

Figure 1: Local Stability of the Equilibrium.

Example 2. The global stability of the equilibrium point of Eq. (1) is shown in Figure 2 under the values $a = 0.3$, $b = 0.7$, $c = 4$, $d = 0.9$, $u_{-6} = 0.1$, $u_{-5} = 1$, $u_{-4} = -3$, $u_{-3} = 2$, $u_{-2} = -4$, $u_{-1} = 5$, $u_0 = -6$.

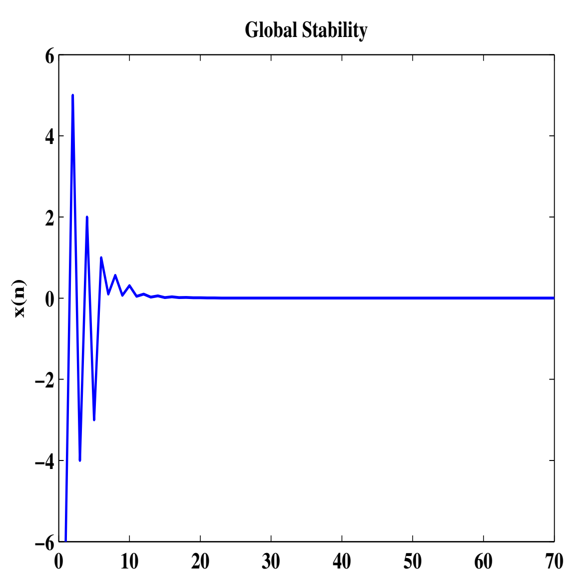

Figure 2: Global Behaviour about the Equilibrium.

Example 3. Figure 3 illustrates the periodicity of the solution of Eq. (12) when we assume that $u_{-6} = 2.8$, $u_{-5} = 3.5$, $u_{-4} = 1.6$, $u_{-3} = 2.5$, $u_{-2} = 1$, $u_{-1} = 3$, $u_0 = 2$.

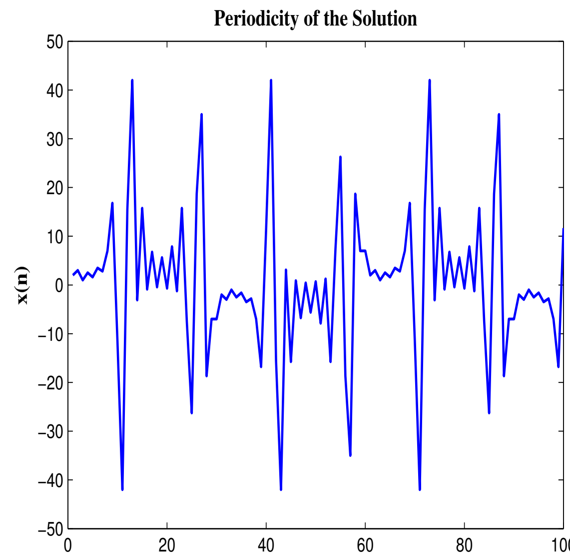

Figure 3: The Periodicity of Eq. (12).

Example 1. The stability of Eq. (12) is depicted in Figure 4 under the values $a = 0.2$, $b = 0.1$, $c = 0.3$, $d = 0.5$, $u_{-6} = 0.2$, $u_{-5} = 2$, $u_{-4} = 1.8$, $u_{-3} = 2.5$, $u_{-2} = 0.5$, $u_{-1} = 1$, $u_0 = 0.2$.

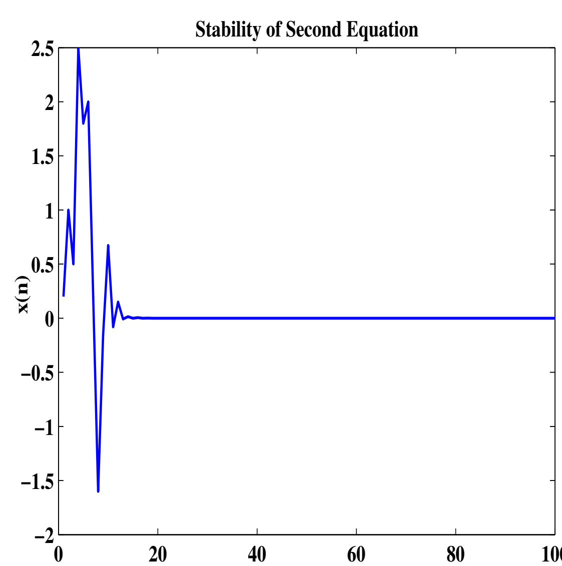

Figure 4: Stability of the Equilibrium.

5 Conclusion

This paper has introduced some new results for tow different equations. The local stability of the obtained equilibrium points are shown. Moreover, we obtained that every solution of the second proposed equation is periodic with period 60. The stability and the periodicity are depicted in some of the presented figures. The used technique in determining the exact solution of the first proposed equation can be straightforwardly applied for equations with high order.

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