

## Research Article

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# Vibrational analysis of an irregular single-walled carbon nanotube incorporating initial stress effects

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**Abstract:** In this work, an attempt is done to apply the eigenvalue approach as well as Donnell thin-shell theory to find out the vibrational analyses of an irregular single-walled carbon (ISWCNT) incorporating initial stress effects. The effects of surface irregularity and initial stresses on natural frequency of vibration of nano materials, especially for single-walled carbon nanotubes (SWCNTs), have not been investigated before, and most of the previous research have been carried for a regular and initial stress-free CNTs. Therefore, it must be emphasized that the vibrations of prestressed irregular SWCNT are novel and applicable for the design of nano oscillators and nanodevices, in which SWCNTs act as the most prevalent nanocomposite structural element. The surface irregularity is assumed in the parabolic form at the surface of SWCNT. A novel equation of motion and frequency equation is derived. The obtained numerical results provide a better representation of the vibration behavior of prestressed ISWCNTs. It has been observed that the presence of either surface irregularity or initial stress has notable effects on the natural frequency of vibration, particularly in the short-length SWCNTs. Numerical results show that the natural frequency of SWCNT decreases with increase in surface irregularity and initial stress parameters. To the authors' best knowledge, the effect of surface irregularity and initial stresses on the vibration behavior of SWCNTs has not yet been studied, and the present work is an attempt to find out this effectiveness. In addition, the results of the present analysis may serve as useful references for the application and the

design of nano oscillators and nanodevices, in which SWCNTs act as the most prevalent nanocomposite structural element.

**Keywords:** vibration analysis, single-walled carbon nanotubes, surface irregularity, initial stresses, Donnell thin-shell theory, eigenvalue approach

## 1 Introduction

The study of vibration analysis in the surfaced irregularity and prestressed single-walled carbon nanotubes (SWCNTs) may give a useful reference to predict frequencies to estimate the expected experimental values for potential application and design of nanoelectronics and nanodevices. Since Iijima [1] discovered the CNTs, they became widely used in scientific and practical applications; they were used in the manufacturing electrodes, in super capacitors and space elevator cables [2]. Moreover, CNTs are used to strengthen the components of nanocomposites as well as are used on a large scale in components of superconducting nanocomposites [3]. The vibrations of CNTs have a considerable importance in nanomechanical applications such as the design of nano oscillators and nanodevices. During the last decade, numerous studies on vibrations of SWCNTs using continuum shell models were performed, as it can be seen from the examples of ref. [4–19]. Based on Euler–Bernoulli beam, Tang and Yang investigated a novel model of fluid-conveying nanotubes made of bidirectional functionally graded materials (FGMs) and presented the dynamic behavior and stability of nanotubes [20]. To study the nonlinear free vibration of the bidirectional (2D) FGMs, Tang *et al.* [21] presented a novel model of Euler–Bernoulli beams. Nonlinear bending, buckling and vibration of bidirectional functionally graded nanobeams were studied by Yang *et al.* [22,23]. A novel model of fluid-conveying nanotubes made of bidirectional FGMs was presented by Tang and Yang [24] to investigate the

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dynamic behaviors and stability of the graded nanotubes. The post-buckling behavior and nonlinear vibration of a fluid-conveying pipe composed of an FGM were analytically studied by Tang and Yang [25]. Tang and Ding [26] presented the nonlinear hydrothermal dynamics of a bidirectional functionally graded beam with coupled transverse and longitudinal displacements. The fluid-conveying pipes made of polymer-like materials are widely applied in engineering fields. The fractional dynamics of the pipes subjected to the excitation of the supporting foundation is studied by Tang *et al.* [27]. SWCNTs, also called graphene nanotubes have properties that make them the only all-purpose additive that can improve specific characteristics of a multitude of materials. Due to the importance of SWCNTs, which are known to be the strongest materials, there are many experimental and theoretical studies carried out about composites that contains SWCNT, as can be seen from the examples given in refs. [28–47]. During the last decade, numerous studies were carried out on the vibration analysis of SWCNTs, in particular, to understand the dynamic behavior of torsional vibration of CNTs, and numerous researchers conducted the computational simulations to study the torsional vibrations of the nanotubes. Gheshlaghi and Hasheminejad [48] studied the free torsional responses of CNT using modified couple stress model to achieve the effect of the size dependency on the response of the nanotube. The torsional vibration of nonlinear homogenous and isotropic nano-cone was studied based on the strain gradient model theory and generalized differential quadrature method [49]. The mechanic behavior of the CNTs embedded in an elastic medium were investigated by Arda and Aydogdu [50] based on the modern elasticity theory. Based on Kelvin–Voigt model, El-Borgi *et al.* [51] reported a novel model to investigate the vibrational behavior of the viscoelastic nanorods. Modern elasticity theory and Maxwell model were used by Zarezadeh *et al.* [52] to study the torsional vibration of functionally graded nanorods embedded in an elastic bulk medium under magnetic field. In addition, some studies focused on the mechanical behavior of CNTs, such as Ru [53], Shen [54] and Thai [55]. However, the models used in these studies could not be applied for predicting the vibrational behavior of prestressed irregular SWCNTs (ISWCNTs), due to technical limitations. CNTs acting as basic elements of nanostructures often occur in compressional initial stresses due to the mismatch between different materials or initially external axial loads. Moreover, the nature frequency of vibration of SWCNTs is influenced by the change in the surface structures of the nanotube. The irregularities in CNTs

constructions may occur due to the consequence of manufacturing defect, reckless servicing, etc. Thus, it is of great importance to deal with surface irregularity to study the vibrational analysis of SWCNTs.

The effects of surface irregularity and initial stresses on the deformation of the static medium were studied by Selim [56]. The vibration analysis of ISWCNTs without initial stresses was investigated by Selim [57]. It was shown that the presence of surface irregularity has a notable effect on the vibration analysis of SWCNTs. The present study is an extension of the previous work to find the effectiveness of both surface irregularity and initial stresses on the natural frequency of vibrations of WCNTs, due to the similarity structures of CNTs with the cylindrical shells. Donnell thin-shell approach is capable of tackling vibrations studied of SWCNTs. The analytical technique used should be simple, easily applicable and yield results robustly. On the other hand, CNTs often suffer from initial compressional stresses due to residual stress, thermal effect, surface effect, mismatch between the material properties of nanoplates and the surrounding medium, initial external load and may be due to any other physical causes. It is obvious that the varying of the surface structures of the nanotube as well as the presence of the initial stresses influences the natural frequency of the nanotubes. Hence, it is of great importance to deal with various constructions to study the vibration behavior of irregular and prestressed CNTs. As can be seen from the above literature summary, to the best of our knowledge, the effect of both the surface irregularity and initial stresses on vibrational analysis of SWCNTs has not yet been investigated. The vibration of CNTs under the effects of surface irregularity and initial stresses is very important in terms of both predicting the results of the experiments and providing useful information for the next generation studies and accurate designs of nanodevices. In this sense, the present study is much important within the framework of the Donnell thin-shell theory; and thus, the results can provide a useful information for the next generation of studies and accurate designs of nanomachines. It is pointed out that the present results can be used to analyze the surface irregularity and initial stresses effects on the natural frequency of vibration of SWCNTs. The numerical results are plotted in graphs for different values of initial stress and surface irregularity parameters and have been compared with the cases of uniform and initial stress-free SWCNTs. In this work, the obtained frequency equation and numerical results may give a useful reference to predict the natural frequency of vibrations of SWCNTs to estimate the expected experimental values for

potential applications and designs of nano oscillators and nanodevices.

## 2 Mathematical formulation of the problem

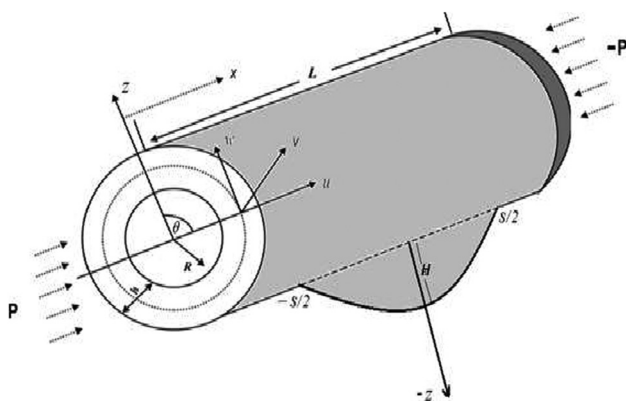
The cylindrical coordinate system is used for describing the effectiveness of surface irregularity and initial stresses on the vibration analysis of SWCNT, which is defined in Figure 1. The  $x$ -coordinate is taken in the axial direction of the nanotube, the  $z$ -coordinate and  $\theta$ -coordinate are taken in the radial and circumferential directions, respectively. The displacements are defined by  $u$ ,  $v$  and  $w$  in the  $x$ -,  $\theta$ - and  $z$ -directions, respectively. The SWCNT has a thickness of  $h$ , length of  $L$  and radius of  $R$ . The compressional initial stress ( $-P$ ) is effected on the  $x$ -direction and the surface irregularity is assumed in the parabola form at the lower boundary of the SWCNT (see Figure 1). The span of surface irregularity and maximum depth of irregularity are denoted by  $s$  and  $H'$ , respectively.

In case of parabolic irregularity, the boundary surface may be described by [58]:

$$z = -(R + h + \varepsilon \delta(x)),$$

$$\delta(x) = \begin{cases} s \left(1 - \frac{4x^2}{s^2}\right) & \text{for } |x| < \frac{s}{2}, \\ 0 & \text{for } x \geq \frac{s}{2} \text{ and } x \leq -\frac{s}{2}, \end{cases} \quad (1)$$

where the maximum amplitude of irregular boundary is  $\varepsilon = \frac{H'}{s} \ll 1$ .



**Figure 1:** Geometry of prestressed irregular single-walled carbon nanotube

The governing equations of the vibration of pre-stressed ( $S_{xx} = -P$ ) ISWCNT can be written as [59]:

$$\left[ A_{11} \frac{\partial^2}{\partial x^2} + \frac{A_{22}(1-\nu)}{2R^2} \frac{\partial^2}{\partial \theta^2} - \rho h \frac{\partial^2}{\partial t^2} \right] u + \left[ \left( \frac{A_{12}}{R} + \frac{A_{22}(1-\nu)}{2R} - \frac{\zeta_x E}{2R(1+\nu)} \right) \frac{\partial^2}{\partial x \partial \theta} \right] v \quad (2)$$

$$+ \left[ \left( \frac{A_{12}}{R} \frac{\partial}{\partial x} + \frac{\zeta_x E}{2R^2(1+\nu)} \frac{\partial^2}{\partial \theta^2} \right) \right] w = 0,$$

$$\left( \frac{A_{12}}{R} + \frac{A_{22}(1-\nu)}{2R} \right) \frac{\partial^2}{\partial x \partial \theta} u + \left[ \left( \frac{A_{22}(1-\nu)}{2} + \frac{A_{22}(1-\nu)h^2}{24R^2} - \frac{\zeta_x E}{2(1+\nu)} \right) \frac{\partial^2}{\partial x^2} + \left( \frac{A_{22}}{R^2} + \frac{h^2 A_{22}}{12R^4} \right) \frac{\partial^2}{\partial \theta^2} - \rho h \frac{\partial^2}{\partial t^2} \right] v \quad (3)$$

$$+ \left[ \left( \frac{A_{22}}{R^2} \frac{\partial}{\partial \theta} - \frac{h^2 A_{22}}{12R^4} \frac{\partial^3}{\partial \theta^3} + \frac{\zeta_x E}{2R(1+\nu)} \frac{\partial^2}{\partial x \partial \theta} - \left( \frac{h^2 A_{22}}{6R^2} + \frac{h^2 A_{12}}{12R^2} \right) \frac{\partial^4}{\partial x^2 \partial \theta^2} \right) \right] w = 0,$$

$$\left[ -\frac{A_{12}}{R^2} \frac{\partial}{\partial x} \right] u + \left[ \left( \frac{h^2 A_{12}}{12R^2} + \frac{A_{22}(1-\nu)h^2}{12R^2} \right) \frac{\partial^3}{\partial x^2 \partial \theta} + \left( \frac{h^2 A_{22}}{12R^4} \right) \frac{\partial^3}{\partial \theta^3} - \left( \frac{A_{22}}{R^2} \right) \frac{\partial}{\partial \theta} \right] v \quad (4)$$

$$- \left[ \frac{h^2 A_{11}}{12} \frac{\partial^4}{\partial x^4} + \left( \frac{h^2 A_{12}}{6R^2} + \frac{A_{22}(1-\nu)h^2}{6R^2} \right) \frac{\partial^4}{\partial x^2 \partial \theta^2} + \frac{h^2 A_{22}}{12R^4} \frac{\partial^4}{\partial \theta^4} + \frac{\zeta_x E}{2(1+\nu)} \frac{\partial^2}{\partial x^2} + \frac{A_{22}}{R^2} + \rho h \frac{\partial^2}{\partial t^2} \right] w = 0,$$

where

$$A_{11} = \left( \frac{(1 + \zeta_x(1-\nu))}{(1-\nu^2)} \right) Eh, \quad (5)$$

$$A_{12} = \left( \frac{\nu + \zeta_x(1-\nu)}{(1-\nu^2)} \right) Eh, \quad (5)$$

$$A_{22} = \frac{Eh}{(1-\nu^2)},$$

$$\zeta_x = \frac{P(1+\nu)}{E},$$

$h$ ,  $R$  and  $\rho$  are the thickness, the radius of the nanotube and the mass density of the CNT, respectively;  $E$ ,  $\nu$  and  $Eh$  are the elastic modulus, the Poisson's ratio and the in-plane rigidity, respectively;  $\rho h$ ,  $P$  and  $t$  are the mass density per unit lateral area, the initial compressional stress and the time, respectively.

In matrix form, the equations (2), (3) and (4) may be expressed as:

$$\begin{bmatrix} \Gamma_{11} + \Delta_{11} & \Gamma_{12} + \Delta_{12} & \Gamma_{13} + \Delta_{13} \\ \Gamma_{21} + \Delta_{21} & \Gamma_{22} + \Delta_{22} & \Gamma_{23} + \Delta_{23} \\ \Gamma_{31} + \Delta_{31} & \Gamma_{32} + \Delta_{32} & \Gamma_{33} + \Delta_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (6)$$

where  $\Gamma_{ij}$  and  $\Delta_{ij}$  ( $i, j = 1, 2, 3$ ) are the differential operators with respect to the coordinates axis  $x$  and  $\theta$ , and are defined by,

$$\Gamma_{11} = \frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2R^2} \frac{\partial^2}{\partial \theta^2} - \beta \rho h \frac{\partial^2}{\partial t^2}, \quad \Gamma_{12} = \frac{1+\nu}{2R} \frac{\partial^2}{\partial x \partial \theta}, \quad (7)$$

$$\Gamma_{13} = \frac{\nu}{R} \frac{\partial}{\partial x}, \quad \Gamma_{21} = \frac{1+\nu}{2R} \frac{\partial^2}{\partial x \partial \theta}, \quad (8)$$

$$\Gamma_{22} = \frac{1-\nu}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2} + \Omega \left( (1-\nu) \frac{\partial^2}{\partial x^2} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2} \right) - \beta \rho h \frac{\partial^2}{\partial t^2}, \quad (9)$$

$$\Gamma_{23} = \frac{1}{R^2} \frac{\partial}{\partial \theta} - \Omega \left( (2-\nu) \frac{\partial^3}{\partial x^2 \partial \theta} + \frac{1}{R^2} \frac{\partial^3}{\partial \theta^3} \right), \quad (10)$$

$$\Gamma_{31} = \frac{-\nu}{R} \frac{\partial}{\partial x},$$

$$\Gamma_{32} = -\frac{1}{R^2} \frac{\partial}{\partial \theta} + \Omega \left( (2-\nu) \frac{\partial^3}{\partial x^2 \partial \theta} + \frac{1}{R^2} \frac{\partial^3}{\partial \theta^3} \right), \quad (11)$$

$$\Gamma_{33} = -\left[ \frac{1}{R^2} + \Omega \left( R^2 \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial \theta^2} + \frac{1}{R^2} \frac{\partial^4}{\partial \theta^4} \right) + \beta \rho h \frac{\partial^2}{\partial t^2} \right], \quad (12)$$

and

$$\Delta_{11} = (1-\nu) \zeta_x \frac{\partial^2}{\partial x^2}, \quad \Delta_{12} = \frac{\zeta_x (1-\nu) (2h-1)}{2R} \frac{\partial^2}{\partial x \partial \theta}, \quad (13)$$

$$\Delta_{21} = (1-\nu) \zeta_x \frac{\partial^2}{\partial x \partial \theta}, \quad \Delta_{13} = \frac{(1-\nu) \zeta_x}{R} \left( \frac{\partial}{\partial x} + \frac{1}{2Rh} \frac{\partial^2}{\partial \theta^2} \right), \quad (14)$$

$$\Delta_{22} = \frac{(1-\nu) \zeta_x}{2h} \frac{\partial^2}{\partial x^2}, \quad \Delta_{23} = \frac{(1-\nu) \zeta_x}{2hR} \left( \frac{\partial^2}{\partial x \partial \theta} - \frac{h^3}{6R} \frac{\partial^3}{\partial x^2 \partial \theta} \right), \quad (15)$$

$$\Delta_{31} = \frac{\zeta_x (1-\nu)}{R} \frac{\partial}{\partial x}, \quad \Delta_{32} = \frac{\zeta_x h^2 (1-\nu)}{12R} \frac{\partial^3}{\partial x^2 \partial \theta}, \quad (16)$$

$$\Delta_{33} = \frac{\zeta_x (1-\nu)}{2h} \left( \frac{\partial^2}{\partial x^2} + \frac{h^3}{6} \frac{\partial^4}{\partial x^4} + \frac{h^3}{3R^2} \frac{\partial^4}{\partial x^2 \partial \theta^2} \right), \quad (17)$$

$\Omega = \frac{h^2}{12R^2}$  is the dimensionless thickness parameter,  $\zeta_x = \frac{P(1+\nu)}{E}$  is the initial stress parameter and  $\beta = \frac{(1-\nu^2)}{Eh}$ .

### 3 Solution procedure

The functions  $u$ ,  $v$  and  $w$  are used to designate their respective displacement functions and  $u^0$  is the midline stretching. Therefore, for modal deformation, the displacements are assumed (for  $|x| \leq s/2$ ) as:

$$u = u^0(x, \theta, t) - z \frac{\partial w(x, \theta, t)}{\partial x}, \quad (18)$$

$$u^0(x, \theta, t) = U_m e^{-i(k_m x - \omega t)} \sin(n\theta) \quad (19)$$

$$v(x, \theta, t) = V_m e^{-i(k_m x - \omega t)} \cos(n\theta), \quad (20)$$

$$w(x, \theta, t) = W_m e^{-i(k_m x - \omega t)} \sin(n\theta), \quad (21)$$

where  $U_m$ ,  $V_m$  and  $W_m$  represent three vibration amplitude coefficients in axial, circumferential and radial directions, respectively.  $m$  and  $n$  are the axial half and the circumferential wave numbers, respectively, where  $k_m$  is the axial wave number related with an end condition. The angular frequency is denoted by  $\omega$  and the natural frequency is  $f = \omega/2\pi$ .

In view of the differential operators in equations (7)–(17) and using the  $u$ ,  $v$  and  $w$  given in equations (18)–(21), the matrix form of the vibration frequency equation of the SWCNTs with surface irregularity under initial compression stresses is as follows:

$$\begin{bmatrix} L_{11} + \alpha_{11} + \eta \omega^2 & L_{12} + \alpha_{12} & \hat{L}_{13} + \alpha_{13} \\ L_{21} + \alpha_{21} & L_{22} + \alpha_{22} + \eta \omega^2 & \hat{L}_{23} + \alpha_{23} \\ L_{31} + \alpha_{31} & L_{32} + \alpha_{32} & \hat{L}_{33} + \alpha_{33} + \eta \omega^2 \end{bmatrix} \begin{bmatrix} U_m \\ V_m \\ W_m \end{bmatrix} = 0. \quad (22)$$

Using the Appendices 1 and 2, the matrix elements  $L_{ij}$ ,  $\hat{L}_{ij}$  and  $\alpha_{ij}$  ( $i, j = 1, 2, 3$ ) can be written as.

$$L_{11} = -k_m^2 - \left( \frac{1-\nu}{2R^2} \right) n^2, \quad L_{12} = ik_m \left( \frac{1+\nu}{2R} \right) n = -L_{21}, \quad (23)$$

$$\hat{L}_{13} = ik_m \left[ \frac{\nu}{R} + \left[ \left( k_m^2 (R + h + \varepsilon \delta(x)) - 16ik_m \frac{\varepsilon x}{s} + \frac{8\varepsilon}{s} \right) + (R + h + \varepsilon \delta(x)) \left( \frac{1-\nu}{2R^2} n^2 + \eta \omega^2 \right) \right] \right], \quad (24)$$

$$\begin{aligned} L_{21} &= -ik_m n \left( \frac{1+\nu}{2R} \right) = -L_{12}, \\ L_{22} &= -k_m^2 \left( \frac{1-\nu}{2} \right) - \frac{n^2}{R^2}, \end{aligned} \quad (25)$$

$$\begin{aligned} L_{23} &= \frac{-n}{R^2} + n \left( \frac{1+\nu}{2R} \right) \\ &\times \left( -k_m^2 (R + h + \varepsilon \delta(x)) + ik_m \frac{8\varepsilon x}{s} \right), \end{aligned} \quad (26)$$

$$L_{31} = -ik_m \left( \frac{\nu}{R} \right), \quad L_{32} = -\frac{n}{R^2}, \quad (27)$$

$$\begin{aligned} \hat{L}_{33} &= \frac{-1}{R^2} - \frac{(1-\nu^2)\alpha}{Eh} \left( k_m^4 + \frac{2}{R^2} k_m^2 n^2 + \frac{n^4}{R^4} \right) \\ &+ \left( \frac{\nu}{R} \right) \left( -k_m^2 (R + h + \varepsilon \delta(x)) + ik_m \frac{8\varepsilon x}{s} \right), \end{aligned} \quad (28)$$

and

$$\begin{aligned} \alpha_{11} &= -\zeta_x k_m^2 (1-\nu), \\ \alpha_{12} &= i \frac{k_m n \zeta_x (2h-1)(1-\nu)}{2R}, \end{aligned} \quad (29)$$

$$\begin{aligned} \alpha_{13} &= \frac{-\zeta_x (1-\nu)}{R} \left( ik_m + \frac{n^2}{2Rh} \right) + i \zeta_x (1-\nu) k_m \\ &\times \left[ \left( k_m^2 (R + h + \varepsilon \delta(x)) + 16ik_m \frac{\varepsilon x}{s} + \frac{8\varepsilon}{s} \right) \right], \end{aligned} \quad (30)$$

$$\begin{aligned} \alpha_{21} &= -i \zeta_x k_m n (1-\nu), \\ \alpha_{22} &= -k_m^2 \frac{\zeta_x (1-\nu)}{2h}, \end{aligned} \quad (31)$$

$$\begin{aligned} \alpha_{23} &= \frac{\zeta_x (1-\nu)}{2hR} \left( -ik_m n + \frac{k_m^2 n h^3}{6R} \right) \\ &+ ik_m \zeta_x (1-\nu) n \left( ik_m (R + h + \varepsilon \delta(x)) + \frac{8\varepsilon x}{s} \right), \end{aligned} \quad (32)$$

$$\begin{aligned} \alpha_{31} &= -ik_m \frac{\zeta_x (1-\nu)}{R}, \\ \alpha_{32} &= k_m^2 \frac{n \zeta_x h^2 (1-\nu)}{12R}, \end{aligned} \quad (33)$$

$$\begin{aligned} \alpha_{33} &= \frac{\zeta_x (1-\nu)}{2h} \left[ -k_m^2 \left( 1 - n^2 \frac{h^3}{3R^2} \right) + \frac{h^3}{6} k_m^4 \right] \\ &+ ik_m \frac{\zeta_x (1-\nu)}{R} \left( ik_m (R + h + \varepsilon \delta(x)) + \frac{8\varepsilon x}{s} \right). \end{aligned} \quad (34)$$

$$\eta = \frac{(1-\nu^2)\rho h}{Eh}. \quad (35)$$

In order to obtain the vibration frequency and associated modes for ISWCNTs, the determinant of the matrix in equation (22) should vanish. Thus, one obtains the following characteristic equation:

$$\det \begin{bmatrix} L_{11} + \alpha_{11} + \eta \omega^2 & L_{12} + \alpha_{12} & \hat{L}_{13} + \alpha_{13} \\ L_{21} + \alpha_{21} & L_{22} + \alpha_{22} + \eta \omega^2 & \hat{L}_{23} + \alpha_{23} \\ L_{31} + \alpha_{31} & L_{32} + \alpha_{32} & \hat{L}_{33} + \alpha_{33} + \eta \omega^2 \end{bmatrix} = 0, \quad (36)$$

where the lowest root of equation (36) indicates that the eigenvalues are associated with vibration frequencies of prestressed ISWCNT and the vector  $(U_m, V_m, W_m)^T$  represents their mode shapes. In the present study, we suppose the nanotube is simply supported in its ends, so the axial wave number is taken in the following form [59]:

$$k_m = \frac{m\pi}{L}. \quad (37)$$

## 4 Particular cases

Case (I): if the effect of initial stresses is absent (i.e.  $\zeta_x = 0$ ), then  $\alpha_{ij} = 0$ , and equation (22) becomes

$$\begin{bmatrix} L_{11} + \eta \omega^2 & L_{12} & \hat{L}_{13} \\ L_{21} & L_{22} + \eta \omega^2 & \hat{L}_{23} \\ L_{31} & L_{32} & \hat{L}_{33} + \eta \omega^2 \end{bmatrix} \begin{bmatrix} U_m \\ V_m \\ W_m \end{bmatrix} = 0, \quad (38)$$

which coincides with the result obtained by Selim [59].

Case (II): if both the effects of surface irregularity and initial stresses are absent (i.e.  $\varepsilon = 0$  and  $\zeta_x = 0$ ), then equation (22) becomes

$$\begin{bmatrix} L_{11} + \eta \omega^2 & L_{12} & L_{13} \\ L_{21} & L_{22} + \eta \omega^2 & L_{23} \\ L_{31} & L_{32} & L_{33} + \eta \omega^2 \end{bmatrix} \begin{bmatrix} U_m \\ V_m \\ W_m \end{bmatrix} = 0, \quad (39)$$

where

$$\begin{aligned} L_{13} &= ik_m \left[ \frac{\nu}{R} \right], \\ L_{23} &= \frac{n}{R^2}, \\ L_{33} &= \frac{-1}{R^2} - \frac{(1-\nu^2)\alpha}{Eh} \left( k_m^4 + \frac{2}{R^2} k_m^2 n^2 + \frac{n^4}{R^4} \right), \end{aligned} \quad (40)$$

which agrees with the result of Hussain [60].



## 5 Numerical results and discussion

In this section, numerical results for vibration of the simply supported SWCNTs under the effects of surface irregularity are carried out using the simulation parameters presented in Table 1 [5].

An examination of the studies concerned with fabrication procedures of CNTs reveals that perfectly straight CNTs without any deviation in geometry and orientation are very difficult to achieve [61]. Therefore, understanding the influences of surface irregularity as well as initial stresses on the vibration analysis of SWCNT may serve as useful references for the applications and designs of nano oscillators and nanodevices, in which SWCNTs act as the most prevalent nanocomposite structural element.

In order to show the effects of compressive initial stresses as well as surface irregularity on the torsional natural frequency of SWCNT, the relation between the torsional frequency and axial wave number is calculated. Numerical computations are carried out for different values of compressive initial stress parameter ( $\zeta_x = 0.2, 0.4$ ) and surface irregularity parameter ( $\varepsilon = 0.28, 0.56$ ). The results of computations are plotted in Figures 2 and 3, for various values of wave number  $k_m$  at first and second modes.

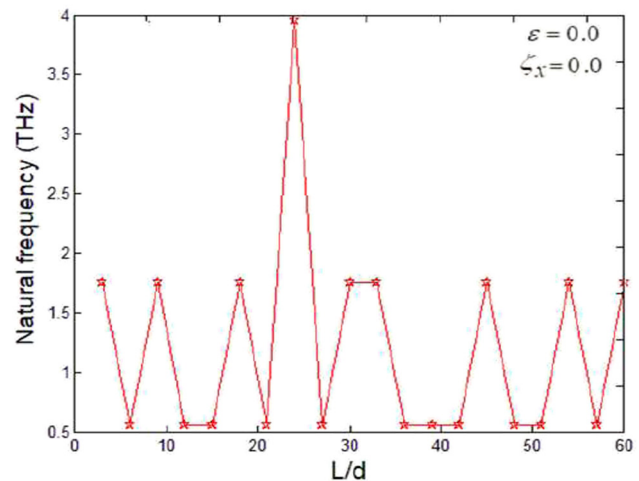
The natural frequency (THz) as a function of  $0 \leq x \leq L$  for the case of simply supported SWCNTs at first vibration mode ( $m = n = 1$ ) has been evaluated based on the Donnell thin cylindrical shell theory using (36). The results obtained are compared and discussed with the case of uniform initial stress-free nanotube ( $\varepsilon = 0.0, \zeta_x = 0.0$ ).

In order to show the effects of surface irregularity and initial stresses on the natural frequency of vibration, a series of computations are carried out for different values of the aspect ratio ( $L/d$ ), irregularity and initial stress parameters. The frequency (THz) data versus ratio of  $L/d$  are plotted in Figures 2–7.

Figure 2 shows the variations of natural frequencies of uniform ( $\varepsilon = 0.0$ ) SWCNT for different values of aspect ratio ( $L/d$ ). It is clearly seen from this figure that the natural frequency of the uniform SWCNT takes values that lie in the range of 0.5–14 terahertz for the aspect ratio intervals 0–60.

**Table 1:** Simulation parameters' armchairs

$h$	$E/\rho$	$u$	$ Eh$	$R$
0.34 nm	$3.6481 \times 10^8 \text{ m}^2/\text{s}^2$	0.2	278.25 GPa nm	0.343322 nm

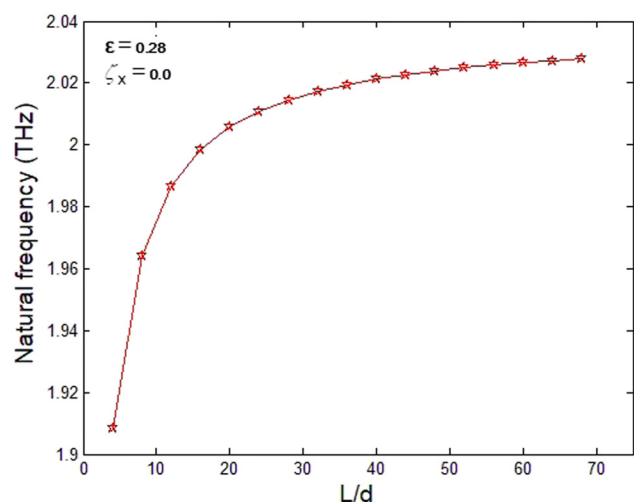


**Figure 2:** Variations in the natural frequency of uniform ( $\varepsilon = 0.0$ ) initial stress-free ( $\zeta = 0.0$ ) single-walled carbon nanotube against aspect ratio ( $L/d$ ).

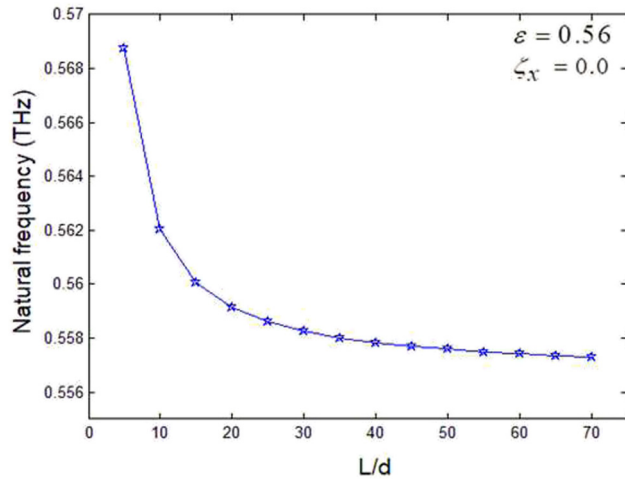
The effects of surface irregularity ( $\varepsilon = 0.28$  and  $\varepsilon = 0.56$ ) on the natural frequency of vibration are demonstrated in Figures 3 and 4.

The effect of surface irregularity parameter ( $\varepsilon = 0.28$ ) on the natural frequency of the nanotube various aspect ratio at ( $\zeta_x = 0.0$ ) is shown Figure 3. From this figure, it can be seen that the natural frequency of SWCNT is quite different, compared with its values at ( $\varepsilon = 0.0$ ). The reason is very clear; and the change is due to the effect of surface irregularity.

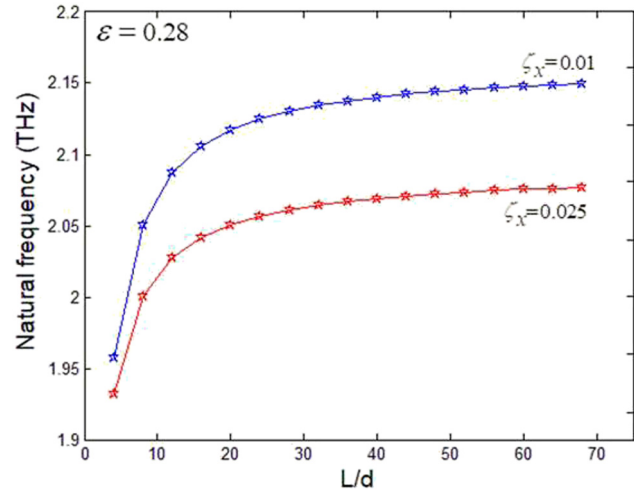
The effect of surface irregularity becomes obvious for the higher values of the irregularity parameter ( $\varepsilon = 0.56$ )



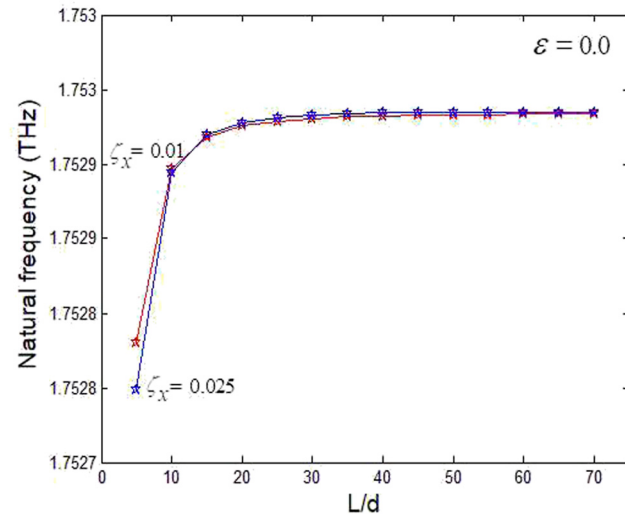
**Figure 3:** Variations in the natural frequency of irregular ( $\varepsilon = 0.28$ ) single-walled carbon nanotube against aspect ratio ( $L/d$ ) at ( $\zeta = 0.0$ ).



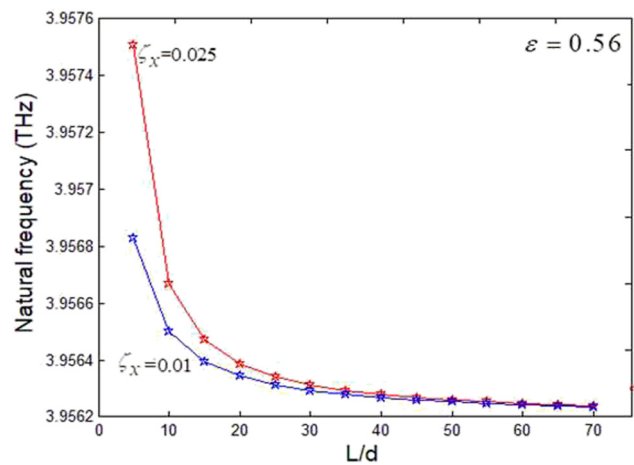
**Figure 4:** Variations in the natural frequency of irregular ( $\varepsilon = 0.56$ ) single-walled carbon nanotube against aspect ratio ( $L/d$ ) at ( $\zeta = 0.0$ ).



**Figure 6:** Variations in the natural frequency of irregular ( $\varepsilon = 0.28$ ) single-walled carbon nanotube against aspect ratio ( $L/d$ ) under initial stresses ( $\zeta = 0.01, 0.025$ ).



**Figure 5:** Variations in the natural frequency of uniform single-walled carbon nanotube against aspect ratio ( $L/d$ ) under initial stresses ( $\zeta = 0.01, 0.025$ ).



**Figure 7:** Variations in the natural frequency of irregular ( $\varepsilon = 0.56$ ) single-walled carbon nanotube against aspect ratio ( $L/d$ ) under initial stresses ( $\zeta = 0.01, 0.025$ ).

as shown in Figure 4. From this figure, one can observe that the natural frequency of nanotube influenced by the increase in the surface irregularity parameter and the curve takes opposite direction compared with the result obtained in Figure 3 at  $\varepsilon = 0.28$ . This means that the increase in irregularity parameter decreases the natural frequency of the vibration and this decrease is most pronounced when the nanotube is long. However, the natural frequency of the nanotube is more affected by increasing the value of irregularity parameter compared with the cases of lower values of this parameter ( $\varepsilon = 0.28$  and  $\varepsilon = 0.0$ ). From Figures 2 and 3, one can get the information that the presence of surface irregularities in the

SWCNTs has notable effects on the natural frequency of the nanotubes.

Figure 5 shows the effects of initial stress on the natural frequency of uniform nanotube with various aspect ratio at ( $\zeta_x = 0.01, 0.025$ ). From Figure 5, it can be seen that small increase in the initial stress parameter (0.015) makes a notable change in the natural frequency of the vibrating nanotube. This means that the natural frequency is independent of the initial stresses present in the nanotube.

The effects of irregularity and initial stress parameters on the natural frequency of SWCNT are shown in Figures 6 and 7.

Figure 6 shows the variations in natural frequency of vibration with respect to aspect ratio ( $L/d$ ) with different values of the initial stress parameter ( $\zeta_x = 0.01, 0.025$ ) at surface irregularity parameter ( $\varepsilon = 0.28$ ). From this figure, it can be observed that the surface irregularity and initial stresses (together) have considerable effects on the natural frequency of the vibrating nanotube. In addition, the values of natural frequency of vibration are quite different from the values shown in Figure 5 for uniform nanotube at the same values of the initial stress parameter ( $\zeta_x = 0.01, 0.025$ ).

The effect of surface irregularity and initial stresses becomes obvious for the higher values of the irregularity parameter ( $\varepsilon = 0.56$ ) as shown in Figure 7. From this figure, we can observe that the natural frequency of nanotube is influenced by the increase in the surface irregularity and initial stress parameters, and the curve takes the opposite direction. This means that the increase in surface irregularity with an increase in initial stress parameter decreases the natural frequency of the vibration, and this decrease is most pronounced when the nanotube is long. However, the natural frequency of the nanotube is more affected by the surface irregularity and initial stresses at shorter nanotubes.

From Figures 6 and 7, one can get the information that the presence of surface irregularity and initial stresses in SWCNTs, has notable effects on the natural frequency of vibration.

From the numerical results, it is observed that the presence of surface irregularity as well as compressive initial stresses in the SWCNTs has notable effects on torsional natural frequency. In addition, we conclude that the increase in compressive initial stress parameter increases the torsional natural frequency of SWCNTs no matter whether the surface irregularity is increasing or not.

## 6 Conclusions

In this article, a novel frequency equation of vibration of pre-stressed ISWCNT based on Donnell thin-shell theory is reported. Taking into account the effects of surface irregularity and initial stresses, vibration of SWCNTs is analyzed and the numerical results are plotted in several figures to examine the dependence of natural frequency of vibration on the surface irregularity and initial stress parameters. Some conclusions are drawn as follows:

- Based on a theoretical model for vibration of SWCNT dependence of the surface irregularity and initial

stresses, novel equation of motion and natural frequency equation are derived.

- The natural frequencies are calculated for various surface irregularity and initial stress parameters, with detailed comparison to the results, which neglect the effects of surface irregularity or the initial stresses on the vibration.
- The presence of surface irregularity and initial stresses has notable effects on the natural frequency of SWCNTs.
- To the authors' best knowledge, the effect of surface irregularity and initial stresses on the vibration behavior of SWCNTs has not yet been studied, and the present work is an attempt to find out this effectiveness, and the results of the present analysis may serve as useful references for the application and design of nano oscillators and nanodevices, in which SWCNTs act as the most prevalent nanocomposite structural element.

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