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MAXIMUMS OF UPPER SEMICONTINUOUS STRONG ŚWIĄTKOWSKI FUNCTIONS

Abstract. In this paper we characterize the family of the maximums of upper semicontinuous strong Świątkowski functions and we show that the family of all upper semicontinuous strong Świątkowski functions is not closed with respect to maximums.

1. Preliminaries

The letters $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. The symbol $\mathrm{I}(a,b)$ denotes the open interval with endpoints a and b. For each $A \subset \mathbb{R}$ we use the symbol $\operatorname{Int} A$ to denote its interior.

Let I be an interval and $f\colon I \to \mathbb{R}$. We say that f is a *Darboux function* if it maps connected sets onto connected sets. We say that f is a *quasi-continuous function* [3] at a point $x \in I$ if for all open sets $U \ni x$ and $V \ni f(x)$ we have $\operatorname{Int}\big(U \cap f^{-1}(V)\big) \neq \emptyset$. The symbols $\mathcal{C}(f)$ and $\mathcal{Q}(f)$ will stand for the set of points of continuity of f and the set of points of quasi-continuity of f, respectively. If $\mathcal{Q}(f) = I$, then we say that f is *quasi-continuous*. The function f is Darboux quasi-continuous if it is both Darboux and quasi-continuous. We say that f is a *strong Świątkowski function* [4] ($f \in \acute{\mathcal{S}}_s$), if whenever $\alpha, \beta \in I$, $\alpha < \beta$, and $y \in \mathrm{I}\big(f(\alpha), f(\beta)\big)$, there is an $x_0 \in (\alpha,\beta) \cap \mathcal{C}(f)$ such that $f(x_0) = y$. The function f is upper semicontinuous strong Świątkowski if it is both upper semicontinuous and strong Świątkowski. The symbol $\mathcal{U}(f)$ denotes $\bigcup\big\{(a,b) : f\restriction(a,b) \in \acute{\mathcal{S}}_s\big\}$.

Let $f\colon \mathbb{R} \to \mathbb{R}$. If $A \subset \mathbb{R}$ and x is a limit point of A, then let

$$\overline{\lim}(f, A, x) = \overline{\lim_{t \to x, t \in A}}\, f(x).$$

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Similarly we define $\overline{\lim}(f, A, x^-)$ and $\overline{\lim}(f, A, x^+)$. Moreover we write $\overline{\lim}(f, x)$ instead of $\overline{\lim}(f, \mathbb{R}, x)$, etc.

2. Introduction

In 1992 T. Natkaniec proved the following result [7, Proposition 3].

Theorem 2.1. *For every function f the following conditions are equivalent*:

a) *there are quasi-continuous functions g_1 and g_2 with* $f = \max\{g_1, g_2\}$,
b) *the set $\mathbb{R} \setminus \mathcal{Q}(f)$ is nowhere dense, and $f(x) \leq \overline{\lim}\big(f, \mathcal{C}(f), x\big)$ for each $x \in \mathbb{R}$.*

(In 1996 this theorem was generalized by J. Borsík for functions defined on regular second countable topological spaces [1].) He remarked also that if a function f can be written as the maximum of Darboux quasi-continuous functions, then

$$f(x) \leq \min\big\{\overline{\lim}(f, \mathcal{C}(f), x^-), \overline{\lim}(f, \mathcal{C}(f), x^+)\big\} \quad \text{for each } x \in \mathbb{R}, \tag{1}$$

and asked whether the following conjecture is true [7, Remark 3].

Conjecture 2.2. If f is a function such that $\mathbb{R} \setminus \mathcal{Q}(f)$ is nowhere dense and condition (1) holds, then there are Darboux quasi-continuous functions g_1 and g_2 with $f = \max\{g_1, g_2\}$.

In 1999 A. Maliszewski showed that this conjecture is false, and proved some facts about the maximums of Darboux quasi-continuous functions [5]. However the problem of characterization of the maximums of Darboux quasi-continuous functions is still open.

In 2002 I proved the following theorem [8, Theorem 4.1].

Theorem 2.3. *For every function f the following conditions are equivalent*:

a) *there are functions $g_1, g_2 \in \acute{\mathcal{S}}_s$ with* $f = \max\{g_1, g_2\}$,
b) *the set $\mathcal{U}(f)$ is dense in $\mathbb{R}$ and*

$$f(x) \leq \min\big\{\overline{\lim}(f, \mathcal{C}(f), x^+), \overline{\lim}(f, \mathcal{C}(f), x^-)\big\} \quad \textit{for each } x \in \mathbb{R}.$$

In this paper we characterize the family of the maximums of upper semicontinuous strong Świątkowski functions (Theorem 4.1), and we show that the family of all upper semicontinuous strong Świątkowski functions is not closed with respect to maximums (Example 4.2).

3. Auxiliary lemmas

Lemma 3.1 is an immediate consequence of [6, Lemma 1].

Lemma 3.1. *Let $I = [x_0, x_1]$ and $f\colon I \to \mathbb{R}$. If $\inf f[I] > -\infty$ and f is lower semicontinuous at x_0 and x_1, then there is a continuous function φ such that $\varphi \le f$ on I and $\varphi(x_i) = f(x_i)$ for $i \in \{0, 1\}$.*

The proofs of next two lemmas we can find in [8, Lemmas 3.2 and 3.3].

Lemma 3.2. *Let $a_0 < a_1 < a_2$. If $f{\restriction}[a_{i-1}, a_i] \in \acute{\mathcal{S}}_s$ for $i \in \{1, 2\}$ and $a_1 \in \mathcal{C}(f)$, then $f{\restriction}[a_0, a_2] \in \acute{\mathcal{S}}_s$.*

Lemma 3.3. *If I is a compact interval and $I \subset \mathcal{U}(f)$, then $f{\restriction}I \in \acute{\mathcal{S}}_s$.*

The proofs of Lemma 3.4, Lemma 3.5 and condition a) of Theorem 4.1 are similar to proofs of [8, Lemmas 3.4, 3.6 and Theorem 4.1], respectively.

Lemma 3.4. *Let $I = [x_0, x_2]$ be an interval. Assume that $f\colon I \to \mathbb{R}$ is upper semicontinuous strong Świątkowski function, $x_0, x_2 \in \mathcal{C}(f)$ and $J \subset (-\infty, \sup f[I])$ is a compact interval. Then there are upper semicontinuous strong Świątkowski functions $g_1, g_2\colon I \to \mathbb{R}$, such that $f = \max\{g_1, g_2\}$ and for $i \in \{1, 2\}$, $g_i[I] \supset J$ and $g_i(x_j) = f(x_j)$ for $j \in \{0, 2\}$.*

Proof. If $\inf f[I] = -\infty$, then $f\big[[x_0, x_2]\big] \supset J$, and we can set $g_1 = g_2 = f$. So, assume $\inf f[I] > -\infty$.

Since $f \in \acute{\mathcal{S}}_s$ and J is a compact interval, there is an $x_1 \in (x_0, x_2) \cap \mathcal{C}(f)$ with $f(x_1) > \max J$. Define

$$f_1(x) = \begin{cases} f(x) & \text{if } x \notin \big\{(x_0 + x_1)/2, (x_1 + x_2)/2\big\}, \\ \min\big\{\min J, f(x)\big\} & \text{otherwise.} \end{cases}$$

We have $\inf f_1[I] > -\infty$ and $x_0, x_1, x_2 \in \mathcal{C}(f_1)$. By Lemma 3.1, there is a continuous function $\varphi\colon I \to \mathbb{R}$ such that $\varphi \le f_1 \le f$ on I and $\varphi(x_i) = f_1(x_i) = f(x_i)$ for $i \in \{0, 1, 2\}$. For $i \in \{1, 2\}$ define

$$g_i(x) = \begin{cases} \varphi(x) & \text{if } x \in [x_{i-1}, x_i], \\ f(x) & \text{if } x \in [x_{2-i}, x_{3-i}]. \end{cases}$$

Clearly $f = \max\{g_1, g_2\}$ and g_1, g_2 are upper semicontinuous. Fix $i \in \{1, 2\}$. By Lemma 3.2 $g_i \in \acute{\mathcal{S}}_s$. Moreover, since

$$\varphi\big((x_{i-1} + x_i)/2\big) \le f_1\big((x_{i-1} + x_i)/2\big) \le \min J$$

and $\varphi(x_1) = f(x_1) > \max J$, we obtain

$$g_i[I] \supset \varphi\big[[x_{i-1}, x_i]\big] \supset \big[\varphi((x_{i-1} + x_i)/2), \varphi(x_1)\big] \supset J.$$

(Recall that φ is continuous.) Clearly $g_i = f$ on $\{x_0, x_2\}$, which completes the proof. ∎

Lemma 3.5. *Let $f\colon \mathbb{R}\to\mathbb{R}$. If the set $\mathcal{U}(f)$ is dense in $\mathbb{R}$ and*

$$\overline{\lim}(f,x^+)=\overline{\lim}(f,\mathcal{C}(f),x^+)=\overline{\lim}(f,\mathcal{C}(f),x^-)=\overline{\lim}(f,x^-)=f(x)$$

for each $x\in\mathbb{R}$, then there are upper semicontinuous functions $g_1,g_2\colon\mathbb{R}\to\mathbb{R}$ such that $f=\max\{g_1,g_2\}$ on $\mathbb{R}$,

$$\mathcal{U}(f)\subset\mathcal{U}(g_1)\cap\mathcal{U}(g_2), \tag{2}$$

and for $i\in\{1,2\}$,

$$\forall_{a\notin\mathcal{U}(f)}\ \forall_{\delta>0}\quad g_i\big[(a,a+\delta)\cap\mathcal{C}(g_i)\big]\supset\big(-\infty,f(a)\big), \tag{3}$$

$$\forall_{a\notin\mathcal{U}(f)}\ \forall_{\delta>0}\quad g_i\big[(a-\delta,a)\cap\mathcal{C}(g_i)\big]\supset\big(-\infty,f(a)\big). \tag{4}$$

Proof. Let $\mathcal{U}(f)=\mathbb{R}$. Then, by assumption, f is upper semicontinuous strong Świątkowski function, and we can set $g_1=g_2=f$. In the opposite case write $\mathcal{U}(f)$ as the union of a family $\mathcal{I}$, consisting of nonoverlapping compact intervals, such that

(5) for each $x\in\mathcal{U}(f)$, there are $I_1,I_2\in\mathcal{I}$ with $x\in\operatorname{Int}(I_1\cup I_2)$,

(6) if $I\in\mathcal{I}$ and x is an endpoint of I, then $x\in\mathcal{C}(f)$.

For each $I\in\mathcal{I}$ define $J_I=[c_I,d_I]$, where

$$d_I=\min\big\{\sup f[I]-r_I,1/r_I\big\},\quad c_I=\min\big\{d_I-1,-1/r_I\big\},$$

and $r_I=\operatorname{dist}\big(I,\mathbb{R}\setminus\mathcal{U}(f)\big)>0$. Obviously $I\subset\mathcal{U}(f)$, whence $f{\restriction}I\in\acute{\mathcal{S}}_s$, and since

$$\overline{\lim}(f,x^+)=\overline{\lim}(f,x^-)=f(x)\quad\text{for each } x\in I,$$

then $f{\restriction}I$ is upper semicontinuous. By Lemma 3.4 there are upper semicontinuous strong Świątkowski functions $g_{1I},g_{2I}\colon I\to\mathbb{R}$ such that $g_{iI}[I]\supset J_I$, $f(x)=g_{iI}(x)$ whenever x is an endpoint of I ($i\in\{1,2\}$), and $f{\restriction}I=\max\big\{g_{1I},g_{2I}\big\}$. For $i\in\{1,2\}$ define

$$g_i(x)=\begin{cases} g_{iI}(x) & \text{if } x\in I, I\in\mathcal{I},\\ f(x) & \text{otherwise.}\end{cases}$$

By (5), (6) and Lemma 3.2, we can easily see that condition (2) holds. Evidently $f=\max\big\{g_1,g_2\big\}$ on $\mathbb{R}$. Fix an $i\in\{1,2\}$. Observe that $g_i(a)\geq\overline{\lim}(g_i,a^-)$ for each $a\notin\mathcal{U}(f)$. Indeed, if there was $a\notin\mathcal{U}(f)$ such that $g_i(a)<\overline{\lim}(g_i,a^-)$, we would have $f(a)<\overline{\lim}(g_i,a^-)\leq\overline{\lim}(f,a^-)$, a contradiction. Similarly, $g_i(a)\geq\overline{\lim}(g_i,a^+)$ for each $a\notin\mathcal{U}(f)$. So, g_i is upper semicontinuous.

Now we will verify that condition (3) holds. Let $a\notin\mathcal{U}(f)$, $\delta>0$, and $y<f(a)$. Choose $y'\in\big(y,f(a)\big)$ and let

$$\delta'=\min\big\{y'-y,\delta,1/(|y|+1)\big\}.$$

There is an interval $I \in \mathcal{I}$ such that $I \subset (a, a+\delta')$ and $\sup f[I] > y'$. Since $r_I < \delta'$, we have

$$d_I = \min\{\sup f[I] - r_I, 1/r_I\} \geq \min\{y' - \delta', |y| + 1\} \geq y$$

and

$$c_I \leq -1/r_I \leq -|y| - 1 < y.$$

Hence

$$y \in J_I = [c_I, d_I] \subset g_{iI}\big[I \cap \mathcal{C}(g_i)\big] = g_i\big[I \cap \mathcal{C}(g_i)\big] \subset g_i\big[(a, a+\delta) \cap \mathcal{C}(g_i)\big].$$

Similarly we can show that condition (4) holds. ∎

4. Main result

Theorem 4.1. *For every function $f : \mathbb{R} \to \mathbb{R}$ the following conditions are equivalent*:

a) *there are upper semicontinuous strong Świątkowski functions g_1, g_2 such that $f = \max\{g_1, g_2\}$*,

b) *the set $\mathcal{U}(f)$ is dense in $\mathbb{R}$ and for each $x \in \mathbb{R}$*

$$\overline{\lim}(f, x^+) = \overline{\lim}(f, \mathcal{C}(f), x^+) = \overline{\lim}(f, \mathcal{C}(f), x^-) = \overline{\lim}(f, x^-) = f(x). \tag{7}$$

Proof. a) ⇒ b). Assume that there are upper semicontinuous strong Świątkowski functions g_1, g_2 with $f = \max\{g_1, g_2\}$. Then by Theorem 2.3, the set $\mathcal{U}(f)$ is dense in $\mathbb{R}$. Now suppose that there is an $x \in \mathbb{R}$ such that condition (7) is not satisfied. Without loss of generality we can consider the following two cases.

Case 1. $\overline{\lim}(f, x^+) < f(x)$.

Then obviously $\overline{\lim}(f, \mathcal{C}(f), x^+) < f(x)$, which contradicts condition b) of Theorem 2.3.

Case 2. $\overline{\lim}(f, \mathcal{C}(f), x^+) > f(x)$.

Then clearly $\overline{\lim}(f, x^+) > f(x)$. Since $f = \max\{g_1, g_2\}$, we have

$$\overline{\lim}(g_1, x^+) = \overline{\lim}(f, x^+) \quad \text{or} \quad \overline{\lim}(g_2, x^+) = \overline{\lim}(f, x^+),$$

and $f(x) \geq g_i(x)$ for $i \in \{1, 2\}$. But g_1, g_2 are upper semicontinuous, a contradiction.

Observe that all other cases are analogous. So, condition (7) holds for each $x \in \mathbb{R}$, which completes the first part of the proof.

b) ⇒ a). By Lemma 3.5, there are upper semicontinuous functions g_1, g_2 such that $f = \max\{g_1, g_2\}$ and conditions (2)–(4) are fulfilled. Fix an $i \in \{1, 2\}$. We will show that $g_i \in \acute{\mathcal{S}}_s$.

Let $\alpha < \beta$ and $y \in \mathrm{I}\big(g_i(\alpha), g_i(\beta)\big)$. Without loss of generality we may assume that $g_i(\alpha) < g_i(\beta)$. If $[\alpha, \beta] \subset \mathcal{U}(f)$, then by (2), $[\alpha, \beta] \subset \mathcal{U}(g_i)$, and by Lemma 3.3, there is an $x \in (\alpha, \beta) \cap \mathcal{C}(g_i)$ with $g_i(x) = y$. So, assume that $[\alpha, \beta] \setminus \mathcal{U}(f) \neq \emptyset$. We consider two cases.

Case 1. $\beta \notin \mathcal{U}(f)$.

By assumption, $y < g_i(\beta) \leq f(\beta)$ and by (4), there is an $x \in (\alpha, \beta) \cap \mathcal{C}(g_i)$ such that $g_i(x) = y$.

Case 2. $\beta \in \mathcal{U}(f)$.

Put $\gamma = \max\big\{[\alpha, \beta] \setminus \mathcal{U}(f)\big\}$. Then $\gamma < \beta$ and $\gamma \notin \mathcal{U}(f)$. By (3), there is an $\eta \in (\gamma, \beta) \cap \mathcal{C}(g_i)$ such that $g_i(\eta) < y$. By (2), we have $[\eta, \beta] \subset \mathcal{U}(f) \subset \mathcal{U}(g_i)$. So by Lemma 3.3, there is an $x \in (\eta, \beta) \cap \mathcal{C}(g_i) \subset (\alpha, \beta) \cap \mathcal{C}(g_i)$ such that $g_i(x) = y$. ■

In 2002 I proved that the maximal class with respect to maximums for the family of strong Świątkowski functions (i.e. the family of all functions whose the maximum with every element of $\acute{\mathcal{S}}_s$ belongs to $\acute{\mathcal{S}}_s$) consists of constant functions only [9, Corollary 3.6]. It is known that the maximum of two upper semicontinuous functions is upper semicontinuous (see e.g. [2, p. 83]). The example below shows that the family of all upper semicontinuous strong Świątkowski functions is not closed with respect to maximums.

Example 4.2. There is an upper semicontinuous function $f\colon \mathbb{R} \to \mathbb{R}$ which is the maximum of two upper semicontinuous strong Świątkowski functions and which is not strong Świątkowski functions.

Construction. Define

$$f(x) = \begin{cases} \sin x^{-1} + x + \operatorname{sgn} x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$

Then 0 is the only point of discontinuity of f. Observe that $f(x) \neq 0 \in \operatorname{Int} f[\mathbb{R}]$ for each $x \in (-1, 1) \cap \mathcal{C}(f)$, which proofs that f is not strong Świątkowski functions. Since the set $\mathcal{U}(f) = \mathbb{R} \setminus \{0\}$ is dense in $\mathbb{R}$ and

$$\overline{\lim}(f, x^+) = \overline{\lim}(f, \mathcal{C}(f), x^+) = \overline{\lim}(f, \mathcal{C}(f), x^-) = \overline{\lim}(f, x^-) = f(x)$$

for each $x \in \mathbb{R}$, the function f is upper semicontinuous, and by Theorem 4.1, there are upper semicontinuous strong Świątkowski functions g_1, g_2 with $f = \max\{g_1, g_2\}$. ■

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