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ON A REFINEMENT OF THE MAJORISATION TYPE INEQUALITY

Abstract. In this note, two mean value theorems are proved by using some recent results by Barnett et al. [N. S. Barnett, P. Cerone, S. S. Dragomir, *Majorisation inequalities for Stieltjes integrals*, Appl. Math. Lett. 22 (2009), 416-421]. A new class of Cauchy type means for two functions is studied. Logarithmic convexity for differences of power means is proved. Monotonicity of Cauchy means is shown.

1. Introduction

Let $x, y : [a,b] \to \mathbb{R}$ be two monotonic non-increasing real functions. The function x is said to *majorise* y, if

$$\int_a^s x(t)\,dt \geq \int_a^s y(t)\,dt \quad \text{for } s \in [a,b]$$

and

$$\int_a^b x(t)\,dt = \int_a^b y(t)\,dt$$

(see [1, p. 417], [3, p. 324]).

The following result is known as *majorisation theorem* for integrals (see [1, p. 417], [3, p. 325]).

Theorem 1.1. [1, 3] *Let $x, y : [a,b] \to I$ be two monotonic non-increasing real functions, where $I \subset \mathbb{R}$ is an interval.*

The function x majorises y iff the inequality

$$\int_a^b F(x(t))\,dt \geq \int_a^b F(y(t))\,dt$$

holds for all continuous convex functions $F : I \to \mathbb{R}$ such that the integrals exist.

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In [1] the following majorisation type theorem for the Stieltjes integral and its refinement have been proved (cf. [2, p. 11], [3, pp. 324-325]).

Theorem 1.2. [1] *Let $F : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous convex function on interval I and let $x, y, p, u : [a, b] \to I$ be real functions such that:*

(i) *x, y, p, u are continuous on $[a, b]$ with $p(t) \geq 0$ for any $t \in [a, b]$;*
(ii) *u is monotonic non-decreasing on $[a, b]$;*
(iii) *p is of bounded variation on $[a, b]$;*
(iv) *y is monotonic non-decreasing (non-increasing) and $x - y$ is monotonic non-decreasing (non-increasing) on $[a, b]$ and*

$$\int_a^b p(t)x(t)\,du(t) = \int_a^b p(t)y(t)\,du(t).$$

Then

$$\int_a^b p(t)F(x(t))\,du(t) \geq \int_a^b p(t)F(y(t))\,du(t). \tag{1}$$

Proposition 1.3. [1] *Let $G : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interior I° of interval I and such that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with the property that $\gamma \leq \frac{d^2G(z)}{dz^2} \leq \Gamma$ for any $z \in I^\circ$, and let $x, y, p, u : [a, b] \to I$ be real functions such that the conditions* (i)–(iv) *of Theorem* 1.2 *are satisfied.*

Then

$$\begin{aligned} \frac{1}{2}\Gamma\int_a^b p(t)\left[x^2(t) - y^2(t)\right] du(t) &\geq \int_a^b p(t)G(x(t))\,du(t) - \int_a^b p(t)G(y(t))\,du(t) \\ &\geq \frac{1}{2}\gamma\int_a^b p(t)\left[x^2(t) - y^2(t)\right] du(t). \end{aligned} \tag{2}$$

Remark 1.4. It was proved in [1, Remark 4] that if statements (i)–(iv) of Theorem 1.2 are valid, then we have

$$\int_a^b p(t)\left[x^2(t) - y^2(t)\right] du(t) \geq 0.$$

By using equality condition for Čebyšev inequality [3, p. 197], we have that

$$\int_a^b p(t)\left[x^2(t) - y^2(t)\right] du(t) = 0 \text{ iff } x(t) - y(t) \text{ or } x(t) + y(t) \text{ is constant.}$$

The purpose of this paper is to introduce and study a new class of Cauchy type means for two functions. To this end, we prove two mean value theorems for Stieltjes integrals by using the above results (see Section 2). In Section 3 we show the logarithmic convexity for differences of power means. We also prove the monotonicity of Cauchy means (see Section 4). Finally, in Section 5 we give some interpretations of the results in probability.

2. Mean value theorems

First we will give a mean value theorem.

Theorem 2.1. *Let $f \in C^2(I)$, where I is a closed interval in $\mathbb{R}$. If x, y, p, u satisfy conditions* (i)–(iv) *from Theorem* 1.2, *then there exists $\xi \in I$ such that*

$$\int_a^b p(t)\,[f(y(t)) - f(x(t))]\,du(t) = \frac{f''(\xi)}{2}\left\{\int_a^b p(t)\,\left[y^2(t) - x^2(t)\right]\,du(t)\right\}. \tag{3}$$

Proof. Since f'' is continuous on I, so $\gamma \le f''(z) \le \Gamma$ for $z \in I$, where $\gamma = \min_{z\in I} f''(z)$ and $\Gamma = \max_{z\in I} f''(z)$. By Proposition 1.3 we get

$$\begin{aligned} \frac{1}{2}\gamma\int_a^b p(t)\,\left[x^2(t) - y^2(t)\right]\,du(t) &\le \int_a^b p(t)\,[f(x(t)) - f(y(t))]\,du(t) \\ &\le \frac{1}{2}\Gamma\int_a^b p(t)\,\left[x^2(t) - y^2(t)\right]du(t). \end{aligned} \tag{4}$$

In consequence, in the case when

$$\int_a^b p(t)[x^2(t) - y^2(t)]\,du(t) = 0,$$

we also have

$$\int_a^b p(t)[f(x(t)) - f(y(t))]\,du(t) = 0.$$

So, it is sufficient to define ξ as a point in I, because both the sides of equality (3) are zeros.

Now, consider the case when

$$\int_a^b p(t)[x^2(t) - y^2(t)]\,du(t) > 0.$$

From (4) we obtain

$$\gamma \le \frac{2\int_a^b p(t)\,[f(x(t)) - f(y(t))]\,du(t)}{\int_a^b p(t)\,[x^2(t) - y^2(t)]\,du(t)} \le \Gamma.$$

Now using the fact that for $\rho \in [\gamma, \Gamma]$ there exists $\xi \in I$ such that $f''(\xi) = \rho$, we get (3). By Remark 1.4 the proof is completed. ∎

Remark 2.2. In the proof of Theorem 2.1 if $\gamma > 0$ and $x(t) - y(t)$ and $x(t) + y(t)$ are non-constants, then

$$\int_a^b p(t)\,[f(x(t)) - f(y(t))]\,du(t) > 0.$$

Theorem 2.3. *Let $f, g \in C^2(I)$, where I is a closed interval in $\mathbb{R}$. If x, y, p, u satisfy conditions* (i)–(iv) *from Theorem* 1.2 *and $x(t) - y(t)$ and $x(t) + y(t)$ are non-constants, then there exists $\xi \in I$ such that*

$$\frac{f''(\xi)}{g''(\xi)} = \frac{\int_a^b p(t)\,[f(y(t)) - f(x(t))]\,du(t)}{\int_a^b p(t)\,[g(y(t)) - g(x(t))]\,du(t)}, \tag{5}$$

provided that the denominators are non-zero.

Proof. Let the function $k \in C^2(I)$ be defined by

$$k = c_1 f - c_2 g,$$

where c_1 and c_2 are defined as

$$c_1 = \int_a^b p(t)\,[g(y(t)) - g(x(t))]\,du(t),$$

$$c_2 = \int_a^b p(t)\,[f(y(t)) - f(x(t))]\,du(t).$$

Then, using Theorem 2.1 with $f = k$, we have

$$0 = \left(\frac{c_1 f''(\xi)}{2} - \frac{c_2 g''(\xi)}{2}\right)\int_a^b p(t)\left[y^2(t) - x^2(t)\right]du(t), \tag{6}$$

because $\int_a^b p(t)\,[k(y(t)) - k(x(t))]\,du(t) = 0$. Since

$$\int_a^b p(t)\left[y^2(t) - x^2(t)\right]du(t) < 0$$

(see Remark 1.4), (6) gives

$$\frac{c_2}{c_1} = \frac{f''(\xi)}{g''(\xi)}.$$

After putting the values of c_1 and c_2, we obtain (5). ∎

Corollary 2.4. *Let $x, y, p, u : [a, b] \to I$ be positive real valued functions such that the conditions* (i)–(iv) *of Theorem* 1.2 *are satisfied, and $x(t) - y(t)$ and $x(t) + y(t)$ are non-constants. Then for $-\infty < w \neq v \neq 0, 1 \neq w < \infty$ there exists $\xi \in I$ such that*

$$\xi^{w-v} = \frac{v(v-1)\int_a^b p(t)\,[y^w(t) - x^w(t)]\,du(t)}{w(w-1)\int_a^b p(t)\,[y^v(t) - x^v(t)]\,du(t)}. \tag{7}$$

Proof. Setting $f(z) = z^w$ and $g(z) = z^v$, $z \in I$, with $0, 1 \neq w \neq v \neq 0, 1$ in Theorem 2.3, we get (7). ∎

Remark 2.5. Since the function $\xi \to \xi^{w-v}$ is invertible, then from (7) we have

$$m \leq \left\{ \frac{v(v-1)\int_a^b p(t)\left[y^w(t) - x^w(t)\right]\, du(t)}{w(w-1)\int_a^b p(t)\left[y^v(t) - x^v(t)\right]\, du(t)} \right\}^{\frac{1}{w-v}} \leq M, \tag{8}$$

where $m = \min\{\min_t\{x(t)\}, \min_t\{y(t)\}\}$, $M = \max\{\max_t\{x(t)\}, \max_t\{y(t)\}\}$ and $I = [m, M]$.

In fact, similar result can also be given for (5). Namely, suppose that $\frac{f''}{g''}$ has inverse function. Then from (5) we have

$$\xi = \left(\frac{f''}{g''}\right)^{-1}\left(\frac{\int_a^b p(t)\left[f(y(t)) - f(x(t))\right]\, du(t)}{\int_a^b p(t)\left[g(y(t)) - g(x(t))\right]\, du(t)}\right). \tag{9}$$

So, the expression on the right hand side of (9) is also a mean of $x(t)$ and $y(t)$.

3. log-convexity for differences of power means

Let us define the function

$$\varphi_s(z) = \begin{cases} \frac{z^s}{s(s-1)}, & s \neq 0, 1; \\ -\log z, & s = 0; \\ z \log z, & s = 1 \end{cases} \qquad \text{for } z \in (0, \infty). \tag{10}$$

It is readily seen that $\varphi_s''(z) = z^{s-2} > 0$ for $z > 0$, that is φ_s is convex on $(0, +\infty)$.

In the sequel the following lemmas will be needed.

Lemma 3.1. [4] *A positive function f is log-convex in Jensen sense on an interval I, that is, for each $s, t \in I$*

$$f(s)f(t) \geq f^2\left(\frac{s+t}{2}\right),$$

if and only if the relation

$$v^2 f(s) + 2vwf\left(\frac{s+t}{2}\right) + w^2 f(t) \geq 0$$

holds for each real v, w and $s, t \in I$.

Lemma 3.2. [3, p. 2] *If f is convex on an interval $I \subseteq \mathbb{R}$, then*

$$(s_3 - s_2)f(s_1) + (s_1 - s_3)f(s_2) + (s_2 - s_1)f(s_3) \geq 0$$

holds for every $s_1, s_2, s_3 \in I$ such that $s_1 < s_2 < s_3$.

We are now in a position to give main result.

Theorem 3.3. *Let $x, y, p, u : [a, b] \to I$ be positive real valued functions such that the conditions* (i)–(iv) *of Theorem* 1.2 *are satisfied.*

If the function

$$\Gamma_s = \int_a^b p(t)\,[\varphi_s(x(t)) - \varphi_s(y(t))]\,du(t), \quad s \in \mathbb{R}, \tag{11}$$

is positive, then it is log-convex, that is for $r, s, t \in \mathbb{R}$ with $r < s < t$, we have

$$(\Gamma_s)^{t-r} \le (\Gamma_r)^{t-s}(\Gamma_t)^{s-r}. \tag{12}$$

Proof. We adapt the technique in [4]. Let us consider the function defined by

$$\mu(z) = v^2\varphi_s(z) + 2vw\varphi_r(z) + w^2\varphi_t(z), \quad z > 0,$$

where φ_s is defined by (10), $r = \frac{s+t}{2}$ and $v, w \in \mathbb{R}$. It is readily seen that

$$\mu''(z) = v^2 z^{s-2} + 2vwz^{r-2} + w^2 z^{t-2} = \left(vz^{\frac{s}{2}-1} + wz^{\frac{t}{2}-1}\right)^2 \ge 0, \quad z > 0.$$

Therefore $\mu(\cdot)$ is convex on $(0, +\infty)$. Using Theorem 1.2 we obtain

$$\int_a^b p(t)\mu(x(t))\,du(t) \ge \int_a^b p(t)\mu(y(t))\,du(t).$$

Hence we get that

$$v^2\Gamma_s + 2vw\Gamma_r + w^2\Gamma_t \ge 0.$$

By Lemma 3.1 we have that $s \to \Gamma_s$, $s \in \mathbb{R}$, is log-convex in Jensen sense. Since Γ_s is continuous for $s \in \mathbb{R}$, therefore it is a log-convex function. By Lemma 3.2 for $-\infty < r < s < t < \infty$ we derive the inequality

$$(t - s)\log\Gamma_r + (r - t)\log\Gamma_s + (s - r)\log\Gamma_t \ge 0,$$

which is equivalent to (12). ■

The following corollary extends a result of Simic [5]. Here we apply Theorem 3.3 for constant function y on $[a, b]$.

Corollary 3.4. *Let $x, y, p, u : [a, b] \to I$ be positive real valued functions with the constant function*

$$y(t) = \frac{1}{\int_a^b p(t)\,du(t)} \int_a^b p(t)x(t)\,du(t) \quad \textit{for } t \in [a, b] \tag{13}$$

such that the conditions (i)–(iv) *of Theorem* 1.2 *are satisfied.*

If the function

$$\mathbb{R} \ni s \to \frac{1}{\int_a^b p(t)\,du(t)} \int_a^b p(t)\varphi_s(x(t))\,du(t) - \varphi_s\left(\frac{1}{\int_a^b p(t)\,du(t)} \int_a^b p(t)x(t)\,du(t)\right)$$

is positive, then it is log-convex.

For interpretations of Theorem 3.3 and Corollary 3.4, see Section 5.

4. Monotonicity of Cauchy means

We give the following definition.

For positive real valued functions $x, y, p, u : [a, b] \to I$ such that conditions (i)–(iv) of Theorem 1.2 are valid, we define

$$M_{w,v} = \left(\frac{\Gamma_w}{\Gamma_v}\right)^{\frac{1}{w-v}} \quad \text{for } w \neq v \tag{14}$$

(see (11)). By Remark 2.5, the expressions $M_{w,v}$ are means. We can extend these means in other cases. Namely, for $w \neq 0, 1$, by limit we have

$$M_{w,w} = \exp\left(\frac{\int_a^b p(t)\left[y^w(t)\log y(t) - x^w(t)\log x(t)\right]\,du(t)}{\int_a^b p(t)\left[y^w(t) - x^w(t)\right]\,du(t)} - \frac{2w-1}{w(w-1)}\right),$$

$$M_{0,0} = \exp\left(\frac{\int_a^b p(t)\left[\log^2 y(t) - \log^2 x(t)\right]\,du(t)}{2\int_a^b p(t)\left[\log y(t) - \log x(t)\right]\,du(t)} + 1\right),$$

and

$$M_{1,1} = \exp\left(\frac{\int_a^b p(t)\left[y(t)\log^2 y(t) - x(t)\log^2 x(t)\right]\,du(t)}{2\int_a^b p(t)\left[y(t)\log y(t) - x(t)\log x(t)\right]\,du(t)} - 1\right).$$

As application to Theorem 3.3 we have the following result (cf. [4, Lemma 2.6]).

Corollary 4.1. *Under the assumptions of Theorem* 3.3, *let* $t, s, w, v \in \mathbb{R}$ *such that* $t \leq w, s \leq v$. *Then the following inequality is valid.*

$$M_{t,s} \leq M_{w,v}. \tag{15}$$

Proof. By Theorem 3.3, Γ_s is log-convex, i.e. $\log \Gamma_s$ is convex. Therefore it holds [3, p. 2] that

$$\frac{\log \Gamma_s - \log \Gamma_t}{s-t} \leq \frac{\log \Gamma_v - \log \Gamma_w}{v-w}$$

with $t \leq w$, $s \leq v$, $t \neq s$, $w \neq v$. Consequently,

$$\log\left(\frac{\Gamma_s}{\Gamma_t}\right)^{\frac{1}{s-t}} \leq \log\left(\frac{\Gamma_v}{\Gamma_w}\right)^{\frac{1}{v-w}}.$$

Hence, by (14), we get (15) for $s \neq t$ and $w \neq v$.

For $s = t$ and/or $w = v$ we have limiting case. ∎

5. Applications

We now give a probabilistic interpretation of Theorem 3.3 in the case $p(t) = 1$ for $t \in [a, b]$.

Let T be a random variable with values in the interval $[a, b]$. Let u be the restriction to $[a, b]$ of the cumulative distribution function of T. Assume that u is continuous. Here we have $\int_a^b p(t)\,du(t) = 1$.

Given a continuous function $x : [a, b] \to (0, \infty)$ we consider the random variable $X = x(T)$. The expectation of X is

$$EX = \int_a^b x(t)\,du(t).$$

For $s \in \mathbb{R}$, the expectation of the random variable $\varphi_s(X) = \varphi_s(x(T))$ is given by

$$E\varphi_s(X) = \int_a^b \varphi_s(x(t))\,du(t).$$

We denote

$$E_s X = \int_a^b (x(t))^s\,du(t).$$

Corollary 5.1. *Under the above notation and assumptions, let $X = x(T)$ and $Y = y(T)$ be two random variable such that $EX = EY$, where $x, y : [a, b] \to (0, \infty)$ are continuous and monotonic non-decreasing (non-increasing) functions on $[a, b]$ and $x - y$ is monotonic non-decreasing (non-increasing) on $[a, b]$.*

If the difference function

$$s \to \Gamma_s = E\varphi_s(X) - E\varphi_s(Y), \quad s \in \mathbb{R}, \tag{16}$$

is positive, then it is log-convex, that is (12) *holds.*

For example, if $r < s < t$ with $r, s, t \neq 0, 1$, then (12) gives

$$\left(\frac{E_s X - E_s Y}{s(s-1)}\right)^{t-r} \leq \left(\frac{E_r X - E_r Y}{r(r-1)}\right)^{t-s} \left(\frac{E_t X - E_t Y}{t(t-1)}\right)^{s-r}.$$

In the case y is given by (13), we obtain $E_s Y = (EX)^s$. So, we get

$$\left(\frac{E_s X - (EX)^s}{s(s-1)}\right)^{t-r} \leq \left(\frac{E_r X - (EX)^r}{r(r-1)}\right)^{t-s} \left(\frac{E_t X - (EX)^t}{t(t-1)}\right)^{s-r}.$$

Furthermore, if in addition $x(t) = t$ for $t \in [a, b]$, then $X = T$ and $E_s X = ET^s$ and therefore

$$\left(\frac{ET^s - (ET)^s}{s(s-1)}\right)^{t-r} \leq \left(\frac{ET^r - (ET)^r}{r(r-1)}\right)^{t-s} \left(\frac{ET^t - (ET)^t}{t(t-1)}\right)^{s-r}$$

(see [5, Proposition 3.3 and p. 7]).

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