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A NEW PROOF OF THE FAITHFULNESS OF THE TEMPERLEY–LIEB REPRESENTATION OF B_3

Abstract. We find a way to calculate the term with lowest exponent in the Kauffman bracket polynomial of a closed 3-string braid. This leads us to a new proof of the faithfulness of the Temperley-Lieb representation of the 3-string braid group B_3. In particular we will prove that for any link with braid index 3 either the coefficient of the lowest degree term or the coefficient of the highest degree term of the Jones polynomial is equal to ± 1.

1. Introduction

The Burau representation $\mu_n : B_n \to M_{n-1}(\mathbb{Z}[t^{\pm 1}])$ is one of the classic representations of the braid group B_n. In the 1990's it was shown this is not faithful for $n \geq 9$ (Moody [11]), $n \geq 6$ (Long and Paton [10]), and $n \geq 5$ (Bigelow [2]). On the other hand it is known to be faithful for $n = 3$ (Bigelow [1], Birman [3]). The case $n = 4$ is still an open question.

The Temperley–Lieb algebra TL_n is defined over $\mathbb{Z}[A^{\pm 1}]$, has dimension $C_n = \frac{1}{n+1}\binom{2n}{n}$, $n-1$ generators $\{U_i^i\}_{i=1}^{n-1}$, and relations:

(TL1) $U_i^i \cdot U_i^i = (-A^{-2} - A^2)U_i^i$,
(TL2) $U_i^i \cdot U_j^j \cdot U_i^i = U_i^i$ for $|i-j| = 1$,
(TL3) $U_i^i \cdot U_j^j = U_j^j \cdot U_i^i$ for $|i-j| > 1$.

In the case $n = 3$ the relation (TL3) may be omitted.

It can be shown that the map $\rho_n \colon B_n \to TL_n$, $\sigma_i \mapsto A + A^{-1}U_i^i$, extends to a representation of the braid group. It is known that the faithfulness of μ_3 and ρ_3 are equivalent (Kędziorek [9]). The question of faithfulness of ρ_n in general is not yet decided.

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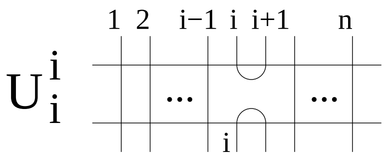

Fig. 1.

We propose a new, direct approach to the question of the faithfulness by analyzing the Kauffman bracket. We will prove the following two theorems.

Theorem 1. *The representation $\rho_3 \colon B_3 \to TL_3$ is faithful*

Theorem 2. *For every link L that may be represented by a closed 3-braid the Jones polynomial V_L of that link has the coefficient equal to ± 1 at at least one end (i.e. either the coefficient of the highest degree term or the coefficient of the lowest degree term is ± 1).*

2. Kauffman bracket

The Kauffman bracket (Kauffman [8]) of an unoriented link diagram L is a Laurent polynomial with integer coefficients $\langle L\rangle(A)$ defined by the following rules:

(1) $\langle \times \rangle = A\langle\,)(\,\rangle + A^{-1}\langle \asymp \rangle$,
(2) $\langle \bigcirc \rangle = 1$
(3) $\langle \bigcirc \sqcup L \rangle = (-A^{-2} - A^{2})\langle L \rangle$.

The Jones polynomial $V_L(t) \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$ of an oriented link L can be defined by substituting $A^2 \mapsto t^{\frac{1}{2}}$ in $(-A^{-3})^{wr(L)}\langle L\rangle(A)$, a variation of the link's Kauffman bracket, where $wr(L)$ is the difference between the number of positive and negative crossings in L. It is well known that the bracket polynomial, and therefore the Jones polynomial, is multiplicative with respect to the connected sum of diagrams/links (see also Jones [6], [7]).

If the Kauffman bracket rules (1)–(3) are applied to a braid, then we obtain a linear combination of the Kauffman diagrams with coefficients in $\mathbb{Z}[A^{\pm 1}]$ rather than a single polynomial. This may be treated as a definition of the Jones representation of B_n into the Temperley–Lieb algebra TL_n (for details on TL_n see Kauffman [8]).

Kauffman states for an unoriented link diagram are defined in [8]. Every state S has its well defined contribution $\langle S, D\rangle$ to the Kauffman bracket $\langle D\rangle$ of the link diagram D. It will be convenient to extend this notion to braids.

Following Kauffman again, we will define the contribution $\rho(S,\beta)$ of a state S to $\rho(\beta)$ in the obvious way:

Suppose the state S was obtained by splitting p crossings positively, and n crossing negatively. The state S itself is a Kauffman diagram K_S, possibly with a number, say c, of simple closed curves. Then the contribution of S to $\rho(\beta)$ is:

$$A^{p-n} \cdot (-A^{-2} - A^2)^c K_S. \tag{2.1}$$

We will also write $\langle S, \beta \rangle$ to denote the contribution of the closure of S to the bracket polynomial $\langle \beta \rangle$ of the closure of β, so that

$$\langle \beta \rangle = \sum_S \langle S, \beta \rangle, \qquad \rho(\beta) = \sum_S \rho(S, \beta). \tag{2.2}$$

We will also use what we propose to call σ_1-*states.* A σ_1-*state* of a braid is a diagram obtained by splitting all crossings, some of them positively, the rest negatively (we will only use it for braids in which σ_1^{-1} does not appear, so-called σ_1-*positive* braids). To stress the difference between Kauffman states and σ_1-states we will use notation S and $\mathbb{S}$ respectively. Like for Kauffman states we will use $\langle \mathbb{S}, \beta \rangle$ and $\rho(\mathbb{S}, \beta)$ to denote the contribution of $\mathbb{S}$ to $\langle \beta \rangle$ and to $\rho(\beta)$ respectively. Of course we have formulas analogous to 2.2.

$$\langle \beta \rangle = \sum_{\mathbb{S}} \langle \mathbb{S}, \beta \rangle, \qquad \rho(\beta) = \sum_{\mathbb{S}} \rho(\mathbb{S}, \beta). \tag{2.3}$$

Two extreme cases of σ_1-states will play a special role in our considerations: $\mathbb{S}_H$ – the horizontal state obtained by smoothing all the σ_1 crossings horizontally, and $\mathbb{S}_V$ – the vertical state obtained by smoothing all σ_1 crossings vertically.

Observation. If $\mathbb{S}$ is a non-vertical σ_1-state then the trivial Kauffman diagram $\mathbf{1}$ does not appear in $\rho(\mathbb{S}, \beta)$ (or more precisely, the coefficient of $\mathbf{1}$ in $\rho(\mathbb{S}, \beta)$ is zero).

3. σ_1-states

It is known (Dehornoy [4], [5]) that every n-string braid has a *reduced* or σ_1-*consistent* form, i.e. such a form that either all exponents of σ_1 are positive, or all are negative. In the case of a closed 3-string braid, the reduced form looks as in Figure 2.

In proving Theorem 1 we may restrict our attention to the case of σ_1-*positive* braids (those in which all occurrences of σ_1 have positive exponents) in $\ker \rho$. Moreover, the following lemma allows for further restriction of the braid's form.

Lemma 1. *Let γ be a positive 3-braid word. Assume that γ contains a $\sigma_1\sigma_2\sigma_1$ or $\sigma_2\sigma_1\sigma_2$ sequence. Then γ is equivalent to a positive braid word of the form $\sigma_2\beta$ of the same length.*

Proof. We can assume that it is the $\sigma_2\sigma_1\sigma_2$ sequence that is present in γ. It is well known that $\sigma_1\sigma_2\sigma_1\sigma_2$ is equivalent as a braid to $\sigma_2\sigma_1\sigma_2\sigma_2$ and

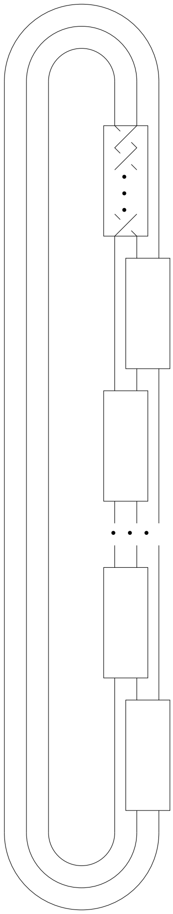

Fig. 2.

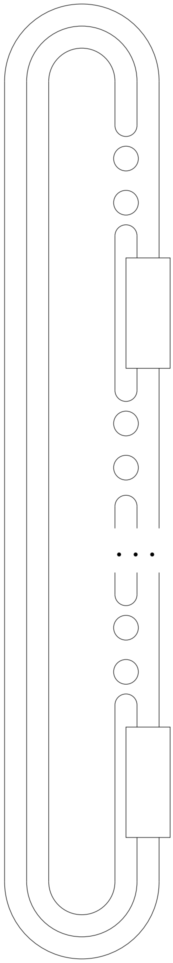

Fig. 3.

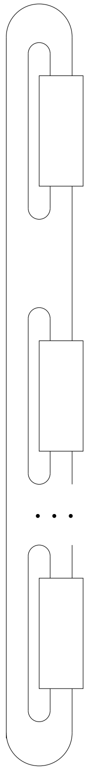

Fig. 4.

$\sigma_2\sigma_2\sigma_1\sigma_2$ is equivalent to $\sigma_2\sigma_1\sigma_2\sigma_1$. It follows that we can drag the sequence $\sigma_2\sigma_1\sigma_2$ gradually to the left staying within the initial isotopy class of the braid. Finally we obtain a braid word as required. ∎

Corollary 1. *Let γ be a σ_1-positive 3-braid word. Assume that γ is of minimum length in its conjugacy class (minimum taken over all σ_1-positive words in the conjugacy class). Assume also that the conjugacy class of γ contains none of the following:*

1. *positive braid,*
2. σ_2^n*,*
3. $\sigma_1^k\sigma_2^n$.

Then γ may be written (as a cyclic word) as $\alpha_1\sigma_2^{-i_1}\ldots\alpha_k\sigma_2^{-i_k}$, where $i_1,\ldots,i_k>0$, $k\geq 2$, in such a way that for every $1\leq i\leq k$ the sequence α_i is positive and contains no isolated σ_2 term, and moreover that α_i contains no isolated σ_1 term, except possibly at the beginning/end (in particular the sequence $\sigma_2\sigma_1\sigma_2$ does not appear in the considered braid).

Proof. This is all obvious, except we need to work on the *no single σ_2 (σ_1)* condition. Assume to the contrary, that we cannot avoid that for some s the term α_s contains a single σ_2. If it appears at either end of α_s then it may be cancelled with a σ_2^{-1} term taken from the preceding $\sigma_2^{-i_{s-1}}$ sequence or from the following $\sigma_2^{-i_s}$ sequence (in the cyclic word and therefore in the same conjugacy class). The result is still σ_1-positive and the length is smaller contradicting the assumption. If α_s contains a single σ_2 (or a single σ_1) somewhere in the middle, then it also contains a $\sigma_1\sigma_2\sigma_1$ (or a $\sigma_2\sigma_1\sigma_2$) sequence. It follows (by Lemma 1) that α_s is equivalent to a positive word beginning with a σ_2 and we proceed as before. ∎

To analyze the bracket polynomial of $\hat{\beta}$ we will consider what we proposed to call σ_1-states of the considered braid. The closure of each σ_1-state may be transformed by regular isotopy (just cancellation of $\sigma_2\sigma_2^{-1}$ and $\sigma_2^{-1}\sigma_2$ terms, in fact) into the disjoint union of a number of free circles with the connected sum of a certain number of standard diagrams of $(2,k)$-torus links, including possibly $k=\pm 1$ and $k=0$ (Figures 3 and 4). We recall the formulas for the bracket of a standard 2-torus link diagram.

Lemma 2. *Let $T_n=\widehat{\sigma^n}$ and $T_{-n}=\widehat{\sigma^{-n}}$ be standard 2-torus link diagrams, and let T_0 be the trivial diagram of the trivial 2-component link. Then:*

$$\begin{aligned}\langle T_0\rangle &= -A^{-2}-A^{2},\\ \langle T_n\rangle &= -A^{n-2}+\sum_{i=0}^{n}(-1)^{n-1-i}A^{-3n+2+4i},\\ \langle T_{-n}\rangle &= -A^{-n+2}+\sum_{i=0}^{n}(-1)^{-n-1-i}A^{3n-2-4i}.\end{aligned}$$

In fact, we only need to know the exponent of the lowest degree term $\deg_{\min}\langle T_n\rangle$. We formulate this for future reference.

Corollary 2.

$$\begin{aligned}\deg_{\min}\langle T_0\rangle &= -2,\\ \deg_{\min}\langle T_n\rangle &= -3n+2, \quad \textit{for } n\geq 2,\\ \deg_{\min}\langle T_1\rangle &= 3,\\ \deg_{\min}\langle T_n\rangle &= n-2 \quad \textit{for } n<0\end{aligned}$$

(unlike with T_1 there is no need to treat the case of T_{-1} separately).

Theorem 1 and Theorem 2 are obviously true for 3-braids of the form σ_2^n and $\sigma_1^k\sigma_2^n$. Also, positive 3-braids will need special attention (later).

For now we will consider 3-braids that are not positive (more precisely, those for which there is no positive braid in the conjugacy class) and not of the form σ_2^n or $\sigma_1^k\sigma_2^n$. We will also assume that these braids are first transformed into the special form introduced in Corollary 1. For a given state $\mathbb{S}$ let us consider $\langle\mathbb{S},\beta\rangle$ in more detail. Let d be the difference of the number of σ_1 crossings smoothed vertically and the number of σ_1 crossings smoothed horizontally to obtain $\mathbb{S}$, and let c be the number of simple closed curves in $\mathbb{S}$. As explained before, what remains when the simple closed curves are removed is the connected sum of a number of standard T_n diagrams. Given such notation we have:

$$\langle\mathbb{S},\beta\rangle = A^d\cdot(-A^{-2}-A^2)^c\cdot\prod\langle T_n\rangle, \tag{3.1}$$

Remark 1. *The coefficient of the lowest degree term in $\langle\mathbb{S},\beta\rangle$ is equal to ±1.*

Proof. It follows immediately from formula (3.1) and Lemma 2, as the right hand side of formula (3.1) is a product of polynomials, each of them obviously having the required property. □

Proposition 1. *Let β be a σ_1-positive 3-braid of the special form described in* Corollary 1 *(and not of the form σ_2^n or $\sigma_1^k\sigma_2^n$) and let $\mathbb{S}_H$ be its horizontal state. Then*

$$\deg_{\min}\langle\mathbb{S}_H,\beta\rangle = \deg_{\min}\langle\beta\rangle, \tag{3.2}$$

while for all the other σ_1-states $\mathbb{S}$ we have

$$\deg_{\min}\langle\mathbb{S}_H,\beta\rangle < \deg_{\min}\langle\mathbb{S},\beta\rangle. \tag{3.3}$$

Moreover, the coefficient of the lowest degree term in $\langle\mathbb{S}_H,\beta\rangle$ is ±1.

Proof. What we really need to prove is the sharp inequality 3.3. The equality 3.2 follows easily, and the last statement is a consequence of 3.2 and Remark 1.

Suppose that a counterexample to the Proposition exists. Such a counterexample would involve a pair $(\beta,\mathbb{S}_0)$ consisting of a braid β and a non-horizontal σ_1-state $\mathbb{S}_0$ of β such that

$$\deg_{\min}\langle\mathbb{S}_0,\beta\rangle \le \deg_{\min}\langle\mathbb{S}_H,\beta\rangle. \tag{3.4}$$

The set of all possible states $\mathbb{S}_0$ with this property for a given braid β may be ordered (partially) by the number of σ_1 crossings smoothed vertically rather than horizontally in the state $\mathbb{S}_0$. Keeping the notation as described above – $(\beta,\mathbb{S}_0)$ is assumed to be a hypothetical minimal counterexample – we formulate the following lemma.

Lemma 3. *If in the state $\mathbb{S}_0$ a certain σ_1 crossing in the sequence*

$$\ldots\sigma_2^i\sigma_1^j\sigma_2^k\ldots$$

is smoothed vertically, then all the j crossings are smoothed vertically.

Proof. Suppose Lemma 3 is false for the braid β. Let us consider another state $\mathbb{S}'$ which is identical to $\mathbb{S}_0$ except that *all* the j crossings of type σ_1 from the considered $\ldots\sigma_2^i\sigma_1^j\sigma_2^k\ldots$ sequence are smoothed horizontally. When we analyze the formulas (like formula (3.1)) for the contribution of $\mathbb{S}_0$ and $\mathbb{S}'$ to $\langle\beta\rangle$ it is clear that in both cases the 2-torus links are exactly the same, the multiplier for $\mathbb{S}'$ is of smaller degree than the one for $\mathbb{S}_0$ and the number of closed curves in $\mathbb{S}'$ is greater than in $\mathbb{S}_0$. Also, the number of σ_1 crossings smoothed vertically in $\mathbb{S}'$ is smaller than in $\mathbb{S}_0$. If this was possible, then $(\beta,\mathbb{S}')$ would be a smaller counterexample to the Proposition than $(\beta,\mathbb{S}_0)$ against assumption. ∎

We are now in position to prove Proposition 1. Using the same notation we can now choose a sequence $\sigma_2^{k_0}\sigma_1^{j_1}\sigma_2^{k_1}\ldots\sigma_1^{j_n}\sigma_2^{k_n}$ such that in the state $\mathbb{S}_0$ all the σ_1 crossings in the given sequence are smoothed vertically and that the considered sequence is of maximum length with respect to this property. By Lemma 3 we know that $n \geq 1$. This is applied to a cyclic word, therefore it is possible that the sequence considered is the whole braid word β (in this case either $k_0 = 0$ or $k_n = 0$). We now want to see if smoothing all the σ_1 crossings in the considered sequence vertically may have the effect of decreasing the $\deg_{\min}$ or of keeping it unchanged. To determine this we analyze how the suitable terms of the following formula change when we pass from $\mathbb{S}_H$ to $\mathbb{S}_0$

$$\langle\mathbb{S},\beta\rangle = A^d\cdot(-A^{-2}-A^2)^c\cdot\prod\langle T_{k_i}\rangle. \tag{3.5}$$

The change of the exponent d is easy to control. It is increased by $\sum j_i$ which has the effect of increasing $\deg_{\min}$ by $2\cdot\sum j_i$. The change of the parameter c (the number of closed curves in the state) is a decrease by $\sum(j_i-1)$ if the considered sequence is not the whole braid or by $-1+\sum(j_i-1)$ in the opposite case (because in this case one additional curve is created from the first string of the braid). While we said *decrease*, in fact the parameter c may stay unchanged (when all the j_i's are equal to 1) or may actually *increase* by 1 (if in addition to $j_i = 1$ for all i, the considered sequence is equal to the whole braid). Accordingly, this causes a suitable change in $\deg_{\min}$ — an increase in most cases, no change in one special case and a decrease by 2 in the extreme case described above. While the change in parameter d has always a favourable effect, this is not so with the change of c, as described above.

Finally, let us discuss the change in $\prod\langle T_{k_i}\rangle$. Here we replace the bracket polynomials of $T_{k_0},\ldots,T_{k_n}$ with the bracket polynomial of $T_{k_0+\cdots+k_n}$. We will consider this case in three stages. First, we will assume that $k_0,\ldots k_n$ are all positive, next we will assume that they are all negative and then we will consider the case when some of them are negative and some are positive.

We start with $k_0,\ldots k_n>0$. Now, it is time to use our assumptions about the special form of the considered braid word. We are assuming that no isolated σ_2 crossing appears in the considered braid word. It follows that we have in fact $k_i\geq 2$ for $i=0,\ldots,n$. According to Corollary 2, the contribution of $\sigma_2^{k_i}$ to $\deg_{\min}$ in $\langle\mathbb{S}_H,\beta\rangle$ is $-3k_i+2$ for $i=0,\ldots,n$. Altogether, the contribution of $\sigma_2^{k_0},\ldots,\sigma_2^{k_n}$ to $\deg_{\min}\langle\mathbb{S}_H,\beta\rangle$ is $-3\cdot(k_0+\ldots+k_n)+2(n+1)$. On the other hand, the contribution of $T_{k_0+\cdots+k_n}$ to $\deg_{\min}\langle\mathbb{S}_0,\beta\rangle$ is $-3\cdot(k_0+\ldots+k_n)+2$. This means a decrease by $2n$ which must be more than compensated if we want to prove that $\mathbb{S}_H$ is responsible for the single term of the lowest degree. But it is easy to see, that the change in the parameter d (described above) is an increase of $2\cdot\sum j_i\geq 2n$. This means that there is no contribution of degree lower than $\deg_{\min}\langle\mathbb{S}_H,\beta\rangle$ to $\langle\beta\rangle$ coming from $\langle\mathbb{S}_0,\beta\rangle$, so $\deg_{\min}\langle\mathbb{S}_H,\beta\rangle\leq\deg_{\min}\langle\mathbb{S}_0,\beta\rangle$. However, we need prove that the sharp inequality holds. To do better we can now use another property of our special form of the considered braid: in the considered situation ($k_0,\ldots k_n>0$) we have $j_i\geq 2$ for $i=1,\ldots,n$ which implies that in this special case the change in the parameter d increases the $\deg_{\min}$ by at least $4n$. Let us observe that there is no need to consider the possible decrease caused by the change in the parameter c — this may only happen when the considered sequence is equal to the whole braid and this is not the case here as the considered braid is assumed not to be positive.

Next, we consider the case of $k_0,\ldots k_n<0$. We assume that $n\geq 1$, so we have at least two negative torus link diagrams involved. An easy calculation shows that grouping the torus link diagrams into one (by smoothing the σ_1's vertically) increases the $\deg_{\min}$ by at least 2. The change in d increases it further by at least 2. Together, it more than compensates the change in the wrong direction caused possibly by the change in c.

What remains is to consider the case when at least one of the $k_i's$ is positive and at least one is negative. We will reorder them, so that all the negative ones appear first. Reordering T_k's in such a way (or any other) has no effect on the final result. What does matter is that the connected sum of T_k diagrams is replaced by $T_{k_0+\ldots+k_n}$. After such an operation it may no longer be true that a $\sigma_2\sigma_1\sigma_2$ sequence does not appear in β. Fortunately, this will not be needed anymore. We will pass from $\mathbb{S}_H$ to $\mathbb{S}_0$ in three stages.

First, we smooth all the σ_1's in the first part (the one grouping the negative σ_2 crossings). This results in an increase of $\deg_{\min}$ or in the worst case (which is that of just one negative k_i in the whole sequence) no change of $\deg_{\min}$. Secondly, we do the same for the part grouping the positive σ_2 crossings. While single σ_1's might now appear in this part of the sequence, it is still true that the contribution to $\deg_{\min}$ is at the worst the same and not smaller. Now, we smooth the final set of σ_1 crossings separating the two parts and obviously the contribution of $\mathbb{S}_0$ to $\deg_{\min}$ is smaller than that of $\mathbb{S}_H$. This is because at this last stage we connect in one box two standard 2-torus link diagrams, one positive, of length at least 2 and the other negative, and this always results in a decrease of $\deg_{\min}$. This completes the proof of Proposition 1. ∎

Proof of Theorem 1. Suppose that there is a non-trivial braid β in $\ker\rho$. First, we will exclude the possibility of β being a positive braid of length $n > 0$. It is well-known that in such case the coefficient of the unit Kauffman diagram $\mathbf{1}$ in $\rho(\beta)$ is A^n, rather than 1, which means that $\beta \notin \ker\rho$. The same argument works for any braid that is conjugate to a positive braid, and more generally, for any braid with non-zero writhe. As mentioned before, there are no non-trivial braids of the form σ_2^n or $\sigma_1^k\sigma_2^n$ in $\ker\rho$. Therefore, what remains is to consider braids of the form described in Corollary 1 with null writhe.

Let m be the number of σ_1 crossings in β. Then the vertical state $\mathbb{S}_V$ is a disjoint union of one free circle and T_{-m} (it is T_{-m} with this specific parameter $-m$ because of the assumption about the sum of exponents in the considered braid being equal to zero). From formula 3.1 we obtain

$$\langle \mathbb{S}_V, \beta\rangle = A^m \cdot (-A^{-2} - A^2) \cdot \langle T_{-m}\rangle. \tag{3.6}$$

Then by Corollary 2

$$\deg_{\min}\langle \mathbb{S}_V, \beta\rangle = m - 2 - m - 2 = -4. \tag{3.7}$$

On the other hand, by Proposition 1,

$$\deg_{\min}\langle \beta\rangle = \deg_{\min}\langle \mathbb{S}_H, \beta\rangle < \deg_{\min}\langle \mathbb{S}_V, \beta\rangle = -4, \tag{3.8}$$

so

$$\deg_{\min}\langle \beta\rangle < -4. \tag{3.9}$$

It follows that $\rho(\beta) \neq \mathbf{1}$. Otherwise we would have $\rho(\beta) = \rho(1)$ (here 1 means the trivial 3-string braid). That would imply that

$$\langle \beta\rangle = \langle 1\rangle = (-A^{-2} - A^2)^2. \tag{3.10}$$

However the minimum degree of the left-hand side is strictly smaller than -4 (3.9), while the minimum degree of the right-hand side is -4. ∎

Proof of Theorem 2. Theorem 2 is checked directly for links represented by braids of the form σ_2^n, $\sigma_1^k\sigma_2^n$. For positive braids one can easily prove that the highest degree term has coefficient 1. Namely, the vertical state contains the unique term of the highest degree (because it is easy to see that when we go from the vertical state to the state with just one crossing smoothed horizontally, then the maximum degree always decreases by 2 and in all subsequent steps it cannot increase). Proposition 1 proves the remaining σ_1-positive cases. The case of negative or σ_1-negative 3-braids is analogous. ∎

Example 1. The assumption that the considered braid is not positive is necessary in Proposition 1. For example the bracket polynomial of the braid $\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1$ is $2A^{-6} + A^2 + A^{10}$.

Example 2. Another example shows that a σ_1-positive 3-braid may have a coefficient of the highest degree term that is not equal to ± 1. In the example below the writhe equals 6. Let us remark that there is no such example with writhe 0 since in such case it is well known that the bracket polynomial is symmetric. The braid $\sigma_1\sigma_2^{-2}\sigma_1^2\sigma_2^5$ has bracket polynomial

$$A^2 - 2A^6 + 3A^{10} - 5A^{14} + 5A^{18} - 5A^{22} + 4A^{26} - 2A^{30} + 2A^{34}.$$

Example 3. We end with an example of a 4-braid showing that the above is not necessarily true for braids with more than 3 strings. The σ_1-positive 4-braid with null writhe

$$\sigma_2\sigma_3\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^2\sigma_3^{-1}\sigma_2^{-2}\sigma_1\sigma_2^{-1}\sigma_1\sigma_3^{-2}\sigma_2^{-2}\sigma_1^2\sigma_2\sigma_3^{-1}\sigma_2^{-2}\sigma_1^2\sigma_3^{-1}$$

has bracket polynomial

$$-2A^{-30}+9A^{-26}-24A^{-22}+45A^{-18}-61A^{-14}+61A^{-10}-45A^{-6}+14A^{-2}$$
$$+19A^2-50A^6+68A^{10}-69A^{14}+54A^{18}-31A^{22}+12A^{26}-2A^{30}.$$

The coefficients of both the term with the lowest exponent and the term with the highest exponent, are not equal to ± 1.

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