DEMONSTRATIO MATHEMATICA
Vol. XLIV No 3 2011

10.1515/dema-2013-0321

Sergey A. Solovyov

EXTENDED-ORDER ALGEBRAS AS A GENERALIZATION OF POSETS

Abstract. Motivated by the recent study of several researchers on *extended-order algebras*, introduced by C. Guido and P. Toto as a possible common framework for the majority of algebraic structures used in many-valued mathematics, the paper focuses on the properties of homomorphisms of the new structures, considering extended-order algebras as a generalization of partially ordered sets. The manuscript also introduces the notion of *extended-relation algebra* providing a new framework for developing the theory of rough sets.

1. Introduction

The notion of *partially ordered set* is undoubtedly one of the cornerstones of modern abstract algebra. Introduced by F. Hausdorff [18] at the beginning of the previous century (notice that the axioms used in the definition of an order relation had already been considered by G. Leibniz around 1690; moreover, G. Cantor [2] presented in 1895 the notion of *totally ordered set*), the concept soon drew the attention of many researchers, who successfully developed the theory of partially ordered sets (or *posets* for short) up to its present state, when it has found a way in almost every area of (not only exact) science. No wonder that a significant amount of time has been spent to provide various generalizations of the concept. In particular, the topic of this paper was motivated by the following three approaches.

In 1974, H. Rasiowa [30] has come out with the notion of *implicative algebra*, introduced as a possible tool for a uniform algebraic treatment of various logics.

2000 *Mathematics Subject Classification*: 06A75, 06A11, 06F99, 18B99.

Key words and phrases: categorical equivalence, completion of a poset, (co)product of objects, coseparator, d-algebra, epimorphism, extended-order algebra, extended-relation algebra, free object, Hilbert algebra, implicative algebra, initial morphism, poset, preordered set, quantale, rough set, $\bigvee$-semilattice.

This research was supported by ESF Project 2009/0223/1DP/1.1.1.2.0/09/APIA/VIAA/008 of the University of Latvia.

Definition 1. An *implicative algebra* is an abstract algebra $(A, \Rightarrow, V)$, where V is a nullary operation and $\Rightarrow$ is a binary operation such that for every $a, b, c \in A$, the following conditions hold:

(1) $a \Rightarrow a = V$;
(2) if $a \Rightarrow b = V$ and $b \Rightarrow c = V$, then $a \Rightarrow c = V$;
(3) if $a \Rightarrow b = V$ and $b \Rightarrow a = V$, then $a = b$;
(4) $a \Rightarrow V = V$.

In 1999, J. Neggers and H. S. Kim [29] introduced the notion of *d-algebra* as yet another generalization of BCK-algebras.

Definition 2. A *d-algebra* is a non-empty set X with a constant 0 and a binary operation $*$ satisfying for every $x, y \in X$ the following axioms:

(1) $x * x = 0$;
(2) $0 * x = 0$;
(3) if $x * y = 0$ and $y * x = 0$, then $x = y$.

A d-algebra $(X, *, 0)$ is called *d-transitive* provided that for every $x, y, z \in X$, $x * y = 0$ and $y * z = 0$ imply $x * z = 0$.

In 2008, C. Guido and P. Toto [17] provided the concept of *weak extended-order algebra*, deemed to serve as a common framework for the majority of algebraic structures used in many-valued mathematics.

Definition 3. A *weak extended-order algebra* (*w-eo algebra*) is a triple $(L, \rightarrow, \top)$, where L is a non-empty set, $L \times L \xrightarrow{\rightarrow} L$ is a binary operation on L, and $\top$ is a distinguished element of L such that the following conditions are satisfied for every $a, b, c \in L$:

(1) $a \rightarrow \top = \top$ (upper bound condition);
(2) $a \rightarrow a = \top$ (reflexivity condition);
(3) if $a \rightarrow b = \top$ and $b \rightarrow a = \top$, then $a = b$ (antisymmetry condition);
(4) if $a \rightarrow b = \top$ and $b \rightarrow c = \top$, then $a \rightarrow c = \top$ (weak transitivity condition).

It is important to underline immediately that the notion of w-eo algebra is completely different from that of *lattice-valued partially ordered set*, which relies on a fuzzification of partial order in the form of a map $X \times X \xrightarrow{R} L$, where X is a set and L is a lattice (possibly) with some additional algebraic structure [26].

The above-mentioned concepts are closely related. In particular, Definitions 1, 3 are equivalent up to the name of the notion. Moreover, an easy effort will convince the reader that Definition 2 is actually their dual analogue [17] (see more on that in Section 6.5 of the paper). What is more important to us is the fact that all three notions bear a close connection to partial order. In particular, the next result is easy to show.

Lemma 4.

(1) *Given a w-eo algebra* $(L, \to, \top)$*, the binary relation* $\leqslant$ *on* L *defined for every* $a, b \in L$ *by*

$$a \leqslant b \text{ iff } a \to b = \top$$

provides a poset $(L, \leqslant, \top)$ *with the upper bound* $\top$ *(this partial order is called the* natural *one in* L*).*

(2) *Given an upper-bounded poset* $(L, \leqslant, \top)$*, every binary operation* $\to$ *on* L*, which extends the relation* $\leqslant$ *(*$a \to b = \top$ *iff* $a \leqslant b$*), provides a w-eo algebra* $(L, \to, \top)$*.*

It was precisely the result of Lemma 4 (already mentioned by H. Rasiowa [30]) that motivated the change of terminology to "extended-order algebra".

An attentive reader will notice immediately that the ambition of C. Guido *et al.* in providing a new notion subsumes that of H. Rasiowa, since many-valued mathematics includes lattice-valued logic, which in its turn incorporates classical logic as a crisp subcase. Being more general in the just mentioned sense, C. Guido and his research team decided to investigate the properties of the binary operation $\to$ of a w-eo algebra $(A, \to, \top)$. The main motivation came from the current trend of starting with a *basic* (or *primitive*) binary operation of multiplication ($\otimes$) and then obtain an implication-like operation ($\to$) as a *derived* one. Consider, for example, the well-known case of quantales [25, 34, 35, 36, 37, 38].

Definition 5. A *quantale* $(Q, \otimes, \bigvee)$ is a $\bigvee$-semilattice $(Q, \bigvee)$ (partially ordered set having arbitrary $\bigvee$) equipped with an associative binary operation $\otimes$ (*multiplication*), which distributes across $\bigvee$ from both sides, i.e., $a \otimes (\bigvee S) = \bigvee_{s \in S}(a \otimes s)$ and $(\bigvee S) \otimes a = \bigvee_{s \in S}(s \otimes a)$ for every $a \in Q$ and every $S \subseteq Q$.

The multiplication operation in a given quantale $(Q, \otimes, \bigvee)$ induces two *residuations*, namely, $a \to_r b = \bigvee\{c \in Q \mid a \otimes c \leqslant b\}$ and $a \to_l b = \bigvee\{c \in Q \mid c \otimes a \leqslant b\}$. Moreover, a special case of the residuations provides two $\otimes$-*pseudocomplementations* (the terminology is not standard): $a^{\perp} = a \to_r \perp$ and ${}^{\perp}a = a \to_l \perp$. In one word, the basic operation $\otimes$ gives rise to a variety of derived ones.

C. Guido *et al.* proposed to go the opposite way, developing their theory accordingly [8, 16, 17] (one must underline here that a similar path has already been taken by, e.g., J. M. Dunn [10]).

Definition 6. A w-eo algebra $(L, \to, \top)$ is called *complete* (*w-ceo algebra*) provided that the set L, equipped with the natural partial order, is a complete lattice. A w-ceo algebra $(L, \to, \top)$ is said to be *right-distributive* (*w-rdceo algebra*) provided that for every $a \in L$ and every $S \subseteq L$, $a \to \bigwedge S = \bigwedge_{s \in S}(a \to s)$.

Given a w-rdceo algebra $(L, \to, \top)$, the operation $\to$ induces a multiplication $\otimes$ on L defined by $a \otimes b = \bigwedge\{c \in L \mid b \leqslant a \to c\}$. Moreover, every w-ceo algebra $(L, \to, \top)$ is equipped with a unary operation $(-)^{\perp}$ defined by $a^{\perp} = a \to \perp$. Altogether, the basic operation $\to$ gives rise to a plentitude of derived ones, whose properties can be investigated through those of $\to$. That was precisely the approach taken up by the team of C. Guido, the main idea being the following: base all algebraic structures of many-valued mathematics on a single binary operation $\to$ obtained as an extension of partial order. To back the challenging goal, an advertisement campaign for the new framework has started, stimulating its applications in many-valued mathematics. The series of papers [12, 13, 14] contains an attempt to build the theories of lattice-valued topology and category theory on w-eo algebras.

It should be noticed, however, that the theory of the new structures itself is still quite far from maturity, due to some negligence of C. Guido *et al.* of their proposed algebras. Indeed, having payed much attention to the structure, they never considered its homomorphisms. Since the modern many-valued mathematics relies heavily on category theory (cf., e.g., lattice-valued topology of [32], which is a *de facto* standard in the fuzzy community), the latter issue is of great importance in the development of every new lattice-valued framework. It is the main purpose of this paper to fill in the gap providing a categorical approach to w-eo algebras, thereby studying properties of homomorphisms of the structures in question. In pursuing the course, we naturally regard w-eo algebras as a generalization of posets (cf. the term "extended-order algebras") and that opens a plentitude of possibilities to define their homomorphisms.

While reading the paper, a cunning reader will notice striking similarities with the theory of *Hilbert algebras* [9] (pointed out to the author by A. Palmigiano), introduced as an algebraic description of the implication connective in the linear intuitionistic logic. Indeed, Hilbert algebras (or *positive implication algebras* in the language of H. Rasiowa [30]) give a further restriction of w-eo algebras.

Definition 7. A w-eo algebra $(L, \to, \top)$ is called a *Hilbert algebra* provided that the following conditions are satisfied for every $a, b, c \in L$:

(1) $a \to (b \to a) = \top$;
(2) $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = \top$.

After careful examination, it appears that our approach to w-eo algebras differs from the theory of Hilbert algebras, being mostly based on the topic of homomorphisms and not the structures themselves. A somewhat closer stand was taken by S. Celani [3] in his notion of *semi-homomorphism* of Hilbert algebras, which has arisen as a generalization of the similar notion of Boolean algebras and was studied in connection with the homomorphisms and deductive systems.

Definition 8. Let $(L, \to, \top)$ and $(M, \to, \top)$ be Hilbert algebras. A *semi-homomorphism* is a map $L \xrightarrow{h} M$ such that:

(1) $h(\top) = \top$;
(2) $h(a \to b) \leqslant h(a) \to h(b)$ for every $a, b \in L$.

The idea was further developed by S. Celani, L. M. Cabrer and D. Montangie in [4], where the category $\mathcal{HS}$ of Hilbert algebras and semi-homomorphisms was considered, with an ultimate goal to provide a topological duality theorem for the structures. The current paper uses a similar definition to obtain particular instances of homomorphisms of w-eo algebras (Definition 13), but in a different context. It will be the topic of our further research to provide an analogue of the results of S. Celani *et al.* in the framework of w-eo algebras.

The paper uses both category theory and algebra, relying more on the former. The necessary categorical background can be found in [1, 19, 27]. For the notions of universal algebra, we recommend [5, 6, 30]. Although we tried to make the paper as much self-contained as possible, some details are still omitted and left for the self-study of the interested reader.

2. Extended-order algebras versus partially ordered sets

This section provides a categorical elaboration of the relation between w-eo algebras and partially ordered sets touched in Lemma 4. We begin with the necessary categorical preliminaries from the theory of posets.

Definition 9. **Pos** is the category, whose objects are partially ordered sets $(X, \leqslant)$, and whose morphisms are order-preserving (*monotone*) maps $(X, \leqslant) \xrightarrow{f} (Y, \leqslant)$.

Definition 10. $\mathbf{Pos}^{\top}$ is the non-full subcategory of **Pos**, whose objects are upper-bounded posets $(X, \leqslant, \top)$, and whose morphisms are $\top$-preserving monotone maps.

Turning to the case of w-eo algebras, one can introduce the respective category of the structures. For convenience sake, from now on, we will use the capital letters A, B, C, *etc.* to denote the underlying sets of the algebras in question, and Greek letters ϕ, φ, ψ, *etc.* to denote the respective homomorphisms.

Definition 11. $\mathbf{WEOAlg}^{\top}$ is the category, whose objects are w-eo algebras $(A, \to, \top)$, and whose morphisms $(A, \to, \top) \xrightarrow{\varphi} (B, \to, \top)$ are maps $A \xrightarrow{\varphi} B$ such that for every $a_1, a_2 \in A$ as well as $\top \in A$, the following conditions hold:

(1) if $a_1 \to a_2 = \top$, then $\varphi(a_1) \to \varphi(a_2) = \top$;
(2) $\varphi(\top) = \top$.

An experienced reader will notice that the category $\mathbf{WEOAlg}^{\top}$ provides a direct generalization of the category $\mathbf{Pos}^{\top}$ (a map between d-algebras satisfying item (1) of Definition 11 with the respective change in the notations is called *order-preserving* in [28]). It appears that there exists an even deeper relation between the categories in question, whose proof relies on straightforward computations and, therefore, is omitted.

Theorem 12.

(1) *There exists a functor* $\mathbf{WEOAlg}^{\top} \xrightarrow{\|-\|} \mathbf{Pos}^{\top}$, $\|(A, \to, \top) \xrightarrow{\varphi} (B, \to, \top)\| = (A, \leqslant, \top) \xrightarrow{\varphi} (B, \leqslant, \top)$, *where* $c_1 \leqslant c_2$ *iff* $c_1 \to c_2 = \top$.
(2) *There exists a functor* $\mathbf{Pos}^{\top} \xrightarrow{F} \mathbf{WEOAlg}^{\top}$, $F((X, \leqslant, \top) \xrightarrow{f} (Y, \leqslant, \top)) = (X, \to, \top) \xrightarrow{f} (Y, \to, \top)$, *where*

$$z_1 \to z_2 = \begin{cases} \top, & z_1 \leqslant z_2 \\ z_2, & \textit{otherwise}. \end{cases}$$

(3) *The functors* $\| - \|$ *and* F *provide an equivalence between the categories* $\mathbf{WEOAlg}^{\top}$ *and* $\mathbf{Pos}^{\top}$ *such that* $\| - \| \circ F = 1_{\mathbf{Pos}^{\top}}$.

Theorem 12 shows a categorical generalization of Lemma 4, taking the case of morphisms in play as well. Moreover, the result can be extended even further, relaxing the rather strong conditions on morphisms of the category $\mathbf{WEOAlg}^{\top}$. For the sake of shortness, from now on, given a w-eo algebra $(A, \to, \top)$ and $a, b \in A$, "$a \to b = \top$" will be occasionally denoted by "$a \leqslant b$".

Definition 13. $\mathbf{WEOAlg}^{\leqslant}$ is the non-full subcategory of $\mathbf{WEOAlg}^{\top}$ having the same objects, and whose morphisms $(A, \to, \top) \xrightarrow{\varphi} (B, \to, \top)$ are maps $A \xrightarrow{\varphi} B$ such that for every $a_1, a_2 \in A$ as well as $\top \in A$, the following conditions hold:

(1) $\varphi(a_1 \to a_2) \leqslant \varphi(a_1) \to \varphi(a_2)$;
(2) $\varphi(\top) = \top$.

The reader should notice that Definition 13 uses the approach of [3, Definition 3.1] (quite a natural one, in fact) recalled in Definition 8. Despite the coincidence, our current context is completely different from that of S. Celani. To point out the difference as well as to distinguish the morphisms of the category $\mathbf{WEOAlg}^{\leqslant}$ from the classical universal algebra approach (Definition 21), we will call them *lax* (*w-eo algebra*) *homomorphisms* employing the terminology of [20].

Definition 14. $\mathbf{WEOAlg}^{\leqslant\rightarrow}$ is the full subcategory of $\mathbf{WEOAlg}^{\leqslant}$, the objects of which are w-eo algebras $(A, \rightarrow, \top)$ satisfying the condition $a \rightarrow (b \rightarrow a) = \top$ for every $a, b \in A$.

It is important to underline that following Definition 7, we do *not* restrict ourselves to the case of Hilbert algebras. With the category $\mathbf{WEOAlg}^{\leqslant\rightarrow}$ in hand, one can generalize Theorem 12 as follows.

Theorem 15.

(1) *There exists the restriction* $\mathbf{WEOAlg}^{\leqslant\rightarrow} \xrightarrow{\|-\|^{\leqslant\rightarrow}} \mathbf{Pos}^{\top}$ *of the functor* $\mathbf{WEOAlg}^{\top} \xrightarrow{\|-\|} \mathbf{Pos}^{\top}$.
(2) *There exists the restriction* $\mathbf{Pos}^{\top} \xrightarrow{F^{\leqslant\rightarrow}} \mathbf{WEOAlg}^{\leqslant\rightarrow}$ *of the functor* $\mathbf{Pos}^{\top} \xrightarrow{F} \mathbf{WEOAlg}^{\top}$.
(3) $F^{\leqslant\rightarrow}$ *is a left-adjoint-right-inverse to* $\| - \|^{\leqslant\rightarrow}$.

Proof. *Ad* (1). Just to give a flavor of the new functor, we show its correctness on morphisms. Given a lax w-eo algebra homomorphism $(A, \rightarrow, \top) \xrightarrow{\varphi} (B, \rightarrow, \top)$ and some $a_1, a_2 \in A$ such that $a_1 \leqslant a_2$, it follows that $a_1 \rightarrow a_2 = \top$ and then $\top = \varphi(\top) = \varphi(a_1 \rightarrow a_2) \leqslant \varphi(a_1) \rightarrow \varphi(a_2)$ yields $\varphi(a_1) \leqslant \varphi(a_2)$. In one word, lax homomorphisms are order-preserving, that generalizes [28, Proposition 5.1].

Ad (2). Show the correctness of the new functor on both objects and morphisms. Given a $\top$-preserving monotone map $(X, \leqslant, \top) \xrightarrow{f} (Y, \leqslant, \top)$, fix $x_1, x_2 \in X$. By the definition of F, it follows that $x_2 \leqslant x_1 \rightarrow x_2$ that yields the desired equality. As for morphisms, suppose that $f(x_1) \rightarrow f(x_2) \neq \top$. Then $f(x_1) \nleqslant f(x_2)$ implies $f(x_1) \rightarrow f(x_2) = f(x_2)$, on one hand, and $x_1 \nleqslant x_2$ provides $x_1 \rightarrow x_2 = x_2$, on the other. Altogether, $f(x_1 \rightarrow x_2) = f(x_2) = f(x_1) \rightarrow f(x_2)$.

Ad (3). Rather straightforward computations show that the map $(X, \leqslant, \top) \xrightarrow{\eta} \|F^{\leqslant\rightarrow}(X, \leqslant, \top)\|^{\leqslant\rightarrow} = (X, \leqslant, \top) \xrightarrow{1_X} (X, \leqslant, \top)$ provides the required $\| - \|^{\leqslant\rightarrow}$-universal arrow, i.e., it has the property that every monotone map $(X, \leqslant, \top) \xrightarrow{f} \|(A, \rightarrow, \top)\|^{\leqslant\rightarrow}$ has a unique homomorphism

$F^{\leqslant\rightarrow}(X,\leqslant,\top) \xrightarrow{\overline{f}} (A,\rightarrow,\top)$ making the triangle

$$\begin{tikzcd}
(X,\leqslant,\top) \arrow[r, "\eta"] \arrow[dr, "f"'] & \|F^{\leqslant\rightarrow}(X,\leqslant,\top)\|^{\leqslant\rightarrow} \arrow[d, dotted, "\|\overline{f}\|^{\leqslant\rightarrow}"] \\
& \|(A,\rightarrow,\top)\|^{\leqslant\rightarrow}
\end{tikzcd}$$

commute. ∎

The new framework transforms the equivalence of Theorem 12 into an adjunction, retaining the embedding of $\mathbf{Pos}^{\top}$ into the category $\mathbf{WEOAlg}^{\leqslant\rightarrow}$. The adjunction obtained allows a certain simplification, under the restriction to a particular subcategory of $\mathbf{Pos}^{\top}$.

Definition 16. $\mathbf{BPos}$ is the non-full subcategory of $\mathbf{Pos}^{\top}$, whose objects are bounded posets $(X,\leqslant,\bot,\top)$, and whose morphisms are monotone maps preserving the bounds.

Definition 17. $\mathbf{WEOAlg}^{\leqslant\bot}$ is the non-full subcategory of $\mathbf{WEOAlg}^{\leqslant}$, whose objects are w-eo algebras $(A,\rightarrow,\top)$ having an element $\bot \in A$ such that $\bot \rightarrow a = \top$ for every $a \in A$, and whose morphisms are $\bot$-preserving lax w-eo algebra homomorphisms.

Theorem 18.

(1) *There exists the restriction* $\mathbf{WEOAlg}^{\leqslant\bot} \xrightarrow{\|-\|^{\leqslant\bot}} \mathbf{BPos}$ *of the functor* $\mathbf{WEOAlg}^{\top} \xrightarrow{\|-\|} \mathbf{Pos}^{\top}$.

(2) *There exists a functor* $\mathbf{BPos} \xrightarrow{G} \mathbf{WEOAlg}^{\leqslant\bot}$, *which is given by* $G((X,\leqslant,\bot,\top) \xrightarrow{f} (Y,\leqslant,\bot,\top)) = (X,\rightarrow,\top) \xrightarrow{f} (Y,\rightarrow,\top)$, *where*

$$z_1 \rightarrow z_2 = \begin{cases} \top, & z_1 \leqslant z_2 \\ \bot, & \textit{otherwise} \end{cases}$$

(this operation is called the natural *one on the respective set).*

(3) *G is a left-adjoint-right-inverse to* $\|-\|^{\leqslant\bot}$.

Notice that having a bottom element in hand, allows one to define the operation $\rightarrow$ in item (2) of Theorem 18 as a characteristic map of the partial order in question, which is indeed a standard way of converting a relation into an operation.

3. Extended-order algebras versus preordered sets

This section is devoted to a generalization of the well-known procedure of making a preordered set partially ordered. For the sake of transparency, we begin by recalling the standard developments.

Definition 19. **Prost** is the category, whose objects are *preordered sets* $(X, \leqslant)$ (the relation $\leqslant$ is reflexive and transitive), and whose morphisms are monotone maps.

It is easy to see that **Pos** is the full subcategory of **Prost**, the embedding functor denoted by E.

Theorem 20. *The embedding* $\mathbf{Pos} \xhookrightarrow{E} \mathbf{Prost}$ *has a left adjoint.*

Proof. Given a preordered set $(X, \leqslant)$, define an equivalence relation $\sim$ on X by $x_1 \sim x_2$ iff $x_1 \leqslant x_2$ and $x_2 \leqslant x_1$. The standard quotient map $(X, \leqslant) \xrightarrow{p} E(X/\sim, \leqslant_\sim)$ provides the required E-universal arrow (cf. *Ad* (3) in the proof of Theorem 15). ∎

The above-mentioned simple procedure gains somewhat in complexity, when turning to the case of w-eo algebras. Start with the definition of *weak extended-preorder algebra*, which has never been mentioned by C. Guido and his collaborators.

Definition 21. **WEPOAlg** is the category, whose objects *weak extended-preorder algebras* (*w-epo algebras*) are triples $(A, \rightarrow, \top)$, where L is a non-empty set, $\rightarrow$ is a binary operation on L, and $\top$ is an element of L such that for every $a, b, c \in L$, the following conditions are satisfied:

(1) $a \rightarrow \top = \top$ (upper bound condition);
(2) $a \rightarrow a = \top$ (reflexivity condition);
(3) if $a \rightarrow b = \top$ and $b \rightarrow c = \top$, then $a \rightarrow c = \top$ (weak transitivity condition).

The morphisms of the category $(A, \rightarrow, \top) \xrightarrow{\varphi} (B, \rightarrow, \top)$ are maps $A \xrightarrow{\varphi} B$ with the property $\varphi(a_1 \rightarrow a_2) = \varphi(a_1) \rightarrow \varphi(a_2)$ for every $a_1, a_2 \in A$.

The following lemma shows a simple (but very important) property of w-epo algebra homomorphisms.

Lemma 22. *Every w-epo algebra homomorphism* $(A, \rightarrow, \top) \xrightarrow{\varphi} (B, \rightarrow, \top)$ *is* $\top$*-preserving.*

Proof. $\varphi(\top) = \varphi(\top \rightarrow \top) = \varphi(\top) \rightarrow \varphi(\top) = \top$. ∎

Moreover, with the above-mentioned classical case in mind, one immediately obtains the next definition.

Definition 23. **WEOAlg** is the full subcategory of **WEPOAlg** of w-eo algebras.

On the other hand, to take the path of Theorem 20, one needs a particular subcategory of **WEPOAlg**.

Definition 24. $\mathbf{WEPOAlg}^*$ is the full subcategory of $\mathbf{WEPOAlg}$, whose objects (*w-epo* algebras*) are those w-epo algebras $(A, \to, \top)$, which satisfy for every $a, b, c, d \in A$ the following conditions:

(1) if $a \to b = \top$, $b \to a = \top$ and $c \to d = \top$, $d \to c = \top$, then $(a \to c) \to (b \to d) = \top$ and $(b \to d) \to (a \to c) = \top$;
(2) if $\top \to (a \to b) = \top$ and $\top \to (b \to c) = \top$, then $\top \to (a \to c) = \top$;
(3) if $\top \to (a \to b) = \top$ and $\top \to (b \to a) = \top$, then $a \to b = \top$ and $b \to a = \top$.

Clearly, $\mathbf{WEOAlg}$ is the full subcategory of $\mathbf{WEPOAlg}^*$, with E standing for the embedding functor.

Theorem 25. *The embedding* $\mathbf{WEOAlg} \xhookrightarrow{E} \mathbf{WEPOAlg}^*$ *has a left adjoint.*

Proof. Given a w-epo* algebra $(A, \to, \top)$, define an equivalence relation $\sim$ on A by $a \sim b$ iff $a \to b = \top$ and $b \to a = \top$. Item (1) of Definition 24 ensures that the relation is a congruence. To continue, let $(A/\sim) \times (A/\sim) \xrightarrow{\rightsquigarrow} (A/\sim)$, $[a] \rightsquigarrow [b] = [a \to b]$, where $[a] = \{c \in A \mid a \sim c\}$ is the congruence class of a. Employing items (2), (3) of Definition 24, one gets a w-eo algebra $(A/\sim, \rightsquigarrow, [\top])$. For example, to show the transitivity axiom, notice that $[a] \rightsquigarrow [b] = [\top]$ and $[b] \rightsquigarrow [c] = [\top]$ imply $\top \to (a \to b) = \top$ and $\top \to (b \to c) = \top$ that yields $\top \to (a \to c) = \top$ by the above item (2). Altogether, $(a \to c) \sim \top$ and, thus, $[a] \rightsquigarrow [c] = [\top]$. Easy computations show that the quotient map $A \xrightarrow{p} (A/\sim)$, $p(a) = [a]$ is the required E-universal arrow for $(A, \to, \top)$. ∎

Notice that by Theorem 25, the conditions of Definition 24 are sufficient for the desired result. It is not difficult to see that they are also the necessary ones due to item (1) of Definition 21.

4. Completion of extended-order algebras

The algebraic structures used in many-valued mathematics are often required to be based on complete lattices. The case of lattice-valued topology provides a good example, since its modern theory relies on the concept of *semi-quantale* [32, 33], which is a $\bigvee$-semilattice equipped with a binary operation (notice that neither associativity nor any relation to partial order is required from the binary operation in question as in Definition 5). The softening of the standard quantale-like conditions is related to the fact that the obtained categories of topological structures are itself topological over their ground categories, ensuring that one is doing topology while working in the new framework. Since w-eo algebras in general come equipped with a partial order only, C. Guido and P. Toto [17] provided a special completion procedure for the structures, generalizing the standard MacNeille completion of partially ordered sets. This paper takes a similar turn, providing a suitable extension of the so-called *completion by cuts*. For convenience of the reader, we recall the essence of the developments.

Definition 26. $\mathbf{CSLat}(\bigvee)$ is the non-full subcategory of $\mathbf{Pos}$, with the embedding functor denoted by E, whose objects are $\bigvee$-semilattices (cf. Definition 5), and whose morphisms are $\bigvee$-preserving maps.

Theorem 27. *The embedding* $\mathbf{CSLat}(\bigvee) \overset{E}{\hookrightarrow} \mathbf{Pos}$ *has a left adjoint.*

Proof. Given a poset $(X, \leqslant)$, let $\mathcal{P}_{\downarrow}(X)$ be the collection of all lower sets of X ($S \subseteq X$ is a *lower set* provided that $s \in S$ and $x \leqslant s$ imply $x \in S$). It is easy to see that $\mathcal{P}_{\downarrow}(X)$ is a $\bigvee$-semilattice, where $\bigvee$ are given by set-theoretic unions. Moreover, the map $X \xrightarrow{\downarrow(-)} \mathcal{P}_{\downarrow}(X)$, $\downarrow x = \{y \in X \mid y \leqslant x\}$ provides an E-universal arrow for $(X, \leqslant)$. ∎

The object part of the functor of Theorem 27 gives the above-mentioned completion by cuts, which in general is different from the MacNeille completion. Turning now to the case of w-eo algebras, we begin with a suitable analogue of the category $\mathbf{Pos}$.

Definition 28. $\mathbf{EOAlg}^{\leqslant}$ is the full subcategory of $\mathbf{WEOAlg}^{\leqslant}$, whose objects *extended-order algebras* (*eo algebras*) are those w-eo algebras $(A, \to, \top)$, which satisfy for every $a, b, c \in A$ the following conditions:

(1) if $a \to b = \top$, then $(b \to c) \to (a \to c) = \top$ (weak antitonic condition in the first variable);
(2) if $a \to b = \top$, then $(c \to a) \to (c \to b) = \top$ (weak isotonic condition in the second variable).

On the next step, using more restrictions, we provide a substitute for the category $\mathbf{CSLat}(\bigvee)$.

Definition 29. $\mathbf{LDEOAlg}^{\leqslant}(\bigvee)$ is the non-full subcategory of $\mathbf{EOAlg}^{\leqslant}$, with the embedding functor denoted by E, whose objects *left-distributive complete eo algebras* (*ldceo algebras*) are eo algebras $(A, \to, \top)$, which are $\bigvee$-semilattices (w.r.t. the natural partial order) satisfying the condition $(\bigvee S) \to a = \bigwedge_{s \in S}(s \to a)$ for every $a \in A$ and every $S \subseteq A$, and whose morphisms additionally are $\bigvee$-preserving.

It should be noticed that the notion of (ld)ceo algebra is due to C. Guido *et al.* [17]. Everything is in its place to provide the main result of the section, which extends Theorem 27.

Theorem 30. *The functor* $\mathbf{LDEOAlg}^{\leqslant}(\bigvee) \xhookrightarrow{E} \mathbf{EOAlg}^{\leqslant}$ *has a left adjoint.*

Proof. Given an eo algebra $(A, \to, \top)$, define $\mathcal{P}_{\downarrow}(A) = \{\downarrow S \mid S \subseteq A\}$, where $\downarrow S = \{a \in A \mid a \to s = \top \text{ for some } s \in S\}$. It should be clear that $\mathcal{P}_{\downarrow}(A) = \mathcal{P}_{\downarrow}(A)$, where the notational difference just draws the attention to the new framework. For $T_1, T_2 \in \mathcal{P}_{\downarrow}(A)$ set $T_1 \rightsquigarrow T_2 = \bigcap_{t_1 \in T_1} \bigcup_{t_2 \in T_2} \downarrow (t_1 \to t_2)$. Given a family $(T_i)_{i \in I} \subseteq \mathcal{P}_{\downarrow}(A)$ let $\bigvee_{i \in I} T_i = \bigcup_{i \in I} T_i$. One has to verify that the triple $(\mathcal{P}_{\downarrow}(A), \rightsquigarrow, A)$ is an ldceo algebra. As an example, show that $T_1 \subseteq T_2$ iff $T_1 \rightsquigarrow T_2 = A$. Assuming the left-hand side inclusion, every $t_1 \in T_1$ satisfies $t_1 \in T_2$ and, thus, $A = \downarrow (t_1 \to t_1) \subseteq \bigcup_{t_2 \in T_2} \downarrow (t_1 \to t_2)$. It follows that $A \subseteq T_1 \rightsquigarrow T_2$. Assuming now the right-hand side equality, every $t_1 \in T_1$ yields $\bigcup_{t_2 \in T_2} \downarrow (t_1 \to t_2) = A$ and, thus, there exists some $t_2 \in T_2$ such that $\top \in \downarrow (t_1 \to t_2)$, yielding $t_1 \to t_2 = \top$. The desired $t_1 \in T_2$ now follows.

Straightforward (but tedious) computations show that the map $A \xrightarrow{\downarrow(-)} \mathcal{P}_{\downarrow}(A)$ provides an E-universal arrow for $(A, \to, \top)$, i.e., every lax eo algebra homomorphism $(A, \to, \top) \xrightarrow{\varphi} E(B, \to, \top)$ has a unique extension $\mathcal{P}_{\downarrow}(A) \xrightarrow{\overline{\varphi}} B$, $\overline{\varphi}(T) = \bigvee \varphi^{\to}(T) = \bigvee_{t \in T} \varphi(t)$. As illustrative examples, we will show two things. Firstly, let us check that the map $\downarrow (-)$ is a lax homomorphism, i.e., $\downarrow (a \to b) = \downarrow a \rightsquigarrow \downarrow b$ for every $a, b \in A$.

Given $c \in \downarrow (a \to b)$, it follows that $c \in \bigcup_{t_2 \in \downarrow b} \downarrow (a \to t_2)$ and, thus, $c \in \downarrow (a \to t_2)$ for some $t_2 \in \downarrow b$, i.e., $c \to (a \to t_2) = \top$ and $t_2 \to b = \top$. The latter equality gives $(a \to t_2) \to (a \to b) = \top$ by item (2) of Definition 28 and, therefore, the former one provides $c \to (a \to b) = \top$, i.e., $c \in \downarrow (a \to b)$.

Given $c \in \downarrow (a \to b)$, it follows that $c \to (a \to b) = \top$. Every $t_1 \in \downarrow a$ provides $t_1 \to a = \top$ and, thus, $(a \to b) \to (t_1 \to b) = \top$ by item (1) of Definition 28, yielding $c \to (t_1 \to b) = \top$. Then $c \in \downarrow (t_1 \to b)$ and, therefore, $c \in \bigcup_{t_2 \in \downarrow b} \downarrow (t_1 \to t_2)$. Altogether, $c \in \bigcap_{t_1 \in T_1} \bigcup_{t_2 \in T_2} \downarrow (t_1 \to t_2)$ and that was to show.

Secondly, let us verify that the extension map $\mathcal{P}_{\downarrow}(A) \xrightarrow{\overline{\varphi}} B$ is a homomorphism as well. It will be enough to show that $\bigvee \varphi^{\to}(\bigcap_{t_1 \in T_1} \bigcup_{t_2 \in T_2} \downarrow (t_1 \to t_2)) \leqslant (\bigvee \varphi^{\to}(T_1)) \to (\bigvee \varphi^{\to}(T_2))$. Notice that $\bigvee \varphi^{\to}(\bigcap_{t_1 \in T_1} \bigcup_{t_2 \in T_2} \downarrow (t_1 \to t_2)) \leqslant \bigvee(\bigcap_{t_1 \in T_1} \bigcup_{t_2 \in T_2} \downarrow \varphi^{\to}(\downarrow (t_1 \to t_2)))$. Choose an element $b \in \bigcap_{t_1 \in T_1} \bigcup_{t_2 \in T_2} \downarrow \varphi^{\to}(\downarrow (t_1 \to t_2))$. Every $t_1 \in T_1$ has $t_2 \in T_2$ such that $b \to \varphi(a) = \top$ and $a \to (t_1 \to t_2) = \top$ for some $a \in A$. The latter equality provides $\top = \varphi(\top) = \varphi(a \to (t_1 \to t_2)) \leqslant \varphi(a) \to \varphi(t_1 \to t_2) \leqslant \varphi(a) \to (\varphi(t_1) \to \varphi(t_2))$ (mind the use of Item (2) of Definition 28) and, thus, $\varphi(a) \to (\varphi(t_1) \to \varphi(t_2)) = \top$. The former one then gives $b \to (\varphi(t_1) \to \varphi(t_2)) = \top$ yielding (again by item (2)) $b \to (\varphi(t_1) \to \bigvee \varphi^{\to}(T_2)) = \top$. Altogether,

$b \to \bigwedge_{t_1 \in T_1}(\varphi(t_1) \to \bigvee \varphi^{\to}(T_2)) = \top$ and then (employing the condition of Definition 29) $b \to (\bigvee \varphi^{\to}(T_1) \to \bigvee \varphi^{\to}(T_2)) = \top$ that was to show. ∎

For the sake of convenience, from now on, the paper uses the following notation. Given a category $\mathbf{C}$ of w-eo algebra related structures, it is assumed that the respective morphisms are defined as in Definition 21 (homomorphisms of the standard universal algebra approach), whereas $\mathbf{C}^{\leqslant}$ is supposed to have the morphisms as in Definition 13 (lax homomorphisms of the poset motivated approach).

The last part of the proof of Theorem 30 shows that the map $A \xrightarrow{\downarrow(-)} \mathcal{P}_{\downarrow}(A)$ actually lies in the category **EOAlg** and that provides a motivation for changing the setting from $\mathbf{EOAlg}^{\leqslant}$ to **EOAlg**. Despite the expectations, the next lemma states that the desired modification is impossible.

Lemma 31. *The adjunction of Theorem* 30 *can not be restricted to the category* **EOAlg**.

Proof. Consider the eo algebra $(\mathbf{2} = \{\bot, \top\}, \to, \top)$ with the natural operation $\to$ (cf. Theorem 18). If the desired restriction is possible, then there exists a homomorphism $\mathcal{P}_{\downarrow}(\mathbf{2}) \xrightarrow{\overline{1_{\mathbf{2}}}} \mathbf{2}$ defined by $\overline{1_{\mathbf{2}}}(T) = \bigvee T$ (the extension of the identity $\mathbf{2} \xrightarrow{1_{\mathbf{2}}} \mathbf{2}$). On the other hand, letting $T_1 = \{\bot\}$ and $T_2 = \varnothing$ provides $\overline{1_{\mathbf{2}}}(T_1 \rightsquigarrow T_2) = \bot < \top = \overline{1_{\mathbf{2}}}(T_1) \to \overline{1_{\mathbf{2}}}(T_2)$. ∎

Having the desired completion in hand, it would be interesting to compare our result with the respective one of C. Guido *et al.* [17]. Their approach is also based on the concept of extended-order algebra recalled in Definition 28. In particular, they constructed the MacNeille completion of a given eo algebra $(A, \to, \top)$ such that the new operation $\rightsquigarrow$ provides an extension of the original one. The construction of Theorem 30 provides a different (sometimes of a bigger cardinality) completion of eo algebras, the additional condition on distributivity in Definition 29 used to extend the result to homomorphisms. In one word, the object part of the new framework simplifies the respective procedure of C. Guido *et al.* On the other hand, in [15] the obtained MacNeille completion is studied w.r.t. the properties it is capable of preserving (reflecting) from the original eo algebra. It will be the topic of our further research to do the same job in our framework.

5. Free extended-order algebras

Every new algebraic structure raises a question on the description of the respective free algebras over sets. While working in the classical framework of universal algebra, where the operations are finitary and set-indexed, there exists the standard procedure for obtaining free algebras, which relies on the well-known term algebra construction. Unlike the classical case, the algebras used in lattice-valued mathematics do not always enjoy availability of free objects. For example, the construct **CLat** of complete lattices and complete lattice homomorphisms never has free lattices over sets with more than two elements. Luckily, the case of w-eo algebras is a classical one and, therefore, the familiar procedure should be at hand. On the other hand, following our viewpoint on the structures as an extension of posets, we will generalize the standard procedure of the latter framework. For convenience of the reader, we start with some preliminaries.

There exists the forgetful functor $\mathbf{Pos} \xrightarrow{|-|} \mathbf{Set}$ defined by $|(X, \leqslant) \xrightarrow{f} (Y, \leqslant)| = X \xrightarrow{f} Y$, which has the following simple property.

Theorem 32. *The functor* $\mathbf{Pos} \xrightarrow{|-|} \mathbf{Set}$ *has a left adjoint.*

Proof. Given a set X, the identity map $X \xrightarrow{1_X} |(X, =)|$ provides a $|-|$-universal arrow for X. ∎

Turning to the framework of w-eo algebras, one obtains the following generalization (cf. Definition 14).

Theorem 33. *The forgetful functor* $\mathbf{WEOAlg}^{\leqslant\rightarrow} \xrightarrow{|-|} \mathbf{Set}$ *has a left adjoint.*

Proof. Given a set X, define $F(X) = X \uplus \{\top\}$ and let

$$x \rightarrow y = \begin{cases} \top, & x = y \\ y, & \text{otherwise.} \end{cases}$$

Straightforward computations show that $(F(X), \rightarrow, \top)$ lies in $\mathbf{WEOAlg}^{\leqslant\rightarrow}$, and the map $X \xrightarrow{\eta} F(X)$, $\eta(x) = x$ is a $|-|$-universal arrow for X. In particular, every map $X \xrightarrow{f} |(A, \rightarrow, \top)|$ has a unique extension to a lax homomorphism $F(X) \xrightarrow{\overline{f}} A$ defined by

$$\overline{f}(x) = \begin{cases} \top, & x = \top \\ f(x), & \text{otherwise.} \end{cases}$$ ∎

It should be noticed here that relaxing the notion of w-eo algebra homomorphism simplifies dramatically the procedure of obtaining free algebras. Moreover, the adjunction of Theorem 33 can be restricted to a particular subcategory of $\mathbf{WEOAlg}^{\leqslant\rightarrow}$, the motivation for which will be given in the next section.

Definition 34. $\mathbf{WEOAlg}^{\leqslant\rightarrow\star}$ is the full subcategory of $\mathbf{WEOAlg}^{\leqslant\rightarrow}$, whose objects $(A, \rightarrow, \top)$ satisfy for every $a, b, c \in A$ the following condition:

($\star$) if $a \to b = \top$ and $a \to c \neq \top$, then $a \to (b \to c) \neq \top$.

Theorem 35. *There exists the restriction of the adjunction of Theorem 33 to the category* $\mathbf{WEOAlg}^{\leqslant\to\star}$.

Proof. The challenge is to show that the free algebra obtained in Theorem 33 belongs to the subcategory in question, and that can be done employing easy computations to check the condition of Definition 34. ∎

As an immediate consequence, one obtains the following standard category-theoretic result.

Corollary 36. *The monomorphism in* $\mathbf{WEOAlg}^{\leqslant\to}$ *and* $\mathbf{WEOAlg}^{\leqslant\to\star}$ *are precisely the homomorphisms with injective underlying maps.*

Other standard categorical properties (e.g., preservation of limits by the respective forgetful functor) also follow. It will be the topic of our further research to investigate the functors of this section more thoroughly.

6. Categorical properties of extended-order algebras

The previous sections of the paper have probably already convinced the reader of the fruitfulness of the categorical approach to w-eo algebras. Up to now, however, we were mostly concerned with the categorical properties of the algebras, motivated by the structures themselves. It is the purpose of this section to move in the proposed direction even further and consider some categorically motivated features of the concept. All of them will clarify several essential properties of the studied categories, which at the end will cast some light on the considered structures themselves.

6.1. Coseparators

The first categorical concept we are going to consider is that of *coseparator*. For convenience of the reader, we recall its definition.

Definition 37. An object C of a category $\mathbf{C}$ is called *coseparator* provided that for every distinct morphisms $B \overset{f}{\underset{g}{\rightrightarrows}} A$, there exists a morphism $A \xrightarrow{h} C$ such that $B \xrightarrow{f} A \xrightarrow{h} C \neq B \xrightarrow{g} A \xrightarrow{h} C$.

Enjoying a rather simple definition, the object in question plays an important role in every category where it exists. In particular, in each category with products, an object C is a coseparator iff every object is a subobject of some power C^I of C. Many standard constructs have a coseparator. For example, the following result shows that one of the main categories of this paper has this object as well.

Lemma 38. *Coseparators in* **Pos** *are precisely the non-discrete (the order is not given by equality) posets.*

The respective result for w-eo algebras follows the pattern of its predecessor (recall Definition 34).

Theorem 39. *The coseparators in* $\mathbf{WEOAlg}^{\leqslant\to\star}$ *are precisely the objects having at least two elements.*

Proof. Given distinct homomorphisms $B \underset{\psi}{\overset{\varphi}{\rightrightarrows}} A$, there exists $b \in B$ such that $\varphi(b) \neq \psi(b)$. Take some $(C, \to, \top)$ in $\mathbf{WEOAlg}^{\leqslant\to\star}$ such that there exists an element $c \in C$ with $c \neq \top$. Define a map $A \xrightarrow{\phi} C$ by

$$\phi(a) = \begin{cases} \top, & \phi(b) \to a = \top \\ c, & \text{otherwise.} \end{cases}$$

It follows that ϕ is a lax homomorphism and, moreover, $\phi \circ \varphi \neq \phi \circ \psi$. ∎

For example, $(\mathbf{2}, \to, \top)$ with the natural operation $\to$ (cf. the proof of Lemma 31) is the simplest coseparator in $\mathbf{WEOAlg}^{\leqslant\to\star}$. Also notice that since every w-eo algebra has the top element $\top$, the condition of indiscreteness for posets translates into the existence of at least one element different from $\top$ (which is then strictly less than $\top$).

6.2. Epimorphisms

The next categorical concept of interest is that of *epimorphism*, whose definition is recalled below.

Definition 40. A morphism $A \xrightarrow{f} B$ of a category $\mathbf{C}$ is said to be an *epimorphism* provided that for all pairs $B \underset{k}{\overset{h}{\rightrightarrows}} C$ of morphisms such that $h \circ f = k \circ f$, it follows that $h = k$.

Initially deemed to generalize the notion of surjective map, epimorphisms in several well-known constructs do not necessarily have the property. On the other hand, even if they do, the proof can appear far from easy. Luckily, the case of partially ordered sets does not provide any difficulty.

Lemma 41. *Epimorphisms in* **Pos** *are precisely the homomorphisms with surjective underlying maps.*

While the proof of Lemma 41 is sufficiently easy, the generalization of the procedure to the case of w-eo algebras gains slightly in complexity.

Theorem 42. *Epimorphisms in the category* $\mathbf{WEOAlg}^{\leqslant\to}$ *are precisely the homomorphisms with surjective underlying maps.*

Proof. Take a non-surjective lax homomorphism $A \xrightarrow{\varphi} B$ and choose some $b_0 \in B \backslash \varphi^{\rightarrow}(A)$ (cf. the notations of the proof of Theorem 30). Define $B_* = B \uplus \{*\}$ and let

$$b_1 \rightarrow_* b_2 = \begin{cases} b_1 \rightarrow_B b_2, & b_1 \neq * \text{ and } b_2 \neq * \\ \top, & (b_1 = * \text{ and } b_2 = *) \text{ or } (b_1 = * \text{ and } b_2 = b_0) \\ *, & b_1 = b_0 \text{ and } b_2 = * \\ b_0 \rightarrow_B b_2, & b_1 = * \text{ and } b_2 \in B \backslash \{b_0, *\} \\ b_1 \rightarrow_B b_0, & b_1 \in B \backslash \{b_0, *\} \text{ and } b_2 = *. \end{cases}$$

Straightforward (but really long) computations show that $(B_*, \rightarrow_*, \top)$ is a $\mathbf{WEOAlg}^{\leqslant \rightarrow}$-object. Moreover, by defining the maps $B \xrightarrow{\psi_1} B_*$, $\psi_1(b) = b$ and $B \xrightarrow{\psi_2} B_*$,

$$\psi_2(b) = \begin{cases} *, & b = b_0 \\ b, & \text{otherwise,} \end{cases}$$

one obtains $\psi_1 \circ \varphi = \psi_2 \circ \varphi$ and $\psi_1 \neq \psi_2$. It follows that every $\mathbf{WEOAlg}^{\leqslant \rightarrow}$-epimorphism must have a surjective underlying map. The converse statement is easy. ∎

In view of Corollary 36, one can ask about the restriction of the result to the category $\mathbf{WEOAlg}^{\leqslant \rightarrow \star}$. The next lemma dismisses the possibility.

Lemma 43. *The* $\mathbf{WEOAlg}^{\leqslant \rightarrow}$*-object* $(B_*, \rightarrow_*, \top)$ *constructed in Theorem* 42 *does not belong to the category* $\mathbf{WEOAlg}^{\leqslant \rightarrow \star}$*.*

Proof. Letting $b = b_0$, $b_1 = \top$, $b_2 = *$ provides $b \rightarrow_* b_1 = \top$, $b \rightarrow_* b_2 \neq \top$, but $b \rightarrow_* (b_1 \rightarrow_* b_2) = \top$. ∎

It will be the topic of our further research to characterize epimorphisms in the category $\mathbf{WEOAlg}^{\leqslant \rightarrow \star}$.

6.3. Initial morphisms

While doing lattice-valued mathematics, many researchers employ the tools of category theory to study their proposed frameworks. The most common procedure is to consider the categories of some newly introduced many-valued structures and then study their properties. It appears, however, that a more convenient framework arises if the categories in question fall into a specific class of, e.g., topological, algebraic or topologically-algebraic categories. For example, lattice-valued topology [31] relies heavily on the respective categories of many-valued structures to be topological over their ground categories. The concept of topological category in its turn depends on the notion of *initial morphism*, whose definition is given below.

Definition 44. Let $(\mathbf{A}, |-|)$ be a concrete category over $\mathbf{X}$. An $\mathbf{A}$-morphism $A \xrightarrow{f} B$ is called *initial* provided that for every $\mathbf{A}$-object C, an $\mathbf{X}$-morphism $|C| \xrightarrow{g} |A|$ is an $\mathbf{A}$-morphism whenever $|C| \xrightarrow{f \circ g} |B|$ is an $\mathbf{A}$-morphism.

A simple example of the concept provides the category **Pos** of partially ordered sets.

Theorem 45. *In the construct* **Pos**, *a morphism* $(X, \leqslant) \xrightarrow{f} (Y, \leqslant)$ *is initial iff the equivalence* $x_1 \leqslant x_2 \Leftrightarrow f(x_1) \leqslant f(x_2)$ *holds.*

Corollary 46. *Initial morphisms in* **Pos** *have injective underlying maps.*

It seems natural to generalize the procedure to the respective category of w-eo algebras (recall the notations introduced before Definition 13).

Theorem 47. *In the construct* $\mathbf{WEOAlg}^{\leqslant\rightarrow}$, *a morphism* $(A, \rightarrow, \top) \xrightarrow{\varphi} (B, \rightarrow, \top)$ *is initial iff* $a_1 \rightarrow a_2 = \bigvee\{a \in A \,|\, \varphi(a_2) \leqslant \varphi(a) \leqslant \varphi(a_1) \rightarrow \varphi(a_2)\}$ *for every* $a_1, a_2 \in A$.

Proof. For the sufficiency, consider a commutative triangle

$$\begin{tikzcd}
|(C,\rightarrow,\top)| \arrow[d, dotted, "f"'] \arrow[dr, "|\psi|"] & \\
|(A,\rightarrow,\top)| \arrow[r, "|\varphi|"'] & |(B,\rightarrow,\top)|,
\end{tikzcd}$$

where $|-|$ is the respective underlying functor. We have to show that the map $C \xrightarrow{f} A$ is a lax homomorphism. For item (2) of Definition 13, $\varphi \circ f(\top) = \psi(\top) = \top = \varphi(\top)$ yields $\top \rightarrow f(\top) = \bigvee\{a \in A \,|\, \varphi \circ f(\top) \leqslant \varphi(a) \leqslant \varphi(\top) \rightarrow \varphi \circ f(\top)\} = \top$ and, thus, $f(\top) = \top$. For item (1) of Definition 13, notice that given $c_1, c_2 \in C$, $\varphi \circ f(c_1 \rightarrow c_2) = \psi(c_1 \rightarrow c_2) \leqslant \psi(c_1) \rightarrow \psi(c_2) = \varphi \circ f(c_1) \rightarrow \varphi \circ f(c_2)$. Moreover, $c_2 \leqslant c_1 \rightarrow c_2$ implies $\varphi \circ f(c_2) = \psi(c_2) \leqslant \psi(c_1 \rightarrow c_2) = \varphi \circ f(c_1 \rightarrow c_2)$ and, therefore, $f(c_1 \rightarrow c_2) \leqslant f(c_1) \rightarrow f(c_2)$.

For the necessity, take a lax homomorphism $(A, \rightarrow, \top) \xrightarrow{\varphi} (B, \rightarrow, \top)$ such that there exist $a_1, a_2 \in A$ with $a_1 \rightarrow a_2 \neq \bigvee\{a \in A \,|\, \varphi(a_2) \leqslant \varphi(a) \leqslant \varphi(a_1) \rightarrow \varphi(a_2)\}$. It follows that there should be some $a \in A$ such that $\varphi(a_2) \leqslant \varphi(a) \leqslant \varphi(a_1) \rightarrow \varphi(a_2)$ and $a \nleqslant a_1 \rightarrow a_2$. Consider the poset $(C, \leqslant)$ given by the following Hasse diagram:

$$\begin{tikzcd}[arrows=-] & \top \arrow[dl] \arrow[dr] & \\ c_1 & & c_3 \arrow[d] \\ & & c_2 \end{tikzcd}$$

and define an operation $\to$ on C by

$$c \to c' = \begin{cases} \top, & c \leqslant c' \\ c_3, & c = c_1 \text{ and } c' = c_2 \\ c', & \text{otherwise.} \end{cases}$$

It follows that $(C, \to, \top)$ is a $\mathbf{WEOAlg}^{\leqslant \to}$-object and, moreover, by defining the maps

$$C \xrightarrow{f} A,\ f(c) = \begin{cases} a_1, & c = c_1 \\ a_2, & c = c_2 \\ a, & c = c_3 \\ \top, & c = \top \end{cases} \quad \text{and} \quad C \xrightarrow{\psi} B,\ \psi(c) = \begin{cases} \varphi(a_1), & c = c_1 \\ \varphi(a_2), & c = c_2 \\ \varphi(a), & c = c_3 \\ \top, & c = \top, \end{cases}$$

one obtains a commutative triangle similar to the above-mentioned one. This time, however, the map $|(C, \to, \top)| \xrightarrow{f} |(A, \to, \top)|$ is no more a lax homomorphism. Indeed, if it is, then $a = f(c_3) = f(c_1 \to c_2) \leqslant f(c_1) \to f(c_2) = a_1 \to a_2$ that contradicts the assumption. ∎

Similar to Corollary 46, one obtains a useful property of initial morphisms.

Corollary 48. *Initial* $\mathbf{WEOAlg}^{\leqslant \to}$*-morphisms have injective underlying maps.*

Proof. Easy computations show that every initial $\mathbf{WEOAlg}^{\leqslant \to}$-morphism $(A, \to, \top) \xrightarrow{\varphi} (B, \to, \top)$ has the property $a_1 \to a_2 = \top$ iff $\varphi(a_1) \to \varphi(a_2) = \top$ for every $a_1, a_2 \in A$. ∎

It is important to observe that the category **Prost** of preordered sets (Definition 19) is topological over its ground category, whereas the category **Pos** is not. It will be the topic of our further investigation to generalize the result for the respective categories of w-eo algebras.

6.4. Products and coproducts of objects

The concept of (co)product of objects of some category plays an important role not only in category theory, but also in universal algebra, which uses particular instances of categories called (*quasi*)*varieties*. While products of algebras are generally easy to define, their respective counterpart – coproducts (sometimes also called *sums*) are quite often more difficult to construct. In the following, we present both constructions in case of w-eo algebras. For the sake of clarity, we start by recalling the respective developments from the theory of partially ordered sets.

Theorem 49. *Given a family $((X_i, \leqslant_i))_{i\in I}$ of partially ordered sets, the cartesian product $\prod_{i\in I} X_i$ (the disjoint union $\biguplus_{i\in I} X_i$) of the underlying sets, equipped with the pointwise structure (the structure given by the disjoint union $\biguplus_{i\in I} \leqslant_i$), provides a (co)product of the family in the category* **Pos**.

Turning to the case of w-eo algebras, the procedures can be generalized as follows.

Theorem 50. *The category* **WEOAlg** *has products of objects.*

Proof. Given a family $((A_i, \rightarrow_i, \top_i))_{i\in I}$ of w-eo algebras, the cartesian product $\prod_{i\in I} A_i$ of the underlying sets, equipped with the pointwise structure, provides the required product in the category **WEOAlg**. ∎

It is easy to see that the construction of Theorem 50 easily applies to the categories $\mathbf{WEOAlg}^{\top}$, $\mathbf{WEOAlg}^{\leqslant}$, $\mathbf{WEOAlg}^{\leqslant\rightarrow}$ and $\mathbf{WEOAlg}^{\leqslant\rightarrow\star}$ as well. On the other hand, the respective coproducts are slightly more demanding.

Theorem 51. *The category* $\mathbf{WEOAlg}^{\leqslant\rightarrow}$ *has coproducts of objects.*

Proof. Given a family $((A_i, \rightarrow_i, \top_i))_{i\in I}$ of $\mathbf{WEOAlg}^{\leqslant\rightarrow}$-objects let $\bigoplus_{i\in I} A_i = (\biguplus_{i\in I}(A_i\backslash\{\top_i\}))\biguplus\{\top\}$ and $\coprod_{i\in I}(A_i, \rightarrow_i, \top_i) = (\bigoplus_{i\in I} A_i, \rightarrow, \top)$, where

$$a \rightarrow b = \begin{cases} \top, & b = \top \\ \top_i \rightarrow_i b, & a = \top \text{ and } b \in A_i \\ a \rightarrow_i b, & a, b \in A_i \text{ for some } i \in I \\ b, & a \in A_i, b \in A_j \text{ and } i \neq j. \end{cases}$$

For every $j \in I$, define $(A_j, \rightarrow_j, \top_j) \xrightarrow{\mu_j} \coprod_{i\in I}(A_i, \rightarrow_i, \top_i)$ by $\mu_j(a) = a$. It is not difficult to see that $((\mu_i)_I, \coprod_{i\in I}(A_i, \rightarrow_i, \top_i))$ provides the required coproduct in the category $\mathbf{WEOAlg}^{\leqslant\rightarrow}$. ∎

One important moment should be underlined here immediately. In [29], J. Neggers and H. S. Kim considered (co)products of d-algebras (Definition 2), which are the duals of w-eo algebras (see the next subsection). While the product construction goes as usual (there is no way to deviate), the respective coproducts (called by the authors sums) raise strong doubts. In particular, given a family $((X, *_i, 0_i))_{i\in I}$ of d-algebras, the authors propose to consider a subset (a d-subalgebra, in fact) $\bigoplus_{i\in I} X_i$ of $\prod_{i\in I} X_i$ consisting of all elements $(x_i)_{i\in I}$ such that the set $\{i \in I \mid x_i \neq 0\}$ is finite. For every $j \in I$, the respective embedding $X_j \xrightarrow{\iota_j} \bigoplus_{i\in I} X_i$ is given by $\iota_j(x) = (x_i)_{i\in I}$, where

$$x_i = \begin{cases} x, & i = j \\ 0, & \text{otherwise.} \end{cases}$$

J. Neggers *et al.* claim $((\iota_i)_I, (\bigoplus_{i\in I} X_i, *, 0))$ (both $*$ and 0 are induced by the respective d-subalgebra structure) to be the coproduct in question. While the construction goes smoothly in the category of $\vee$-semilattices (cf. Definition 26, but mind that $\vee$ now are finite), its generalization to d-algebras is problematic. The sticking point is the condition that for every family of d-algebra homomorphisms $((X_i, *_i, 0_i) \xrightarrow{f_i} (X, *, 0))_{i\in I}$, there should exist a d-algebra homomorphism $(\bigoplus_{i\in I} X_i, *, 0) \xrightarrow{[f_i]_I} (X, *, 0)$ such that $[f_i]_I \circ \iota_i = f_i$ for every $i \in I$. In case of $\vee$-semilattices, the homomorphism in question can be easily defined by $[f_i]_I((x_i)_I) = \bigvee_{i\in I} x_i$, which is correct due to the finiteness condition on the set $\bigoplus_{i\in I} X_i$. The case of d-algebras, however, does not allow the definition, since the operation $*$ in general is *not* associative. The authors themselves never provide any hint on the map in question, that leaves their claim unjustified. Moreover, our construction differs dramatically from the respective one of J. Neggers and H. S. Kim.

6.5. Dual w-eo algebras

The theory of partially ordered sets is particularly useful because of the availability of the so-called *duality principle*. Indeed, every poset $(X, \leqslant)$ has its dual $(X, \leqslant^o)$, where $x \leqslant^o y$ iff $y \leqslant x$. The operation $(-)^o$ is in fact a functor $\mathbf{Pos} \xrightarrow{(-)^o} \mathbf{Pos}$, with the property $(-)^o \circ (-)^o = 1_{\mathbf{Pos}}$. It appears that the developments can be easily extended to the case of w-eo algebras. We have already noticed in Introduction (following C. Guido *et al.* [17]) that d-transitive d-algebras of [29] provide a dual analogue of w-eo algebras. It is the purpose of this section to elaborate the result in its full extent.

Definition 52. Given a w-eo algebra $(A, \to, \top)$, its *dual* (denoted by $(A, \to, \top)^o$) is the triple $(A, \rightsquigarrow, \bot)$, where $\bot = \top$ and $a \rightsquigarrow b = b \to a$ for every $a, b \in A$.

Having a new concept in hand, it is useful to consider some of its simple features.

Lemma 53. *Every dual w-eo algebra $(A, \to, \top)^o$ has the following properties:*

(1) $\bot \rightsquigarrow a = \bot$;
(2) $a \rightsquigarrow a = \bot$;

(3) *if* $a \rightsquigarrow b = \bot$ *and* $b \rightsquigarrow a = \bot$*, then* $a = b$*;*
(4) *if* $a \rightsquigarrow b = \bot$ *and* $b \rightsquigarrow c = \bot$*, then* $a \rightsquigarrow c = \bot$*;*

and every lax homomorphism $(A, \to, \top) \xrightarrow{\varphi} (B, \to, \top)$ *satisfies the conditions:*

(1) $\varphi(a_1 \rightsquigarrow a_2) = \varphi(a_1) \rightsquigarrow \varphi(a_2)$ *for every* $a_1, a_2 \in A$*;*
(2) $\varphi(\bot) = \bot$.

Lemma 53 gives rise to a new category $\mathbf{WEOAlg}^o$ of *dual w-eo algebras*, which is isomorphic to the category of d-transitive d-algebras provided (implicitly) by J. Neggers and H. S. Kim [29]. Notice, however, that the codomain of the functor $(-)^o$ is no more the category **WEOAlg**. On the other hand, the equality $(-)^o \circ (-)^o = 1_{\mathbf{WEOAlg}}$ is still true.

7. Conclusion

In this paper, we introduced several approaches to homomorphisms of w-eo algebras, based on different categories of the structures. The two main frameworks (both having w-eo algebras as objects) are as follows:

(1) the category **WEOAlg**, the morphisms of which $(A, \to, \top) \xrightarrow{\varphi} (B, \to, \top)$ are maps $A \xrightarrow{\varphi} B$ such that $\varphi(a_1 \to a_2) = \varphi(a_1) \to \varphi(a_2)$ for every $a_1, a_2 \in A$. The additional property $\varphi(\top) = \top$ comes as a consequence.
(2) the category $\mathbf{WEOAlg}^{\leqslant}$, the morphisms of which $(A, \to, \top) \xrightarrow{\varphi} (B, \to, \top)$ are maps $A \xrightarrow{\varphi} B$ such that $\varphi(a_1 \to a_2) \leqslant \varphi(a_1) \to \varphi(a_2)$ for every $a_1, a_2 \in A$, and also $\varphi(\top) = \top$ (the latter condition does not follow automatically).

The first approach backs the algebraic viewpoint on w-eo algebras, whereas the second one considers w-eo algebras as an extension of posets. In the paper, we have assumed the second viewpoint on the structures as the one which seems to be the most appropriate to their essence (cf. the term "extended-order"). In particular, the current manuscript considered several subcategories of the category $\mathbf{WEOAlg}^{\leqslant}$, with the aim to provide a suitable framework to match different properties of the category **Pos**. The plentitude of the subcategories available motivates the following problems.

Problem 54. What is the best subcategory of $\mathbf{WEOAlg}^{\leqslant}$ for obtaining a "convenient" analogue of the category **Pos**?

Notice that by "convenient" subcategory we mean a category, whose properties allow one to restore the majority of the features of the category **Pos**.

Problem 55. Does there exist a better starting point than the category $\mathbf{WEOAlg}^{\leqslant}$?

A better starting point should share more properties of the category **Pos** than the category $\mathbf{WEOAlg}^{\leqslant}$ does and (probably) be more user-friendly.

As the last remark, we would like to point out that the overall developments of the paper suggest a more general structure called *extended-relation algebra*, which could be defined as follows.

Definition 56. An *extended-relation algebra* (*er algebra*) is a triple $(L, \rightarrow, \top)$, where L is a non-empty set, $L \times L \xrightarrow{\rightarrow} L$ is a binary operation on L, and $\top$ is a distinguished element of L.

Notice the lack of the term "weak" in the name of the new concept as being non-appropriate in the current setting. Backed by the results of the manuscript, we do not require the existence of an additional element $\perp$. On the other hand, we do need a distinguished element $\top$, to consider specific properties of er algebras (e.g., the extension of a partial order). A theory could be developed then, which substitutes particular relations with operations, and studies possible interconnections between the properties of the former and the latter. For example, one can consider different combinations of the following characteristics of binary relations [23].

Definition 57. A binary relation R on a set X is called:

(1) *connected*, if for every $x \in X$, there exists $y \in X$ such that xRy;
(2) *reflexive*, if xRx for every $x \in X$;
(3) *symmetric*, if xRy implies yRx for every $x, y \in X$;
(4) *antisymmetric*, if xRy and yRx imply $x = y$ for every $x, y \in X$;
(5) *transitive*, if xRy and yRz imply xRz for every $x, y, z \in X$.

If R is reflexive and symmetric, it is called a *tolerance relation*, and if R is reflexive and transitive, it is called a *preorder* (or a *quasi-order*). If R is both a tolerance and a preorder, then it is an *equivalence relation*.

It is easy to see that the concept of er algebra provides a framework for incorporating all the notions of Definition 57 (and many others as well). For example, a good application field for the new algebras is the theory of *rough sets* [23], where the structures could potentially serve as a good starting point for a generalization of the developments that would streamline the existing results and cast new light on various sticking points. For convenience of the reader, we briefly recall the standard approach.

Definition 58. Let R be a binary relation on a set X. Given $x \in X$, define $R(x) = \{y \in X \mid x\,R\,y\}$. For every $S \subseteq X$ let $S^{\blacktriangledown} = \{x \in X \mid R(x) \subseteq S\}$ and $S^{\blacktriangle} = \{x \in X \mid R(x) \bigcap S \neq \varnothing\}$. The pair $\mathcal{R}(S) = (S^{\blacktriangledown}, S^{\blacktriangle})$ is called the *rough set* of S. The set $\mathcal{R}(X) = \{\mathcal{R}(S) \mid S \subseteq X\}$ is called then the *set of all rough sets of* X. $\mathcal{R}(X)$ is a bounded poset, with the partial order given by $(S^{\blacktriangledown}, S^{\blacktriangle}) \leqslant (T^{\blacktriangledown}, T^{\blacktriangle})$ iff $S^{\blacktriangledown} \subseteq T^{\blacktriangledown}$ and $S^{\blacktriangle} \subseteq T^{\blacktriangle}$, whereas the lower (resp. upper) bound is $\mathcal{R}(\varnothing) = (\varnothing^{\blacktriangledown}, \varnothing)$ (resp. $\mathcal{R}(X) = (X, X^{\blacktriangle})$).

One of the most challenging questions is to characterize the algebraic structure of the above-mentioned poset $(\mathfrak{R}(X), \leqslant, \mathcal{R}(\varnothing), \mathcal{R}(X))$ generated by a particular type of relation R [7, 11, 21, 22, 24] (cf. Definition 57). On the other hand, one can start with an er algebra $(A, \rightarrow, \top)$ and define $\uparrow a = \{b \in A \mid a \rightarrow b = \top\}$ for every $a \in A$. Substituting $R(x)$ with $\uparrow a$ in the procedures of Definition 58, one obtains the poset $(\mathfrak{R}(A), \leqslant, \mathcal{R}(\varnothing), \mathcal{R}(A))$. The following problem then arises immediately.

Problem 59. What additional structure is given to the poset $(\mathfrak{R}(A), \leqslant, \mathcal{R}(\varnothing), \mathcal{R}(A))$ by the operation $\rightarrow$, and how their properties are related?

It will be the topic of our further research to approach the theory of er algebras more thoroughly.

Acknowledgements. The author is grateful to the Department of Mathematics “Ennio De Giorgi” of the University of Salento in Lecce, Italy for the repeated opportunity of spending a month at the university, during which period the draft of the manuscript was prepared.

References

[1] J. Adámek, H. Herrlich, G. E. Strecker, *Abstract and Concrete Categories: The Joy of Cats*, Dover Publications (Mineola, New York), 2009.

[2] G. Cantor, *Beitrage zur Begründung der transfiniten Mengenlehre I*, Math. Ann. 46 (1895), 481–512.

[3] S. Celani, *A note on homomorphisms of Hilbert algebras*, Int. J. Math. Math. Sci. 29(1) (2002), 55–61.

[4] S. A. Celani, L. M. Cabrer, D. Montangie, *Representation and duality for Hilbert algebras*, Cent. Eur. J. Math. 7(3) (2009), 463–478.

[5] I. Chajda, R. Halaš, J. Kühr, *Semilattice Structures*, vol. 30 of Research and Exposition in Mathematics, Heldermann Verlag, 2007.

[6] P. M. Cohn, *Universal Algebra*, D. Reidel Publ. Comp., 1981.

[7] S. D. Comer, *An algebraic approach to the approximation of information*, Ann. Soc. Math. Pol., Ser. IV, Fundam. Inf. 14(4) (1991), 492–502.

[8] M. E. Della Stella, *Extended-order algebras and fuzzy implicators*, in: Asbtracts of ManyVal’10, Università degli Studi dell’Insubria, Varese, Italy, 2010.

[9] A. Diego, *Sur les algèbres de Hilbert*, Collection Logique Math. Sér. A No. 21, Paris, 1966.

[10] J. M. Dunn, *Partial gaggles applied to logics with restricted structural rules*, Schroeder-Heister, Peter (ed.) et al., Substructural logics. Seminar for natural-language processing systems of the University of Tübingen, Germany, on 7 - 8 October 1990. Oxford: Clarendon Press. Stud. Log. Comput. 2 (1994), 63–108.

[11] M. Gehrke, E. Walker, *On the structure of rough sets*, Bull. Polish Acad. Sci. Math. 40(3) (1992), 235–245.

[12] C. Guido, *Lattice-valued categories*, in: E. P. Klement, S. E. Rodabaugh, L. N. Stout (eds.), Abstracts of the 29th Linz Seminar on Fuzzy Set Theory, Johannes Kepler Universität, Linz, 2008.

[13] C. Guido, *Attachment between fuzzy points and fuzzy sets*, in: U. Bodenhofer, B. De Baets, E. P. Klement, S. Saminger-Platz (eds.), Abstracts of the 30th Linz Seminar on Fuzzy Set Theory, Johannes Kepler Universität, Linz, 2009.

[14] C. Guido, *Fuzzy points and attachment*, Fuzzy Sets and Systems 161(16) (2010), 2150–2165.

[15] C. Guido, M. E. Della Stella, *Associativity, commutativity and symmetry in residuated structures*, submitted.

[16] C. Guido, M. E. Della Stella, *Associativity, commutativity and symmetry in residuated structures*, in: P. Cintula, E. P. Klement, L. N. Stout (eds.), Abstracts of the 31st Linz Seminar on Fuzzy Set Theory, Johannes Kepler Universität, Linz, 2010.

[17] C. Guido, P. Toto, *Extended-order algebras*, J. Appl. Log. 6(4) (2008), 609–626.

[18] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914.

[19] H. Herrlich, G. E. Strecker, *Category Theory*, vol. 1 of Sigma Series in Pure Mathematics, 3rd ed., Heldermann Verlag, 2007.

[20] D. Hofmann, *Topological theories and closed objects*, Adv. Math. 215(2) (2007), 789–824.

[21] J. Järvinen, *Approximations of rough sets based on tolerances*, Lecture Notes in Comput. Sci. 2005 (2001), 182–189.

[22] J. Järvinen, *The ordered set of rough sets*, Lecture Notes in Comput. Sci. 3066 (2004), 49–58.

[23] J. Järvinen, *Lattice Theory for Rough Sets*, in: J. F. Peters (ed.), Transactions on Rough Sets VI, vol. 4374 of Lecture Notes in Computer Science, Springer, 2007, pp. 400–498.

[24] J. Järvinen, S. Radeleczki, L. Veres, *Rough sets determined by quasiorders*, Order 26(4) (2009), 337–355.

[25] D. Kruml, J. Paseka, *Algebraic and Categorical Aspects of Quantales*, in: M. Hazewinkel (ed.), Handbook of Algebra, vol. 5, Elsevier, 2008, pp. 323–362.

[26] H. Lai, D. Zhang, *Fuzzy preorder and fuzzy topology*, Fuzzy Sets and Systems 157 (2006), 1865–1885.

[27] S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed., Springer-Verlag, 1998.

[28] J. Neggers, A. Dvurečenskij, H. S. Kim, *On d-fuzzy functions in d-algebras*, Found. Phys. 30(10) (2000), 1807–1816.

[29] J. Neggers, H. S. Kim, *On d-algebras*, Math. Slovaca 49(1) (1999), 19–26.

[30] H. Rasiowa, *An Algebraic Approach to Non-Classical Logics*, vol. 78 of Studies in Logics and the Foundations of Mathematics, North-Holland, Amsterdam, 1974.

[31] S. E. Rodabaugh, *Categorical Foundations of Variable-Basis Fuzzy Topology*, in: U. Höhle, S. E. Rodabaugh (eds.), Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory, vol. 3 of The Handbooks of Fuzzy Sets Series, Dordrecht: Kluwer Academic Publishers, 1999, pp. 273–388.

[32] S. E. Rodabaugh, *Relationship of algebraic theories to powerset theories and fuzzy topological theories for lattice-valued mathematics*, Int. J. Math. Math. Sci. 2007 (2007), 1–71.

[33] S. E. Rodabaugh, *Relationship of algebraic theories to powersets over objects in* **Set** *and* **Set**$\times$**C**, Fuzzy Sets and Systems 161(3) (2010), 453–470.

[34] K. I. Rosenthal, Quantales and Their Applications, vol. 234 of Pitman Research Notes in Mathematics, Addison Wesley Longman, 1990.
[35] S. Solovyov, *On the category* Q-**Mod**, Algebra Universalis 58 (2008), 35–58.
[36] S. Solovyov, *A representation theorem for quantale algebras*, Contr. Gen. Alg. 18 (2008), 189–198.
[37] S. Solovyov, *From quantale algebroids to topological spaces: fixed- and variable-basis approaches*, Fuzzy Sets and Systems 161(9) (2010), 1270–1287.
[38] S. Solovyov, *On fuzzification of the notion of quantaloid*, Kybernetika 46(6) (2010), 1025–1048.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LATVIA
Zellu iela 8
LV-1002 RIGA, LATVIA

INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY OF LATVIA
Raina bulvaris 29
LV-1459 RIGA, LATVIA
E-mail: sergejs.solovjovs@lu.lv, sergejs.solovjovs@lumii.lv

Received October 3, 2010; revised version February 18, 2011.