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REPRESENTATION OF MODALS

Abstract. The main aim of this paper is to describe the free objects in arbitrary varieties of modals (semilattice ordered idempotent and entropic algebras) and give some new representations of modals.

1. Introduction

Algebras considered in this paper are modes and modals. Such algebras were introduced and investigated in detail by A. Romanowska and J. D. H. Smith ([14], [15]). *Modes* (M, Ω) are characterized by two basic properties. They are *idempotent*, in the sense that each singleton is a subalgebra, and *entropic*, i.e. any two of their operations commute. The two properties may also be expressed by means of identities:

$$\omega(x, \dots, x) \approx x, \qquad \text{(idempotent law)}, \tag{1.1}$$

$$\omega(\phi(x_{11}, \dots, x_{n1}), \dots, \phi(x_{1m}, \dots, x_{nm})) \approx \phi(\omega(x_{11}, \dots, x_{1m}), \dots, \omega(x_{n1}, \dots, x_{nm})), \qquad \text{(entropic law)}, \tag{1.2}$$

for every m-ary $\omega \in \Omega$ and n-ary $\phi \in \Omega$.

An operation $f : A^n \to A$ is said to *distribute over a binary operation* $+$ on a set A if and only if for any $1 \leq i \leq n$ and $x_1, \dots, x_i, y_i, \dots, x_n \in A$:

$$f(x_1, \dots, x_i + y_i, \dots, x_n) = f(x_1, \dots, x_i, \dots, x_n) + f(x_1, \dots, y_i, \dots, x_n). \tag{1.3}$$

A *modal* is an algebra $(M, \Omega, +)$ such that (M, Ω) is a mode, $(M, +)$ is a (join) semilattice (with semilattice order $\leq$, i.e. $x \leq y \Leftrightarrow x + y = y$) and

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the operations $\omega \in \Omega$ distribute over $+$. The name "modal" was intended both to refer to the relationship with modes and to suggest the analogy with modules.

Examples of modals include distributive lattices, dissemilattices (see [8]) –algebras $(M, \cdot, +)$ with two semilattice structures $(M, \cdot)$ and $(M, +)$ in which the operation $\cdot$ distributes over the operation $+$, the algebra $(\mathbb{R}, \underline{I}^0, \max)$ defined on the set of real numbers, where $\underline{I}^0$ is the set of the following binary operations: $\underline{p} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $(x, y) \mapsto (1 - p)x + py$, for each $p \in (0, 1) \subset \mathbb{R}$, and semilattice modes (see [6]) - modes with a semilattice derived operation.

Let $(M, \Omega, +)$ be a modal generated by a non-empty set $X \subseteq M$. The subalgebra of Ω-reduct (M, Ω) generated by a set X will be called the *full Ω-mode subreduct* (*of a modal* $(M, \Omega, +)$) *relative to* X and it will be denoted by $(\langle X \rangle_\Omega, \Omega)$.

Given a mode variety $\mathcal{V}$, a modal $(M, \Omega, +)$ is called a $\mathcal{V}$-*modal* if the mode reduct (M, Ω) of the modal lies in $\mathcal{V}$. A. Romanowska and J. D. H. Smith ([14], [12]) described the free $\mathcal{V}$-modals in the case $\mathcal{V}$ is a variety of modes defined by linear identities. (We call a term t *linear*, if every variable occurs in t at most once. An identity $t \approx u$ is called *linear*, if both terms t and u are linear.)

The main aim of this paper is to describe the free objects in arbitrary varieties of modals and give some new representations of modals.

The paper is organized as follows. In Section 2, we recall basic definitions and results concerning modals and extended power algebras of modes. In Section 3 we broaden the result of A. Romanowska and J. D. H. Smith and describe the free objects in an arbitrary variety $\mathcal{M}$ of $\mathcal{V}$-modals and in the quasivariety of Ω-subreducts of modals in $\mathcal{M}$. In Section 4 we apply these results to differential modals. In Section 5 we describe the class $\mathcal{MV}$ of all modals such that for each $(M, \Omega, +) \in \mathcal{MV}$ there exists a non-empty set X of generators such that $(\langle X \rangle_\Omega, \Omega) \in \mathcal{V}$. In particular, we show that each modal in the class $\mathcal{MV}$ is a homomorphic image of the algebra of finitely generated non-empty subalgebras of some free $\mathcal{V}$-mode. We also investigate identities satisfied by modals and we present a necessary and sufficient condition for a modal to satisfy some non-linear identity. In Section 6 we present a certain representation of $\mathcal{V}$-modals based on extended power algebras of modes. We conclude the paper with a list of open problems.

Throughout the paper, $\mathcal{V}$ will denote a variety of Ω-modes. We assume that a set of generators of any algebra is non-empty.

The set of all equivalence classes of a relation $\varrho \subseteq A \times A$ is denoted by A^ϱ. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0.

2. Modals and extended power algebras of modes

The fundamental elementary properties of modals $(M, \Omega, +)$ were proved by A. Romanowska and J. D. H. Smith in [14] and may be summarized in the following three lemmas.

Lemma 2.1. (Monotonicity Lemma) *Each n-ary basic operation $\omega \in \Omega$, $\omega : (M^n, \leq) \to (M, \leq)$ is monotone (as a mapping).*

Lemma 2.2. (Convexity Lemma) *For each positive integer r, an n-ary basic operation $\omega \in \Omega$ and elements $x_{ij} \in M$ for $1 \leq i \leq n$, $1 \leq j \leq r$:*

$$\omega(x_{11}, \ldots, x_{n1}) + \ldots + \omega(x_{1r}, \ldots, x_{nr}) \leq$$
$$\omega(x_{11} + \ldots + x_{1r}, \ldots, x_{n1} + \ldots + x_{nr}).$$

Lemma 2.3. (Sum-Superiority Lemma) *For each n-ary basic operation $\omega \in \Omega$ and elements $x_1, \ldots, x_n \in M$, one has*

$$\omega(x_1, \ldots, x_n) \leq x_1 + \ldots + x_n.$$

If a modal $(M, \Omega, +)$ satisfies also the following law:

$$\omega(x_1 + y_1, \ldots, x_n + y_n) \approx \omega(x_1, \ldots, x_n) + \omega(y_1, \ldots, y_n) \tag{2.1}$$

for each n-ary operation $\omega \in \Omega$ and $x_1, \ldots, x_n$, $y_1, \ldots, y_n \in M$, then $(M, \Omega, +)$ is a mode. We will call such algebras *entropic modals*. An example is given by semilattice modes investigated by K. Kearnes [6].

For a given set A denote by $\mathcal{P}_{>0}A$ the family of all non-empty subsets of A. For any n-ary operation $\omega : A^n \to A$ we define *the complex operation* $\omega : \mathcal{P}_{>0}A^n \to \mathcal{P}_{>0}A$ in the following way:

$$\omega(A_1, \ldots, A_n) := \{\omega(a_1, \ldots, a_n) \mid a_i \in A_i\},$$

where $\emptyset \neq A_1, \ldots, A_n \subseteq A$. *The power* (*complex* or *global*) *algebra* of an algebra (A, Ω) is the algebra $(\mathcal{P}_{>0}A, \Omega)$.

The set $\mathcal{P}_{>0}A$ also carries a join semilattice structure under the set-theoretical union $\cup$. B. Jónsson and A. Tarski proved in [5] that complex operations distribute over the union $\cup$. By adding $\cup$ to the set of basic operations we obtain *the extended power algebra* $(\mathcal{P}_{>0}A, \Omega, \cup)$. The algebra $(\mathcal{P}^{<\omega}_{>0}A, \Omega, \cup)$ of all finite non-empty subsets of A is a subalgebra of the extended power algebra $(\mathcal{P}_{>0}A, \Omega, \cup)$.

As was shown by A. Romanowska and J. D. H. Smith in [14], for a given mode (M, Ω), the sets MS of non-empty subalgebras and MP of finitely generated non-empty subalgebras of (M, Ω) have a mode structure under the ω-complex operations and are subalgebras of the power algebra $(\mathcal{P}_{>0}M, \Omega)$. Moreover, the modes (MS, Ω) and (MP, Ω) satisfy each linear identity true in (M, Ω).

A. Romanowska and J. D. H. Smith also proved that for a given mode (M, Ω), the sets MS and MP have an additional (join) semilattice structure $+$ obtained by setting

$$A_1 + A_2 := \langle A_1 \cup A_2 \rangle, \tag{2.2}$$

for any $A_1, A_2 \in MS$, where $\langle X \rangle$ denotes the subalgebra of (M, Ω) generated by the set X.

These two structures, mode and semilattice, are related by distributive laws (1.3). In this way, we obtain algebras $(MS, \Omega, +)$ and $(MP, \Omega, +)$ that provide basic examples of modals. Further examples of modals are given by results of [10].

Example 2.4. [10] Let γ be a congruence relation on the extended power algebra $(\mathcal{P}_{>0}M, \Omega, \cup)$ of a mode (M, Ω), such that the quotient $(\mathcal{P}_{>0}M^{\gamma}, \Omega)$ is idempotent. Then the quotient algebra $(\mathcal{P}_{>0}M^{\gamma}, \Omega, \cup)$ is a modal.

3. Free modals

A. Romanowska and J. D. H. Smith proved the following universality property for modals crucial for our next results.

Lemma 3.1. [14] *Let (A, Ω) be a mode and $(M, \Omega, +)$ a modal. Then each mode homomorphism $h : (A, \Omega) \to (M, \Omega)$ can be extended to a unique modal homomorphism*

$$\overline{h} : (AP, \Omega, +) \to (M, \Omega, +); \quad \overline{h}(S) \mapsto \sum_{x \in X} h(x),$$

where (S, Ω) is a subalgebra of (A, Ω) generated by a finite set X.

Recall that a modal $(M, \Omega, +)$ is called a $\mathcal{V}$-modal if $(M, \Omega) \in \mathcal{V}$. Let $\mathcal{M}_{\mathcal{V}}$ denote the variety of all $\mathcal{V}$-modals.

Theorem 3.2. (Universality Property for Modals) *Let $(F_{\mathcal{V}}(X), \Omega)$ be the free $\mathcal{V}$-mode over a set X and let $(M, \Omega, +) \in \mathcal{M}_{\mathcal{V}}$. Then each mapping $h : X \to M$ can be extended to a unique modal homomorphism $\overline{h} : F_{\mathcal{V}}(X)P \to M$, such that $\overline{h}/_X = h$.*

Proof. Let $(M, \Omega, +) \in \mathcal{M}_{\mathcal{V}}$ and X be a set. By assumption, $(M, \Omega) \in \mathcal{V}$. So any mapping $h : X \to M$ may be uniquely extended to a mode homomorphism $\overline{h} : (F_{\mathcal{V}}(X), \Omega) \to (M, \Omega)$. By Lemma 3.1, the Ω-mode homomorphism $\overline{h}$ may be extended to a unique modal homomorphism $\overline{\overline{h}} : (F_{\mathcal{V}}(X)P, \Omega, +) \to (M, \Omega, +)$. ∎

Note that the modal $(F_{\mathcal{V}}(X)P, \Omega, +)$ is generated by the set $\{\{x\} \mid x \in X\}$. Hence, if $(F_{\mathcal{V}}(X)P, \Omega, +) \in \mathcal{M}_{\mathcal{V}}$, then it is, up to isomorphism, the unique algebra in $\mathcal{M}_{\mathcal{V}}$ generated by a set X, with the universal mapping property.

Corollary 3.3. *The modal $(F_{\mathcal{V}}(X)P, \Omega, +)$ is free over a set X in the variety $\mathcal{M}_{\mathcal{V}}$ if and only if $(F_{\mathcal{V}}(X)P, \Omega, +) \in \mathcal{M}_{\mathcal{V}}$.*

For a variety $\mathcal{V}$ of modes let $\mathcal{V}^*$ be its *linearization*, the idempotent variety defined by the linear identities satisfied in $\mathcal{V}$. Obviously, $\mathcal{V}^*$ is a variety of modes, and contains $\mathcal{V}$ as a subvariety.

It is well known (see for example [15]) that for any variety $\mathcal{V}$ of modes, the variety generated by the class $\{(AS, \Omega) \mid (A, \Omega) \in \mathcal{V}\}$ is included in $\mathcal{V}^*$. So by Corollary 3.3 we immediately obtain the following result proved earlier by A. Romanowska and J. D. H. Smith.

Theorem 3.4. [14] *The modal $(F_{\mathcal{V}^*}(X)P, \Omega, +)$ is free over a set X in the variety $\mathcal{M}_{\mathcal{V}^*}$.*

Let $\mathcal{M}$ be a non-trivial subvariety of $\mathcal{M}_{\mathcal{V}}$ and X be a set. By [15, Chapter 3.3] the congruence

$$\Theta_{\mathcal{M}}(X) := \bigcap \{\phi \in Con(F_{\mathcal{V}}(X)P, \Omega, +) \mid (F_{\mathcal{V}}(X)P^{\phi}, \Omega, +) \in \mathcal{M}\}$$

is the so-called $\mathcal{M}$-*replica congruence* of $(F_{\mathcal{V}}(X)P, \Omega, +)$ and $(F_{\mathcal{V}}(X)P^{\Theta_{\mathcal{M}}(X)}, \Omega, +)$ is called the $\mathcal{M}$-*replica* of $(F_{\mathcal{V}}(X)P, \Omega, +)$.

Let $(M, \Omega, +) \in \mathcal{M}$. By the universality property of replication (see [15, Lemma 3.3.1.]), for each modal homomorphism $\overline{h} : F_{\mathcal{V}}(X)P \to M$, there is a unique modal homomorphism $\overline{\overline{h}} : F_{\mathcal{V}}(X)P^{\Theta_{\mathcal{M}}(X)} \to M$ such that $\overline{h} = \overline{\overline{h}} \circ nat\Theta_{\mathcal{M}}(X)$, where $nat\Theta_{\mathcal{M}}(X)$ is the natural projection onto the quotient $F_{\mathcal{V}}(X)P^{\Theta_{\mathcal{M}}(X)}$. Hence, by Theorem 3.2, the universality property for $(F_{\mathcal{V}}(X)P, \Omega, +)$ yields the following commuting diagram for any mapping $h : X \to M$:

$$\begin{tikzcd}
X \arrow[r, hook, "i"] \arrow[dr, "h"'] & (F_{\mathcal{V}}(X)P, \Omega, +) \arrow[r, "nat\Theta_{\mathcal{M}}(X)"] \arrow[d, "\overline{h}"] & (F_{\mathcal{V}}(X)P^{\Theta_{\mathcal{M}}(X)}, \Omega, +) \arrow[dl, "\overline{\overline{h}}"] \\
 & (M, \Omega, +) &
\end{tikzcd}$$

As a result, we obtain

Theorem 3.5. *The $\mathcal{M}$-replica of the algebra $(F_{\mathcal{V}}(X)P, \Omega, +)$ is free over a set X in the variety $\mathcal{M} \subseteq \mathcal{M}_{\mathcal{V}}$.*

Corollary 3.6. *Let $(F_{\mathcal{V}}(X)P, \Omega, +) \in \mathcal{M}$. Then it is free in $\mathcal{M} \subseteq \mathcal{M}_{\mathcal{V}}$ over a set X.*

Example 3.7. Let $\mathcal{SL}$ be the variety of semilattices and $\mathcal{DL}$ denote the variety of distributive lattices. Each distributive lattice may be considered as a modal $(M, \cdot, +)$, where $(M, \cdot)$ is a semilattice and the additional identity $x + xy = x$ is satisfied. By Theorem 3.5, the free distributive lattice on a set

X is isomorphic to the $\mathcal{DL}$-replica of the algebra $(F_{\mathcal{SL}}(X)P, \cdot, +)$, where $(F_{\mathcal{SL}}(X), \cdot)$ is the free semilattice over X. By results of [10] and [14] we can describe the congruence $\Theta_{\mathcal{DL}}(X)$ in more detail. It is the congruence ($\alpha_{\{\cdot\}}$ in notation of [10]) such, that for finite subsets $A, B \subseteq F_{\mathcal{SL}}(X)$, $(A, B) \in \Theta_{\mathcal{DL}}(X)$ iff $F_{\mathcal{SL}}(X) \cdot A = F_{\mathcal{SL}}(X) \cdot B$ (or equivalently A and B generate the same *sink* or *ideal*). In this way, we obtain the following

Corollary 3.8. [14] *The free distributive lattice on a set X is the modal of finitely generated non-empty sinks of the free semilattice $(F_{\mathcal{SL}}(X), \cdot)$ on X.*

Let (A, Γ) be an algebra of a given type $\tau : \Gamma \to \mathbb{N}$. Denote by $\mathfrak{B}\Gamma$ a set of derived (or term) operations of Γ and let $\Omega \subseteq \mathfrak{B}\Gamma$. An algebra (A, Ω) is said to be a *reduct* (*Ω-reduct*) of the algebra (A, Γ). A subalgebra of a reduct of (A, Γ) is called a *subreduct*.

It is well known that the subreducts of algebras in a given quasivariety again form a quasivariety (see [7]). Let $\mathcal{Q}$ be a quasivariety of Γ-algebras of a given type $\tau : \Gamma \to \mathbb{N}$. Consider the quasivariety $\mathcal{Q}_\Omega$ of Ω-algebras isomorphic to Ω-subreducts of $\mathcal{Q}$-algebras.

Theorem 3.9. *The free $\mathcal{Q}_\Omega$-algebra $(F_{\mathcal{Q}_\Omega}(X), \Omega)$ over X is isomorphic to the Ω-subreduct $(\langle X\rangle_\Omega, \Omega)$, generated by X, of the free $\mathcal{Q}$-algebra $(F_{\mathcal{Q}}(X), \Gamma)$.*

Theorem 3.9 was formulated in [11], but only for subreducts of affine spaces. It seems to have appeared also in other papers but again for particular algebras not in general context. For the sake of completeness we give here the self-contained proof of this result. Nevertheless, we use exactly the same methods as in [11].

Proof. The universality property for the free algebra $(F_{\mathcal{Q}}(X), \Gamma)$:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "h"'] & (F_{\mathcal{Q}}(X), \Gamma) \arrow[d, "\overline{h}"] \\ & (A, \Gamma) \end{tikzcd}$$

assures commutativity of the following diagram for each Ω-reduct of $(A, \Gamma) \in \mathcal{Q}$ and a mapping $h : X \to A$:

$$\begin{tikzcd} X \arrow[r, hook, "e"] \arrow[dr, "h"'] & \langle X\rangle_\Omega \arrow[r, hook, "i"] & (F_{\mathcal{Q}}(X), \Omega) \arrow[dl, "\overline{h}_\Omega"] \\ & (A, \Omega) & \end{tikzcd}$$

Here $\overline{h}_\Omega$ is the restriction of the uniquely defined Γ-homomorphism $\overline{h} : F_{\mathcal{Q}}(X) \to A$ to the Ω-reduct.

Now let (B, Ω) be a subalgebra of (A, Ω). We want to show that for any mapping $h' : X \to B$, there is a uniquely defined Ω-homomorphism $\overline{h}' : \langle X\rangle_\Omega \to (B, \Omega)$ such that $\overline{h}' \circ e = h'$. Let $j : B \to A$; $x \mapsto x$ be the embedding of (B, Ω) into (A, Ω). Take h equal to $j \circ h'$. Then $\overline{h}_\Omega$ maps $F_{\mathcal{Q}}(X)$ into $j(B)$, and similarly $\overline{h}_\Omega \circ i$ maps $\langle X \rangle_\Omega$ into $j(B)$ (see the diagram below). Define $\overline{h}' := j^{-1} \circ \overline{h}_\Omega \circ i$. It is easy to see that $\overline{h}'$ satisfies the required properties. Hence $\langle X \rangle_\Omega$ is the free $\mathcal{Q}_\Omega$-algebra $(F_{\mathcal{Q}_\Omega}(X), \Omega)$.

$$\begin{tikzcd}
X \arrow[r, hook, "e"] \arrow[dr, "h'"'] \arrow[ddr, bend right, "h"'] & \langle X \rangle_\Omega \arrow[r, hook, "i"] \arrow[d, "\overline{h}'"] & (F_{\mathcal{Q}}(X), \Omega) \arrow[ddl, "\overline{h}_\Omega"] \\
& B \arrow[d, "j"] & \\
& j(B) &
\end{tikzcd}$$

■

Free algebras in a quasivariety $\mathcal{Q}_\Omega$ are also free in the variety $V(\mathcal{Q}_\Omega)$ generated by $\mathcal{Q}_\Omega$ (see [7]).

Let $(M, \Omega, +)$ be a modal generated by a set $X \subseteq M$. Denote by $(\langle X \rangle_\Omega, \Omega)$ the subalgebra of the Ω-reduct (M, Ω) generated by the set X. The algebra $(\langle X \rangle_\Omega, \Omega)$ is necessarily a submode of (M, Ω) and contains all elements from $(M, \Omega, +)$ obtained as results of operations from $\mathfrak{B}\Omega$ on the set X. We will call it the *full Ω-mode subreduct* (*of a modal* $(M, \Omega, +)$) *relative to* X.

As a corollary one obtains a characterization of free algebras in the quasivariety of Ω-subreducts of modals in a given variety of modals.

Corollary 3.10. *Let $\mathcal{M}$ be a variety of modals. Let $\mathcal{M}_\Omega$ be a quasivariety of Ω-subreducts of modals in $\mathcal{M}$. Then the free $\mathcal{M}_\Omega$-mode $(F_{\mathcal{M}_\Omega}(X), \Omega)$ over X is isomorphic to the full Ω-subreduct $(\langle X \rangle_\Omega, \Omega)$ of the free $\mathcal{M}$-modal $(F_{\mathcal{M}}(X), \Omega, +)$.*

4. Free differential modals

In this section we will illustrate the results of Section 3 and describe free objects in some varieties of differential modals. Further examples will be presented in Section 5.

Definition 4.1. A *differential groupoid* is a mode groupoid $(D, \cdot)$ satisfying the additional linear identity:

$$x(yz) \approx xy.$$

Each proper non-trivial subvariety of the variety $\mathcal{D}$ of differential groupoids (see [13]) is relatively based by a unique identity of the form

$$(\ldots((x \underbrace{y)y)\ldots)y}_{i-\text{times}} =: xy^i \approx xy^{i+j} \tag{4.1}$$

for some $i \in \mathbb{N}$ and positive integer j. Denote such a variety by $\mathcal{D}_{i,i+j}$.

Let $(F_{\mathcal{D}}(\{x,y\}),\cdot)$ be the free algebra on two generators x and y in the variety of differential groupoids. As was shown in [13] each element in $(F_{\mathcal{D}}(\{x,y\}),\cdot)$ may be expressed as xy^k or yx^n for some $k,n\in\mathbb{N}$ ($xy^0:=x$ and $yx^0:=y$). So the subalgebras of $(F_{\mathcal{D}}(\{x,y\}),\cdot)$ are any finite or infinite subsets of sets $\{x,xy,xy^2,xy^3,\ldots\}$ or $\{y,yx,yx^2,yx^3,\ldots\}$ or subalgebras generated by two elements xy^k and yx^n for some $k,n\in\mathbb{N}$, i.e.:

$$\langle\{xy^k,yx^n\}\rangle=\{xy^k,xy^{k+1},xy^{k+2},\ldots\}\cup\{yx^n,yx^{n+1},yx^{n+2},\ldots\}.$$

A *differential modal* is a modal whose mode reduct is a differential groupoid.

Example 4.2. Consider the free differential groupoid $(F_{\mathcal{D}_{i,i+j}}(\{x,y\}),\cdot)$ on two generators x and y in the subvariety $\mathcal{D}_{i,i+j}$, for $j>1$. The following two sets:

$$\{x\}F_{\mathcal{D}_{i,i+j}}(\{x,y\})^i=\{x,xy,\ldots,xy^i\}$$

and

$$\{x\}F_{\mathcal{D}_{i,i+j}}(\{x,y\})^{i+j}=\{x,xy,\ldots,xy^i,xy^{i+1},\ldots,xy^{i+j-1}\}$$

are different.

By Corollary 3.3 the algebra $(F_{\mathcal{D}_{i,i+j}}(X)P,\cdot,+)$, for any set X with more than one element, is not free in the variety $\mathcal{M}_{\mathcal{D}_{i,i+j}}$, where $j>1$.

Example 4.3. In particular, consider the free differential groupoid

$$(F_{\mathcal{D}_{0,2}}(\{x,y\}),\cdot)$$

on two generators x and y in the subvariety $\mathcal{D}_{0,2}$ defined by the identity:

$$x\approx(xy)y.$$

Such a groupoid has 4 elements $\{x,y,xy,yx\}$, the following multiplication table:

$\cdot$	x	xy	y	yx
x	x	x	xy	xy
xy	xy	xy	x	x
y	yx	yx	y	y
yx	y	y	yx	yx

and 7 subalgebras, namely: $a:=\{x\}$, $b:=\{y\}$, $c:=\{xy\}$, $d:=\{yx\}$, $e:=\{x,xy\}$, $f:=\{y,yx\}$, $g:=\{x,y,xy,yx\}$.

The multiplication table of the mode reduct of $(F_{\mathcal{D}_{0,2}}(\{x,y\})P,\cdot,+)$ is the following:

$\cdot$	a	b	c	d	e	f	g
a	a	c	a	c	a	c	e
b	d	b	d	b	d	b	f
c	c	a	c	a	c	a	e
d	b	d	b	d	b	d	f
e	e	e	e	e	e	e	e
f	f	f	f	f	f	f	f
g	g	g	g	g	g	g	g

By [15], the modal $(F_{\mathcal{D}_{0,2}}(\{x,y\})P,\cdot,+) \in \mathcal{M}_{\mathcal{D}}$, but it is not a $\mathcal{D}_{0,2}$-modal:

$$(ag)g = (\{x\}\{x,y,xy,yx\})\{x,y,xy,yx\} = \{x,xy\} = e \neq \{x\} = a.$$

By Corollary 3.3, $(F_{\mathcal{D}_{0,2}}(\{x,y\})P,\cdot,+)$ is not free in the variety $\mathcal{M}_{\mathcal{D}_{0,2}}$. But it satisfies the non-linear identity: $xy \approx ((xy)y)y$, hence

$$(F_{\mathcal{D}_{0,2}}(\{x,y\})P,\cdot,+) \in \mathcal{M}_{\mathcal{D}_{1,3}}.$$

Example 4.4. Consider the free differential groupoid $(F_{\mathcal{D}_{i,i+1}}(\{x,y,z\}),\cdot)$ on three generators x, y and z in the subvariety $\mathcal{D}_{i,i+1}$. Let $(A,\cdot) := \langle\{y,z\}\rangle$ be the subalgebra of $(F_{\mathcal{D}_{i,i+1}}(\{x,y,z\}),\cdot)$ generated by y and z. It is easy to check that $\{x\}A^i = \{xy^i, xy^{i-1}z, \ldots, xyz^{i-1}, xz^i\}$ and $\{x\}A^{i+1} = \{xy^i, xy^iz, \ldots, xy^2z^{i-1}, xyz^i, xz^i\}$. Hence, for any set X with at least three elements, the algebra $(F_{\mathcal{D}_{i,i+1}}(X)P,\cdot,+)$ is not free in the variety $\mathcal{M}_{\mathcal{D}_{i,i+1}}$.

Theorem 4.5. *Let $i \in \mathbb{N}$. The modal $(F_{\mathcal{D}_{i,i+1}}(\{x,y\})P,\cdot,+)$ is free in the variety $\mathcal{M}_{\mathcal{D}_{i,i+1}}$.*

Proof. Let $(F_{\mathcal{D}_{i,i+1}}(\{x,y\}),\cdot)$ be the free differential mode on two generators x and y in the subvariety $\mathcal{D}_{i,i+1}$. Any subalgebra of $(F_{\mathcal{D}_{i,i+1}}(\{x,y\}),\cdot)$ has one of the three following forms:

$A_1 = \langle\{xy^{k_1},\ldots,xy^{k_n}\}\rangle = \{xy^{k_1}, xy^{k_2},\ldots,xy^{k_n}\}$, for $i \geq k_1,\ldots,k_n \in \mathbb{N}$,
$A_2 = \langle\{xy^k, yx^l\}\rangle$, for $i \geq k, l \in \mathbb{N}$,
$A_3 = \langle\{yx^{l_1}, yx^{l_2},\ldots,yx^{l_m}\}\rangle = \{yx^{l_1}, yx^{l_2},\ldots,yx^{l_m}\}$, for $i \geq l_1,\ldots,l_m \in \mathbb{N}$.
Hence, $\{xy^{k_1},\ldots,xy^{k_n}\}\{xy^{l_1},\ldots,xy^{l_m}\} = \{xy^{k_1},\ldots,xy^{k_n}\}$;
$\{xy^{k_1},\ldots,xy^{k_n}\}\langle\{xy^k,yx^l\}\rangle^i =$
$\{xy^{k_1},\ldots,xy^{k_n},xy^{k_1+1},\ldots,xy^{k_n+1},\ldots,xy^{k_1+i},\ldots,xy^{k_n+i}\} =$
$\{xy^{k_1},\ldots,xy^{k_n},xy^{k_1+1},\ldots,xy^{k_n+1},\ldots,xy^{k_1+i+1},\ldots,xy^{k_n+i+1}\} =$
$\{xy^{k_1},\ldots,xy^{k_n}\}\langle\{xy^k,yx^l\}\rangle^{i+1}$;
$\{xy^{k_1},\ldots,xy^{k_n}\}\{yx^{l_1},\ldots,yx^{l_m}\}^i = \{xy^{k_1+i},\ldots,xy^{k_n+i}\} =$
$\{xy^{k_1+i+1},\ldots,xy^{k_n+i+1}\} = \{xy^{k_1},\ldots,xy^{k_n}\}\{yx^{l_1},\ldots,yx^{l_m}\}^{i+1}$;
$\langle\{xy^k,yx^l\}\rangle\langle\{xy^s,yx^t\}\rangle = \langle\{xy^k,xy^{k+1},yx^l,yx^{l+1}\}\rangle = \langle\{xy^k,yx^l\}\rangle$;

$\langle\{xy^k, yx^l\}\rangle\{yx^{l_1}, yx^{l_2}, \dots, yx^{l_m}\}^i = \langle\{xy^{k+i}, yx^l\}\rangle = \langle\{xy^{k+i+1}, yx^l\}\rangle =$
$\langle\{xy^k, yx^l\}\rangle\{yx^{l_1}, yx^{l_2}, \dots, yx^{l_m}\}^{i+1}$.

It follows that the algebra $(F_{\mathcal{D}_{i,i+1}}(\{x,y\})P, \cdot, +)$ satisfies the identity (4.1) for $j = 1$, and is free in the variety $\mathcal{M}_{\mathcal{D}_{i,i+1}}$ by Corollary 3.3. ∎

Example 4.6. The free differential groupoid $(F_{\mathcal{D}_{1,2}}(\{x,y\}), \cdot)$ on two generators x and y also has four elements: $\{x, y, xy, yx\}$, the following multiplication table:

$\cdot$	x	xy	y	yx
x	x	x	xy	xy
xy	xy	xy	xy	xy
y	yx	yx	y	y
yx	yx	yx	yx	yx

and ten subalgebras: $a := \{x\}$, $b := \{y\}$, $c := \{xy\}$, $d := \{yx\}$, $e := \{x, xy\}$, $f := \{y, yx\}$, $g := \{xy, yx\}$, $h := \{x, xy, yx\}$, $i := \{y, xy, yx\}$ and $j := \{x, y, xy, yx\}$. Its semilattice reduct is in Figure 1.

The multiplication table of the mode reduct of $(F_{\mathcal{D}_{1,2}}(\{x,y\})P, \cdot, +)$ is the following:

$\cdot$	a	b	c	d	e	f	g	h	i	j
a	a	c	a	c	a	c	e	e	e	e
b	d	b	d	b	d	b	f	f	f	f
c	c	c	c	c	c	c	c	c	c	c
d	d	d	d	d	d	d	d	d	d	d
e	e	c	e	c	e	c	e	e	e	e
f	d	f	d	f	d	f	f	f	f	f
g	g	g	g	g	g	g	g	g	g	g
h	h	g	h	g	h	g	h	h	h	h
i	g	i	g	i	g	i	i	i	i	i
j	h	i	h	i	h	i	j	j	j	j

It is not difficult to check that $(F_{\mathcal{D}_{1,2}}(\{x,y\})P, \cdot, +) \in \mathcal{M}_{\mathcal{D}_{1,2}}$. By Corollary 3.3, $(F_{\mathcal{D}_{1,2}}(\{x,y\})P, \cdot, +)$ is free in the variety $\mathcal{M}_{\mathcal{D}_{1,2}}$.

Let $\mathcal{MV}$ denote the class of all modals such that for each $(M, \Omega, +) \in \mathcal{MV}$ there exists a set X of generators, such that its full Ω-mode subreduct relative to X lies in $\mathcal{V}$.

Corollary 3.10 raises the question whether for the free $\mathcal{M}$-modal generated by a set X, for a given variety $\mathcal{M} \subseteq \mathcal{MV}$, its full Ω-mode subreduct

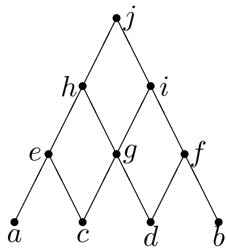

Fig. 1. Semilattice reduct of $(F_{\mathcal{D}_{1,2}}(\{x, y\})P, \cdot, +)$

relative to X is a free mode in $\mathcal{V}$. By Theorem 3.4 this is the case for varieties $\mathcal{M}_{\mathcal{V}^*}$, where $\mathcal{V}^*$ is the linearization of $\mathcal{V}$. The following example gives a negative answer in general.

Example 4.7. Let $(M, \cdot, +)$ be a modal with operations $\cdot$ and $+$ defined as follows:

$\cdot$	x	y	z	u	t
x	x	x	x	x	x
y	z	y	y	y	u
z	y	z	z	z	u
u	u	u	u	u	u
t	t	t	t	t	t

$+$	x	y	z	u	t
x	x	t	t	t	t
y	t	y	u	u	t
z	t	u	z	u	t
u	t	u	u	u	t
t	t	t	t	t	t

.

Consider the variety $\mathcal{M} \subseteq \mathcal{MD}_{1,3}$ of modals generated by the modal $(M, \cdot, +)$. Obviously, $(M, \cdot, +)$ is a free modal on two generators x and y in the variety $\mathcal{M}$. Now the free mode $(F_{\mathcal{M}_{\{\cdot\}}}(\{x, y\}), \cdot)$ in the quasivariety $\mathcal{M}_{\{\cdot\}}$ of $\{\cdot\}$-subreducts of modals from $\mathcal{M}$ is isomorphic to the full subreduct $(\langle\{x, y\}\rangle_{\{\cdot\}}, \cdot)$ defined as follows:

$\cdot$	x	y	z
x	x	x	x
y	z	y	y
z	y	z	z

The free mode $(F_{\mathcal{M}_{\{\cdot\}}}(\{x, y\}), \cdot)$ is also free in the variety $V(\mathcal{M}_{\{\cdot\}})$ generated by the quasivariety $\mathcal{M}_{\{\cdot\}}$, but it is not free in the variety $\mathcal{D}_{1,3}$, although $(\langle\{x, y\}\rangle_{\{\cdot\}}, \cdot)$ lies in this variety. This shows that $V(\mathcal{M}_{\{\cdot\}}) \neq \mathcal{D}_{1,3}$.

5. The class $\mathcal{MV}$ of modals

In this section we will describe in more detail modals in the class $\mathcal{MV}$.

Example 5.1. Let $(M, \Omega) = \langle Y \rangle$ be a mode generated by a set $Y \subseteq M$. The set $\mathfrak{X} := \{\{y\} \mid y \in Y\}$ generates the modal $(MP, \Omega, +)$ and

$(\langle\mathfrak{X}\rangle_\Omega, \Omega) \cong (M, \Omega)$. If (M, Ω) lies in a variety $\mathcal{V}$ then $(MP, \Omega, +)$ belongs to the class $\mathcal{MV}$. In particular for the free mode $(F_\mathcal{V}(X), \Omega)$ in a variety $\mathcal{V}$ over a set X, the modal $(F_\mathcal{V}(X)P, \Omega, +)$ is generated by the set $\{\{x\} \mid x \in X\}$. Hence $\langle\{\{x\} \mid x \in X\}\rangle_\Omega \cong (\langle X\rangle_\Omega, \Omega) = (F_\mathcal{V}(X), \Omega)$ and $(F_\mathcal{V}(X)P, \Omega, +)$ belongs to the class $\mathcal{MV}$.

Let $(M, \Omega, +)$ be a modal generated by a set $X \subseteq M$. An element $r \in M$ is said to be in *disjunctive form* if it is a join of a finite number of elements from $\langle X\rangle_\Omega$. The following lemma shows that each element in a modal may be expressed in such form.

Lemma 5.2. (Disjunctive Form Lemma) *Let $(M, \Omega, +)$ be a modal generated by a set $X \subseteq M$. For each $r \in M$, there exist $r_1, \ldots, r_p \in \langle X\rangle_\Omega$ such that $r = r_1 + \ldots + r_p$.*

Proof. The proof goes by induction on the minimal number m of occurrences of the semilattice operation $+$ in the expression of r as a modal word in the alphabet X.

Consider $r = r_1$ with $r_1 \in \langle X\rangle_\Omega$. Hence, the result holds for $m = 0$. Now suppose that the hypothesis is established for $m > 0$ and let $r \in M$ be an element in which the semilattice operation $+$ occurs $m + 1$ times. Let $r = r_1 + r_2$, for some $r_1, r_2 \in M$. By induction hypothesis there are $r_{11}, \ldots, r_{1k}, r_{21}, \ldots, r_{2n} \in \langle X\rangle_\Omega$ such that

$$r = r_1 + r_2 = r_{11} + \ldots + r_{1k} + r_{21} + \ldots + r_{2n}.$$

Otherwise, $r = \omega(r_1, \ldots, r_k + s_k, \ldots, r_n)$ for some $\omega \in \Omega$ and $r_1, \ldots, r_k, \ldots, r_n$, $s_k \in M$. Then, by distributivity we have

$$r = \omega(r_1, \ldots, r_k + s_k, \ldots, r_n) = \omega(r_1, \ldots, r_k, \ldots, r_n) + \omega(r_1, \ldots, s_k, \ldots, r_n).$$

Because $\omega(r_1, \ldots, r_k, \ldots, r_n), \omega(r_1, \ldots, s_k, \ldots, r_n) \in M$, this completes the inductive proof. ∎

Corollary 5.3. *Let $(M, \Omega, +)$ be a modal generated by a set $X \subseteq M$. There is a set $Y \subseteq M$ of generators of the semilattice $(M, +)$ such that $Y \subseteq \langle X\rangle_\Omega$.*

Theorem 5.4. *Let $\mathcal{V}$ be a variety of Ω-modes satisfying an identity $t \approx u$ and $\mathcal{M} \subseteq \mathcal{MV}$ be a variety of modals $(M, \Omega, +)$ such that the word operation $t : M^n \to M$ distributes over the operation $+$. Then the identity $t \approx u$ is satisfied in $\mathcal{M}$ if and only if the word operation $u : M^n \to M$ distributes over the operation $+$.*

Proof. Let $(M, \Omega, +) \in \mathcal{M}$ and let the word operation $t : M^n \to M$ distribute over the operation $+$. Because the variety $\mathcal{M}$ is, by assumption, included in $\mathcal{MV}$, there exists a set X of generators of $(M, \Omega, +)$, such that its full Ω-mode subreduct relative to X belongs to the variety $\mathcal{V}$. Hence, the identity $t \approx u$ is also true in $(\langle X\rangle_\Omega, \Omega)$.

Suppose first that the word operation $u : M^n \to M$ distribute over the operation $+$ and let $r_1, \dots, r_n \in M$. By the Disjunctive Form Lemma 5.2 there exist $r_{11}, \dots, r_{1k_1}, \dots, r_{n1}, \dots, r_{nk_n} \in \langle X\rangle_\Omega$ such that for each $1 \le i \le n$, $r_i = r_{i1} + \ldots + r_{ik_i}$. Then, by distributivity of operations $t : M^n \to M$ and $u : M^n \to M$

$$\begin{aligned} t(r_1, \dots, r_n) &= t(r_{11} + \ldots + r_{1k_1}, \dots, r_{n1} + \ldots + r_{nk_n}) \\ &= \sum_{\substack{1\le i\le n \\ a_i\in\{r_{i1},\dots,r_{ik_i}\}}} t(a_1, \dots, a_n) \approx \sum_{\substack{1\le i\le n \\ a_i\in\{r_{i1},\dots,r_{ik_i}\}}} u(a_1, \dots, a_n) \\ &= u(r_{11} + \ldots + r_{1k_1}, \dots, r_{n1} + \ldots + r_{nk_n}) \\ &= u(r_1, \dots, r_n). \end{aligned}$$

The converse implication is obvious. ∎

Lemma 5.5. *Let $(M, \Omega, +)$ be a modal and let t be an n-ary linear Ω-term. The word operation $t : M^n \to M$ distributes over the operation $+$.*

Proof. The proof will go by induction on the minimal number m of occurrences of (symbols of) the basic Ω-operations in the corresponding linear Ω-term.

By definition of a modal, the lemma is certainly true for $m = 1$. Now suppose that the hypothesis is established for $m > 1$. Let $t(x_{11}, \dots, x_{kp_k}) = \omega(\nu_1(x_{11}, \dots, x_{1p_1}), \dots, \nu_k(x_{k1}, \dots, x_{kp_k}))$ be a linear Ω-term, for some $\omega \in \Omega$, different variable symbols $x_{11}, \dots, x_{1p_1}, \dots, x_{k1}, \dots, x_{kp_k}$ and linear Ω-words $\nu_1, \dots, \nu_k$, in which the basic Ω-operations occur $m + 1$ times.

By induction hypothesis, the Ω-word operations $\nu_i : M^{p_i} \to M$, for $1 \le i \le k$, distribute over the operation $+$. This implies that for any

$$\begin{aligned} & x_{11}, \dots, x_{1p_1}, \dots, x_{i1}, \dots, x_{ij}, y_{ij}, \dots, x_{ip_i}, \dots, x_{k1}, \dots, x_{kp_k} \in M, \\ & t(x_{11}, \dots, x_{ij}+y_{ij}, \dots, x_{kp_k}) \\ &= \omega(\nu_1(x_{11}, \dots, x_{1p_1}), \dots, \nu_i(x_{i1}, \dots, x_{ij}+y_{ij}, \dots, x_{ip_i}), \dots, \nu_k(x_{k1}, \dots, x_{kp_k})) \\ &= \omega(\nu_1(x_{11}, \dots, x_{1p_1}), \dots, \nu_i(x_{i1}, \dots, x_{ij}, \dots, x_{ip_i}) \\ &\quad +\nu_i(x_{i1}, \dots, y_{ij}, \dots, x_{ip_i}), \dots, \nu_k(x_{k1}, \dots, x_{kp_k})) \\ &= \omega(\nu_1(x_{11}, \dots, x_{1p_1}), \dots, \nu_i(x_{i1}, \dots, x_{ij}, \dots, x_{ip_i}), \dots, \nu_k(x_{k1}, \dots, x_{kp_k})) \\ &\quad +\omega(\nu_1(x_{11}, \dots, x_{1p_1}), \dots, \nu_i(x_{i1}, \dots, y_{ij}, \dots, x_{ip_i}), \dots, \nu_k(x_{k1}, \dots, x_{kp_k})) \\ &= t(x_{11}, \dots, x_{ij}, \dots, x_{kp_k})+t(x_{11}, \dots, y_{ij}, \dots, x_{kp_k}), \end{aligned}$$

what finishes the proof. ∎

Corollary 5.6. *Let $\mathcal{V}$ be a variety of modes satisfying an identity $t \approx u$, where t is linear. The identity $t \approx u$ is true in a variety $\mathcal{M} \subseteq \mathcal{MV}$ of* *modals if and only if the word operation $u : M^n \to M$ distributes over the operation $+$.*

Corollary 5.7. *A variety $\mathcal{M} \subseteq \mathcal{MV}$ of modals satisfies each linear identity true in $\mathcal{V}$.*

Theorem 5.8. *Let $j \in \mathbb{N}$. The free modal on n generators in the variety $\mathcal{M}_{\mathcal{D}_{0,j}}$ is isomorphic to the modal with the free semilattice on n generators as the semilattice reduct and left-zero semigroup as the groupoid reduct.*

Proof. Let $(F_{\mathcal{D}_{0,j}}(A), \cdot)$ be the free differential mode on n generators $A := \{a_1, \dots, a_n\}$ in the subvariety $\mathcal{D}_{0,j}$. By results of [13] each element in $F_{\mathcal{D}_{0,j}}(A)$ may be expressed as $a_i a_1^{k_1} \dots a_{i-1}^{k_{i-1}} a_{i+1}^{k_{i+1}} \dots a_n^{k_n}$, for $1 \le i \le n$ and $0 \le k_1, \dots, k_n \le j-1$. Moreover, a subalgebra of $(F_{\mathcal{D}_{0,j}}(A), \cdot)$ is generated by any non-empty subset of A.

On the other hand, by Corollary 5.6, each modal in the variety $\mathcal{M}_{\mathcal{D}_{0,j}}$ has to satisfy the identity

$$x(y + y_1)^j \approx xy^j + xy_1^j. \tag{5.1}$$

Let $(M, \cdot, +) \in \mathcal{M}_{\mathcal{D}_{0,j}}$ be a modal generated by a set X. Let $x, x_1, x_2, y, y_1, \dots, y_m, z_1, \dots, z_s \in \langle X \rangle_{\{\cdot\}}$. Note that in particular $(\langle X \rangle_{\{\cdot\}}, \cdot) \in \mathcal{D}_{0,j}$. The identity (5.1) implies the following:

$$\begin{aligned}
x(y+y_1)^j = \sum_{\substack{k+l=j\\0\le k,l\le j}} xy^k y_1^l \approx xy^j + xy_1^j &\approx x\\
&\Rightarrow x + xy + xy^2 + \dots + xy^{j-1} \approx x\\
&\Rightarrow xy^k \approx x, \text{ for } 0 \le k \le j-1\\
&\Rightarrow xy_1^{k_1} \dots y_m^{k_m} \approx x, \text{ for } 0 \le k_1, \dots, k_m \le j-1\\
&\Rightarrow \sum_{i=1}^{p} xy_1^{k_{1,i}} \dots y_m^{k_{m,i}} \approx x\\
&\Rightarrow \sum_{i=1}^{p} x_1 y_1^{k_{1,i}} \dots y_m^{k_{m,i}} + \sum_{j=1}^{r} x_2 z_1^{l_{1,j}} \dots z_s^{k_{s,j}} \approx x_1 + x_2.
\end{aligned}$$

This shows that the $+$-reduct of the $\mathcal{M}_{\mathcal{D}_{0,j}}$-replica of $(F_{\mathcal{D}_{0,j}}(A)P, \cdot, +)$ is isomorphic to a semilattice of all non-empty subsets of an n-element set and in fact is isomorphic to the free semilattice on n generators. Note also, that groupoid reduct of the $\mathcal{M}_{\mathcal{D}_{0,j}}$-replica of $(F_{\mathcal{D}_{0,j}}(A)P, \cdot, +)$ belongs to the variety $\mathcal{D}_{0,1}$. ■

The variety $\mathcal{D}_{0,j}$ is locally finite, so Theorem 5.8 implies the following:

Corollary 5.9. *The free modal over the set X in the variety $\mathcal{M}_{\mathcal{D}_{0,j}}$ is isomorphic to the modal with the semilattice of all non-empty finite subsets of X as the semilattice reduct and a left-zero semigroup as the groupoid reduct.*

Example 5.10. We showed in Example 4.3 that $(F_{\mathcal{D}_{0,2}}(\{x,y\})P,\cdot,+)$ is not free in the variety $\mathcal{M}_{\mathcal{D}_{0,2}}$. Using the "Universal Algebra Calculator" [3] written by R. Freese, E. Kiss and M. Valeriote, we can check that the modal $(F_{\mathcal{D}_{0,2}}(\{x,y\})P,\cdot,+)$ has 7 congruences. Only 4 of them contain the $\mathcal{M}_{\mathcal{D}_{0,2}}$-replica congruence $\Theta_{\mathcal{M}_{\mathcal{D}_{0,2}}}(\{x,y\})$. In fact, the relation $\Theta_{\mathcal{M}_{\mathcal{D}_{0,2}}}(\{x,y\})$ has three classes: $A := \{a,c,e\}$, $B := \{b,d,f\}$, $A+B := \{g\}$.

The three-element quotient algebra $(\{A,B,A+B\},\cdot,+) \in \mathcal{M}_{\mathcal{D}_{0,2}}$ is the free differential modal on two generators in $\mathcal{M}_{\mathcal{D}_{0,2}}$.

By Theorem 5.4, Lemma 5.5 and Example 5.1 we immediately obtain the well-known result that the algebra of finitely generated subalgebras of a given mode (M,Ω) satisfies each linear identity true in (M,Ω).

Corollary 5.11. *For any variety $\mathcal{V}$ of modes,*

$$\mathcal{M}_{\mathcal{V}} \subseteq \mathcal{M}\mathcal{V} \subseteq \mathcal{M}_{\mathcal{V}^*}.$$

In particular, $\mathcal{M}\mathcal{V}^* = \mathcal{M}_{\mathcal{V}^*}$.

In general, the class $\mathcal{M}\mathcal{V}$ is not a variety.

Example 5.12. Consider the modal $(F_{\mathcal{D}_{0,2}}(x,y)P,\cdot,+) \in \mathcal{M}\mathcal{D}_{0,2}$. Using the same notation as in Example 4.3 we can observe that the only sets of generators of the subalgebra $(\{a,e,g\},\cdot,+)$ of $(F_{\mathcal{D}_{0,2}}(x,y)P,\cdot,+)$ are $\{a,e,g\}$ and $\{a,g\}$. But the mode $(\langle\{a,g\}\rangle_{\{\cdot\}},\cdot) = (\{a,e,g\},\cdot)$ is not in the variety $\mathcal{D}_{0,2}$. This shows that the class $\mathcal{M}\mathcal{D}_{0,2}$ is not closed under subalgebras.

Lemma 5.13. *The class $\mathcal{M}\mathcal{V}$ is closed under homomorphic images and finite subdirect products.*

Proof. Let $(M,\Omega,+) \in \mathcal{M}\mathcal{V}$ be a modal generated by a set X such that $(\langle X\rangle_\Omega,\Omega) \in \mathcal{V}$ and let $h : (M,\Omega,+) \to (A,\Omega,+)$ be a surjective modal homomorphism. It is clear that $\langle h(X)\rangle_\Omega = h(\langle X\rangle_\Omega)$, $(h(\langle X\rangle_\Omega),\Omega) \in \mathcal{V}$ and $A = h(M) = h(\langle X\rangle) = \langle h(X)\rangle$. This implies that a generating set of $(A,\Omega,+)$ is included in $h(X)$ and $(A,\Omega,+) \in \mathcal{M}\mathcal{V}$.

For each $i = 1,\dots,n$, let $(M_i,\Omega,+) \in \mathcal{M}\mathcal{V}$ be a modal generated by a set $X_i \subseteq M_i$ such that $(\langle X_i\rangle_\Omega,\Omega) \in \mathcal{V}$ and let $(M,\Omega,+)$ be a subdirect product of the algebras $(M_i,\Omega,+)$. Let $a = (a_1,\dots,a_n) \in M$. By assumption, for each $i = 1,\dots,n$, $a_i \in \pi_i(M) = M_i = \langle X_i\rangle$, where π_i is the projection operator onto the i-th factor. By the Disjunctive Form Lemma 5.2, there are $t_{i1},\dots,t_{ik_i} \in \langle X_i\rangle_\Omega$ such that $a_i = \sum_{j=1}^{k_i} t_{ij}$. Then by idempotency of the operation $+$ we have

$$a=(a_1,\dots,a_n)=\Big(\sum_{j=1}^{k_1}t_{1j},\dots,\sum_{j=1}^{k_n}t_{nj}\Big)=\sum_{i=1}^{n}\sum_{j=1}^{k_i}(t_{11},t_{21},\dots,t_{ij},\dots,t_{n1}).$$

This shows that there is a set $X\subseteq\prod_{i=1}^{n}\langle X_i\rangle_{\Omega}\subseteq M$ which generates the modal $(M,\Omega,+)$ and $(\langle X\rangle_{\Omega},\Omega)\in\mathcal{V}$. ∎

Classes of algebras closed under homomorphic images and finite subdirect products were investigated by L. Shemetkov and A. Skiba under the name *formation of algebras* (see [16]).

The next example shows that the class $\mathcal{MV}$ is not closed under arbitrary products.

Example 5.14. Let $X:=\{x_n\mid n\in\mathbb{N}\}$ be an infinite countable set and let $(F_{\mathcal{D}_{0,2}}(X),\cdot)$ be the free $\mathcal{D}_{0,2}$ - differential mode over X. By Example 5.1, the set $\mathfrak{X}:=\{\{x_n\}\mid n\in\mathbb{N}\}$ generates the modal $(F_{\mathcal{D}_{0,2}}(X)P,\cdot,+)$ and

$$(F_{\mathcal{D}_{0,2}}(X),\cdot)\cong(\{\{a\}\mid a\in F_{\mathcal{D}_{0,2}}(X)\},\cdot)=(\langle\mathfrak{X}\rangle_{\{\cdot\}},\cdot)\in\mathcal{D}_{0,2}.$$

Let $I:=\mathbb{N}$ and for each $i\in I$, let $(M_i,\cdot,+):=(F_{\mathcal{D}_{0,2}}(X)P,\cdot,+)$. By Definition (2.2) it is clear that for each $n\in I$,

$$\begin{aligned}\langle\{x_1,\dots,x_n\}\rangle&=\{x_1\}+\dots+\{x_n\}=\{x_1x_2^{\varepsilon_{1,2}}\dots x_n^{\varepsilon_{1,n}}\}+\dots+\\&+\{x_ix_1^{\varepsilon_{i,1}}\dots x_{i-1}^{\varepsilon_{i,i-1}}x_{i+1}^{\varepsilon_{i,i+1}}\dots x_n^{\varepsilon_{i,n}}\}+\dots+\{x_nx_1^{\varepsilon_{n,1}}\dots x_{n-1}^{\varepsilon_{n,n-1}}\},\end{aligned}$$

where $\varepsilon_{k,j}=0$ or $\varepsilon_{k,j}=1$. Moreover, for $x_{n+1}\notin\{x_1,\dots,x_n\}$

$$\{x_1\}+\dots+\{x_n\}=(F_{\mathcal{D}_{0,2}}(\{x_1,\dots,x_n\}),\cdot)\neq(F_{\mathcal{D}_{0,2}}(\{x_1,\dots,x_n,x_{n+1}\}),\cdot).$$

This implies that for the element

$$m:=(\{x_1\},\{x_1\}+\{x_2\},\dots,\{x_1\}+\dots+\{x_n\},\dots)\in\prod_{i\in I}M_i$$

there is no $\{\cdot,+\}$-word t such that $m=t(a_1,\dots,a_k)$, and $a_1,\dots,a_k\in\prod_{i\in I}\langle\mathfrak{X}\rangle_{\{\cdot\}}$. Hence, $\prod_{i\in I}\langle\mathfrak{X}\rangle_{\{\cdot\}}$ does not generate the modal $(\prod_{i\in I}M_i,\cdot,+)$.

Without loss of generality we can assume that for any set $\mathfrak{Z}$ of generators of the modal $(\prod_{i\in I}M_i,\cdot,+)$, there is some $n\in I$, such that $\pi_n(\mathfrak{Z})$ includes subalgebras $(A,\cdot):=\langle\{x_1\}\rangle$ and $(B,\cdot):=\langle\{x_2,x_3,\dots,x_k\}\rangle$ of $(F_{\mathcal{D}_{0,2}}(X),\cdot)$. Thus

$$AB^2=\{x_1\}\cup\{x_1yz\mid y\neq z\in\{x_2,x_3,\dots,x_k\}\}=\{x_1\}=A\ \Leftrightarrow\ x_1y=x_1z,$$

what is impossible, because $x_2,x_3,\dots,x_k$ are free generators of $(F_{\mathcal{D}_{0,2}}(X),\cdot)$.

It follows that for any set $\mathfrak{Z}$ of generators of the modal $(\prod_{i\in I}M_i,\cdot,+)$, there is some $n\in I$, such that $(\langle\pi_n(\mathfrak{Z})\rangle_{\{\cdot\}},\cdot)\notin\mathcal{D}_{0,2}$.

Lemma 5.15. *Let $(M, \Omega, +) \in \mathcal{MV}$ be a modal generated by a set $Y \subseteq M$, such that $(\langle Y\rangle_\Omega, \Omega) \in \mathcal{V}$. Then each mapping $h : X \to Y$ can be uniquely extended to a modal homomorphism $\overline{h} : (F_\mathcal{V}(X)P, \Omega, +) \to (M, \Omega, +)$.*

Proof. Obviously, any mapping $h : X \to Y \hookrightarrow \langle Y\rangle_\Omega$ can be uniquely extended to a mode homomorphism $\overline{h} = \iota\widetilde{h} : (F_\mathcal{V}(X), \Omega) \to (M, \Omega)$, which is a composition of the unique homomorphism $\widetilde{h} : (F_\mathcal{V}(X), \Omega) \to (\langle Y\rangle_\Omega, \Omega)$ and the canonical embedding $\iota : (\langle Y\rangle_\Omega, \Omega) \to (M, \Omega)$. By Lemma 3.1, the Ω-mode homomorphism $\overline{h}$ may be extended to a unique modal homomorphism $\overline{\overline{h}} : (F_\mathcal{V}(X)P, \Omega, +) \to (M, \Omega, +)$. ∎

Theorem 5.16. *Each modal $(M, \Omega, +) \in \mathcal{MV}$ generated by a set X is a homomorphic image of $(F_\mathcal{V}(X)P, \Omega, +)$.*

Proof. Let $(\langle X\rangle_\Omega, \Omega) \in \mathcal{V}$ be the full Ω-mode subreduct of $(M, \Omega, +) \in \mathcal{MV}$. By Lemma 5.15 the composite $X \xrightarrow{id} X \hookrightarrow \langle X\rangle_\Omega$ can be uniquely extended to a modal homomorphism $\overline{\overline{id}} : (F_\mathcal{V}(X)P, \Omega, +) \to (M, \Omega, +)$, where $\overline{\overline{id}}$ is an extension of the composition of the Ω-mode homomorphism $\widetilde{id} : (F_\mathcal{V}(X), \Omega) \to (\langle X\rangle_\Omega, \Omega)$ and the embedding $\iota : (\langle X\rangle_\Omega, \Omega) \to (M, \Omega)$. Note that $\langle X\rangle_\Omega \subseteq F_\mathcal{V}(X)$ and $\widetilde{id}/_{\langle X\rangle_\Omega}$ is in fact the identity homomorphism on $(\langle X\rangle_\Omega, \Omega)$.

Moreover, for each subalgebra (A, Ω) of $(F_\mathcal{V}(X), \Omega)$ finitely generated by a set S, $\overline{\overline{id}}(A) = \sum_{s\in S} \widetilde{id}(s) \in M$. By the Disjunctive Form Lemma 5.2, each element $m \in M$ may be expressed in a form $m = m_1 + \ldots + m_r$ for some $m_1, \ldots, m_r \in \langle X\rangle_\Omega$. Let (A, Ω) be a subalgebra of $(F_\mathcal{V}(X), \Omega)$ generated by the set $\{m_1, \ldots, m_r\}$. Then $\overline{\overline{id}}(A) = \sum_{i=1}^r \widetilde{id}(m_i) = \sum_{i=1}^r id(m_i) = m$. This shows that the mapping $\overline{\overline{id}}$ is a surjective modal homomorphism and $(M, \Omega, +) = \overline{\overline{id}}((F_\mathcal{V}(X)P, \Omega, +))$. ∎

Corollary 5.17. *For any variety $\mathcal{V}$ of modes*

$$\mathcal{MV} = H\{(F_\mathcal{V}(X)P, \Omega, +) \mid X \neq \emptyset\}.$$

Proof. By Theorem 5.16, the inclusion $\mathcal{MV} \subseteq H\{(F_\mathcal{V}(X)P, \Omega, +) \mid X \neq \emptyset\}$ is obvious. On the other hand, by Example 5.1, for each set X, the modal $(F_\mathcal{V}(X)P, \Omega, +) \in \mathcal{MV}$. Additionally, by Lemma 5.13, the class $\mathcal{MV}$ is closed under homomorphic images, so the inclusion

$$\mathcal{MV} \supseteq H\{(F_\mathcal{V}(X)P, \Omega, +) \mid X \neq \emptyset\}$$

is also clear. ∎

Example 5.18. Let $(M, \cdot, +)$ be the modal described in Example 4.7. As a set of generators take the set $X = \{x, y\}$. The full $\{\cdot\}$-mode subreduct $(\langle\{x,y\}\rangle_{\{\cdot\}},\cdot)$ is a differential groupoid which lies in the subvariety $\mathcal{D}_{0,2}$. By Theorem 5.16 the modal $(M,\cdot,+)$ is a homomorphic image of the modal $(F_{\mathcal{D}_{0,2}}(\{x,y\})P,\cdot,+)$ with $x=\overline{\overline{id}}(\{x\})=\overline{\overline{id}}(\{xy\})=\overline{\overline{id}}(\{x,xy\})$, $y=\overline{\overline{id}}(\{y\})$, $z=\overline{\overline{id}}(\{yx\})$, $u=z+y=\overline{\overline{id}}(\{y,yx\})$ and $t=z+x=\overline{\overline{id}}(\{x,y,xy,yx\})$.

It is easy to see that in any entropic modal $(M,\Omega,+)$ each Ω-word operation distributes over the operation $+$. Hence, by Theorem 5.4 any variety $\mathcal{M}\subseteq\mathcal{MV}$ of entropic modals satisfies each identity true in $\mathcal{V}$.

Corollary 5.19. *The entropic modals in the class $\mathcal{MV}$ form a subvariety of $\mathcal{M}_{\mathcal{V}}$.*

Example 5.20. Consider the variety $\mathcal{M}$ of entropic differential modals. It was proved in [17] that each differential entropic modal satisfies the identity $xy=(xy)y$, which means that $\mathcal{M}\subseteq\mathcal{M}_{\mathcal{D}_{1,2}}$.

In Example 4.6 we showed that the algebra $(F_{\mathcal{D}_{1,2}}(\{x,y\})P,\cdot,+)\in\mathcal{M}_{\mathcal{D}_{1,2}}$, but it is easy to see that $(a+d)(b+c)=h\neq g=ab+dc$. So, it is not entropic, hence it is not free in $\mathcal{M}$.

It was also shown in [17] that the free differential entropic modal on $\{x,y\}$ is isomorphic to the algebra $(\{x,y,xy,yx,x+y\},\cdot,+)$, where $x+xy=xy$, $y+yx=yx$, $x+yx=y+xy=xy+yx=x+y$ and the groupoid operation is defined in the following way:

$\cdot$	x	y	xy	yx	$x+y$
x	x	xy	x	xy	xy
y	yx	y	yx	y	yx
xy	xy	xy	xy	xy	xy
yx	yx	yx	yx	yx	yx
$x+y$	$x+y$	$x+y$	$x+y$	$x+y$	$x+y$

Using once again the "Universal Algebra Calculator", we can obtain that the modal $(F_{\mathcal{D}_{1,2}}(\{x,y\})P,\cdot,+)$ has 45 congruences. Only 9 of them contain the $\mathcal{M}$-replica congruence $\Theta_{\mathcal{M}}(\{x,y\})$. The relation $\Theta_{\mathcal{M}}(\{x,y\})$ has five congruence classes: $\{x\}$, $\{y\}$, $\{xy,x+xy\}$, $\{yx,y+yx\}$ and $\{xy+yx,x+yx,y+xy,x+y\}$.

Hence, in fact the algebra $(\{x,y,xy,yx,x+y\},\cdot,+)$ is isomorphic to the $\mathcal{M}$-replica of $(F_{\mathcal{D}_{1,2}}(\{x,y\})P,\cdot,+)$. In particular, by Theorem 5.16

$$(\{x,y,xy,yx,x+y\},\cdot,+)=\overline{\overline{id}}(F_{\mathcal{D}_{1,2}}(\{x,y\})P,\cdot,+),$$

where

$$x = \overline{\overline{id}}(\{x\}), \qquad y = \overline{\overline{id}}(\{y\}),$$
$$xy = \overline{\overline{id}}(\{xy\}) = \overline{\overline{id}}(\{x, xy\}), \qquad yx = \overline{\overline{id}}(\{yx\}) = \overline{\overline{id}}(\{y, yx\}),$$
$$x + y = \overline{\overline{id}}(\{xy, yx\}) = \overline{\overline{id}}(\{x, xy, yx\}) = \overline{\overline{id}}(\{y, xy, yx\}) = \overline{\overline{id}}(\{x, y, xy, yx\}).$$

6. Representation theorem for modals

For any variety $\mathcal{V}$ of modes, Theorem 3.1 serves to represent general $\mathcal{V}^*$-modals as quotients of subalgebra modals.

Theorem 6.1. (Modal Representation Theorem [12]) *For any variety $\mathcal{V}$ of Ω-modes, each $\mathcal{V}^*$-modal $(M, \Omega, +)$ is a quotient of the modal of finitely generated non-empty submodes of its mode reduct (M, Ω).*

Lemma 5.15 gives a representation of general modals in any class $\mathcal{MV}$, based on extended power algebras.

Let (M, Ω) be a mode and $X, Y \subseteq M$. We can define on the set $\mathcal{P}_{>0}M$ a relation α in the following way: $X \,\alpha\, Y \;\Leftrightarrow\; \langle X \rangle = \langle Y \rangle$.

Let $Con_{id}(\mathcal{P}^{<\omega}_{>0}M)$ denote the lattice of congruence relations γ on the extended power algebra $(\mathcal{P}^{<\omega}_{>0}M, \Omega, \cup)$ of a mode (M, Ω), such that the quotient $(\mathcal{P}^{<\omega}_{>0}M^{\gamma}, \Omega)$ is idempotent. The relation α is the least element in this lattice. (For all details see [10].)

Theorem 6.2. [10] *Let (M, Ω) be a mode. The quotient algebra $(\mathcal{P}_{>0}M^{\alpha}$, $\Omega, \cup)$ is isomorphic to the modal $(MS, \Omega, +)$ of all non-empty subalgebras of (M, Ω) and the quotient algebra $(\mathcal{P}^{<\omega}_{>0}M^{\alpha}, \Omega, \cup)$ is isomorphic to the modal $(MP, \Omega, +)$ of all finitely generated subalgebras. The isomorphism is given by the following mapping: $h : \mathcal{P}_{>0}M^{\alpha} \to MS;\quad X^{\alpha} \mapsto \langle X \rangle_{\Omega}$.*

Theorem 6.3. (Power Representation Theorem) *Each modal $(M, \Omega, +)$ generated by a set X is a quotient of a subalgebra of the extended power algebra of the full mode subreduct $(\langle X \rangle_{\Omega}, \Omega)$.*

Proof. Consider the following mapping:

$$h : \mathcal{P}^{<\omega}_{>0}\langle X \rangle^{\alpha}_{\Omega} \to M; \quad \{m_1, \ldots, m_p\}^{\alpha} \mapsto m_1 + \ldots + m_p.$$

Assume that for some $m_1, \ldots, m_p, s_1, \ldots, s_k \in \langle X \rangle_{\Omega}$,

$$\{m_1, \ldots, m_p\}^{\alpha} = \{s_1, \ldots, s_k\}^{\alpha}.$$

So by the definition of the relation α,

$$\langle \{m_1, \ldots, m_p\} \rangle = \langle \{s_1, \ldots, s_k\} \rangle.$$

This means that for each element $m_i \in \langle \{s_1, \ldots, s_k\} \rangle$, $m_i = q(x_1, \ldots, x_l)$ for some l-ary Ω-term q and $\{x_1, \ldots, x_l\} \subseteq \{s_1, \ldots, s_k\}$.

According to the Sum-Superiority Lemma 2.3 one has that

$$m_i = q(x_1, \ldots, x_l) \leq x_1 + \ldots + x_l \leq s_1 + \ldots + s_k.$$

Thus $m_1 + \ldots + m_p \leq s_1 + \ldots + s_k$. The reverse inequality is obtained by symmetry, whence $m_1 + \ldots + m_p = s_1 + \ldots + s_k$ and the mapping h is well defined.

For classes $\{m_{11}, \ldots, m_{1p_1}\}^\alpha, \ldots, \{m_{n1}, \ldots, m_{np_n}\}^\alpha \in \mathcal{P}^{<\omega}_{>0}\langle X\rangle^\alpha_\Omega$ and an n-ary operation $\omega \in \Omega$ we obtain

$$\begin{aligned}
&h(\omega(\{m_{11}, \ldots, m_{1p_1}\}^\alpha, \ldots, \{m_{n1}, \ldots, m_{np_n}\}^\alpha))\\
&\qquad = h(\omega(\{m_{11}, \ldots, m_{1p_1}\}, \ldots, \{m_{n1}, \ldots, m_{np_n}\})^\alpha)\\
&\qquad = h(\{\omega(a_1, \ldots, a_n) \mid a_i \in \{m_{i1}, \ldots, m_{ip_i}\}\}^\alpha)\\
&\qquad = \sum_{\substack{1\leq i\leq n\\ a_i\in\{m_{i1},\ldots,m_{ip_i}\}}} \omega(a_1, \ldots, a_n)\\
&\qquad = \omega(m_{11} + \ldots + m_{1p_1}, \ldots, m_{n1} + \ldots + m_{np_n})\\
&\qquad = \omega(h(\{m_{11}, \ldots, m_{1p_1}\}^\alpha), \ldots, h(\{m_{n1}, \ldots, m_{np_n}\}^\alpha)).
\end{aligned}$$

Hence h is an Ω-homomorphism. Moreover

$$\begin{aligned}
&h(\{m_{11}, \ldots, m_{1p_1}\}^\alpha \cup \{m_{21}, \ldots, m_{2p_2}\}^\alpha)\\
&\qquad = h((\{m_{11}, \ldots, m_{1p_1}\} \cup \{m_{21}, \ldots, m_{2p_2}\})^\alpha)\\
&\qquad = h(\{m_{11}, \ldots, m_{1p_1}, m_{21}, \ldots, m_{2p_2}\}^\alpha)\\
&\qquad = m_{11} + \ldots + m_{1p_1} + m_{21} + \ldots + m_{2p_2}\\
&\qquad = h(\{m_{11}, \ldots, m_{1p_1}\}^\alpha) + h(\{m_{21}, \ldots, m_{2p_2}\}^\alpha),
\end{aligned}$$

which shows that h is a semilattice homomorphism.

By the Disjunctive Form Lemma 5.2, for each $m \in M$ there exist $m_1, \ldots, m_p \in \langle X\rangle_\Omega$ such that $m = m_1 + \ldots + m_p = h(\{m_1, \ldots, m_p\}^\alpha)$. Hence h is a surjection and

$$(M, \Omega, +) = h((\mathcal{P}^{<\omega}_{>0}\langle X\rangle^\alpha_\Omega, \Omega, \cup)) \cong ((\mathcal{P}^{<\omega}_{>0}\langle X\rangle^\alpha_\Omega)^{ker h}, \Omega, \cup).$$

By the Second Isomorphism Theorem $(M, \Omega, +) \cong (\mathcal{P}^{<\omega}_{>0}\langle X\rangle^\beta_\Omega, \Omega, \cup)$, where $\beta \in Con_{id}(\mathcal{P}^{<\omega}_{>0}\langle X\rangle_\Omega, \Omega, \cup)$. This implies $(M, \Omega, +) \in HS(\mathcal{P}_{>0}\langle X\rangle_\Omega, \Omega, \cup)$. ■

By Theorem 6.2 we immediately obtain the next corollary.

Corollary 6.4. *Each modal $(M, \Omega, +)$ generated by a set X is a homomorphic image of $(\langle X\rangle_\Omega P, \Omega, +)$.*

Example 6.5. The modal $(M, \cdot, +)$ given in Example 4.7 is isomorphic to the modal $(\langle\{x, y\}\rangle_{\{\cdot\}} P, \cdot, +)$ by the isomorphism h given by: $x = h(\{x\})$, $y = h(\{y\})$, $z = h(\{z\})$, $u = h(\{y, z\})$ and $t = h(\{x, y, z\})$.

7. Open problems

As shown in [9] and [1], it is usually very difficult to determine which identities (apart from idempotent and linear ones) holding in a mode (M,Ω), also hold in the algebra (MS,Ω). This is in contrast with the power algebra of subsets. According to results of G. Grätzer and H. Lakser in [4], the class of all power algebras of algebras from a fixed variety $\mathcal{V}$ preserves precisely the consequences of linear identities holding in $\mathcal{V}$. There is no such characterization for algebras in the class of all algebras of subalgebras of $\mathcal{V}$-algebras. However there is a conjecture formulated in the following problem.

Problem 7.1. *Is it true that for a variety $\mathcal{V}$ of modes, the class of algebras of subalgebras of $\mathcal{V}$-algebras satisfies precisely the consequences of the idempotent and linear identities true in $\mathcal{V}$?*

As Theorem 5.4 shows, settling this conjecture is closely related to the general structure of modals.

In [17], the lattice of varieties of differential entropic modals was described.

Problem 7.2. *Describe the structure of other classes of modals using their relationship with the structure of power algebras.*

By Corollary 5.11, for a variety $\mathcal{V}$ of modes defined by linear identities, the class $\mathcal{MV}$ is a variety and $\mathcal{MV} = \mathcal{M}_{\mathcal{V}}$. Is the converse true?

Problem 7.3. *Is it true that for a variety $\mathcal{V}$ of modes the equality $\mathcal{MV} = \mathcal{M}_{\mathcal{V}}$ implies that $\mathcal{V}$ is defined by linear identities?*

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