DEMONSTRATIO MATHEMATICA
Vol. XLIV No 3 2011

10.1515/dema-2013-0320

Joel Berman

UPPER BOUNDS ON THE SIZES OF FINITELY GENERATED ALGEBRAS

Abstract. We present an upper bound for the cardinality of any n-generated algebra in a locally finite variety $\mathcal{V}$ of algebras. This upper bound depends only on some fundamental numerical invariants of the n-generated subdirectly irreducible algebras in $\mathcal{V}$. A theorem characterizing those varieties that contain algebras whose cardinalities achieve the upper bound is proved. Several explicit methods for computing the exact values of these invariants are described. The final section contains detailed concrete examples illustrating applications of the characterization theorem and of the various methods for computing the upper bound.

1. Introduction

Let $\mathbf{B}$ be an algebra in a locally finite variety $\mathcal{V}$ with $\mathbf{B}$ generated by a finite set $X = \{x_1, \ldots, x_n\}$. We wish to provide an upper bound on the cardinality of $\mathbf{B}$ in terms of n and some general parameters for $\mathcal{V}$. These parameters will be based on numerical invariants involving the n-generated subdirectly irreducible algebras in $\mathcal{V}$. In Section 2 the upper bound is presented and the varieties and algebras for which this upper bound is obtained are characterized. Section 3 contains a number of methods for actually computing these upper bounds for a given algebra or variety. Section 4 contains applications and examples of the material presented in the earlier sections of the paper.

In general our definitions and notation for algebraic notions follows that in [9] or [16]. For algebras $\mathbf{A}$ and $\mathbf{B}$ we write $\mathbf{A} \triangleleft \mathbf{B}$ ($\mathbf{A} \mathrel{\tilde{\triangleleft}} \mathbf{B}$) to denote that $\mathbf{A}$ is (isomorphic to) a subalgebra of $\mathbf{B}$. For a set A, the equivalence relation $A \times A$ is denoted 1_A and the diagonal relation is denoted 0_A. So the top and bottom elements of the congruence lattice $\mathsf{Con}\,\mathbf{A}$ are 1_A and 0_A.

2000 *Mathematics Subject Classification*: 08B20, 08B26, 06A07, 05A15.
Key words and phrases: arithmetical variety, free algebra, valuation, semisimple, finitely generated, Möbius function.

Stirling numbers appear in several places in this paper. Let $S(p,q)$ denote the *Stirling number of the second kind*, that is, the number of ways to partition a set of p elements into q disjoint nonempty subsets. Thus, $q!S(p,q)$ is the number of onto functions from a p-element set to a q-element set. An inclusion-exclusion argument gives

$$S(p,q) = \frac{1}{q!}\sum_{i=0}^{q}(-1)^i\binom{q}{i}(q-i)^p.$$

Suppose C is the upper triangular k by k matrix with entries $C_{ij} = S(j,i)$. The matrix C has 1's on its main diagonal and hence is invertible. Let $s(i,j)$ denote Stirling numbers of the first kind. From the standard identity

$$\sum_{r=1}^{k} S(i,r)s(r,j) = \delta_{ij} \tag{1}$$

it follows that the inverse of C is the upper triangular matrix M with $M_{ij} = s(j,i)$. Such matrices are often referred to as Stirling matrices, e.g., [10].

2. The upper bound and when it is obtained

Let $\mathcal{S}_n$ be a transversal with respect to isomorphism of the at most n-generated subdirectly irreducible algebras in a locally finite variety $\mathcal{V}$. For a given algebra $\mathbf{B} \in \mathcal{V}$ generated by a set X with $|X| = n$ let $\beta_1, \ldots, \beta_m$ denote those congruence relations of $\mathbf{B}$ that are meet irreducible in $\mathsf{Con}\,\mathbf{B}$. Thus, each $\mathbf{B}/\beta_i$ is subdirectly irreducible and every homomorphism h from $\mathbf{B}$ onto a subdirectly irreducible algebra has $\ker h = \beta_i$ for some i. Let $\mathbf{B}_1, \ldots, \mathbf{B}_m$ be such that $\mathbf{B}/\beta_i \cong \mathbf{B}_i \in \mathcal{S}_n$ and let k_i be an isomorphism from $\mathbf{B}/\beta_i$ onto $\mathbf{B}_i$. Then $\mathbf{B}$ is subdirectly embedded in the product of the $\mathbf{B}_i$ by

$$k : \mathbf{B} \rightarrowtail \prod_{i=1}^{m}\mathbf{B}_i \quad \text{with} \quad k(b) = (\ldots, k_i(b/\beta_i), \ldots).$$

Thus, we have the upper bound

$$|B| \le \prod_{i=1}^{m}|B_i|. \tag{2}$$

Suppose $h_i : \mathbf{B} \to \mathbf{B}_i$ is the onto homomorphism given by $h_i(b) = k_i(b/\beta_i)$. Note that $\ker(h_i) = \beta_i$. Let $u_i : X \to B_i$ be defined by $u_i(x) = h_i(x)$ for all $x \in X$. The set $u_i(X)$ generates the algebra $\mathbf{B}_i$. Each u_i is an example of a *valuation*.

Definition 2.1. Let X be a set, $\mathbf{A}$ an algebra with universe A and $f \in A^X$.

1. The subalgebra of $\mathbf{A}$ generated by $f(X)$ is denoted $\mathbf{Alg}(f)$.
2. If $f(X)$ generates $\mathbf{A}$, then f is a *valuation.* The set of all valuations in A^X is $\mathrm{val}(X, \mathbf{A})$. If no set of size $|X|$ generates $\mathbf{A}$, then $\mathrm{val}(X, \mathbf{A}) = \emptyset$.
3. For $\mathcal{K}$ a class of algebras, $\mathrm{val}(X, \mathcal{K})$ denotes the collection of all $f \in \mathrm{val}(X, \mathbf{A})$ for $\mathbf{A} \in \mathcal{K}$.

The paper [3] explores the way $\mathrm{val}(X, \mathbf{A})$ and the cardinality of this set may be used to describe and understand the structure of free algebras in the variety generated by $\mathbf{A}$.

By using the notion of a valuations the upper bound in (2) becomes

$$|B| \leq \prod_{i=1}^{m} |\mathbf{Alg}(u_i)|. \tag{3}$$

In the present paper we explore the implications of this inequality and characterize those varieties that contain algebras $\mathbf{B}$ for which the inequality in (3) is an equality.

Definition 2.2. For a variety $\mathcal{V}$, two valuations v and w in $\mathrm{val}(X, \mathcal{V})$ are called *equivalent*, denoted $v \sim w$, if there exists an isomorphism k of $\mathbf{Alg}(v)$ onto $\mathbf{Alg}(w)$ such that $k(v(x)) = w(x)$ for all $x \in X$.
The equivalence class of a valuation f with respect to the equivalence relation $\sim$ is denoted $f/\sim$.

The following is easily proved.

Proposition 2.3. *Suppose* $\mathbf{B}$ *is generated by* X *and that* $h_i : \mathbf{B} \to \mathbf{B}_i$, $i = 1, 2$ *are onto homomorphisms. There exists an isomorphism* f *from* $\mathbf{B}_1$ *onto* $\mathbf{B}_2$ *with* $fh_1(x) = h_2(x)$ *for all* $x \in X$ *if and only if* $\ker h_1 = \ker h_2$ *in* $\mathsf{Con}\,\mathbf{B}$.

From this proposition, if in (3) the valuations u_i and u_j are such that $u_i \sim u_j$, then $i = j$, since the congruences β_i are all pairwise distinct.

Let U_n be a transversal with respect to the equivalence relation $\sim$ of $\mathrm{val}(X, \mathcal{S}_n)$. Then the value of m in (3) is bounded above by $|U_n|$. The next lemma provides a formula for the cardinality of U_n. The lemma is an easy consequence of Burnside's Lemma but we present a direct proof.

Lemma 2.4. *Let* $\mathbf{B}$ *be an algebra generated by* X *with* $f \in \mathrm{val}(X, \mathbf{B})$. *There is a bijection between* $\mathrm{Aut}\,\mathbf{B}$ *and* $\{g \in \mathrm{val}(X, \mathbf{B}) : g \sim f\}$.

Proof. Consider the map $k \mapsto kf$ for k ranging over $\mathrm{Aut}\,\mathbf{B}$. The function kf is in $\mathrm{val}(X, \mathbf{B})$. If $g \sim f$, then $g = kf$ for some $k \in \mathrm{Aut}\,\mathbf{B}$ and $kf \in f/\sim$ by Definition 2.2. So the map is onto $f/\sim$. Let $k_1 f = k_2 f$ and suppose $b \in B$.

Since $f(X)$ generates $\mathbf{B}$ there is a term p for which $b = p(f(x_1), \dots, f(x_n))$. Then $k_1(b) = p(k_1(f(x_1)), \dots, k_1(f(x_n))) = p(k_2(f(x_1)), \dots, k_2(f(x_n))) = k_2(b)$. ∎

Corollary 2.5. *If $\mathbf{B}$ is a finite algebra generated by a set X, then the equivalence relation $\sim$ on $\operatorname{val}(X, \mathbf{B})$ has all of its equivalence classes of cardinality $|\operatorname{Aut}\mathbf{B}|$. Hence the transversal of $\operatorname{val}(X, \mathbf{B})$ with respect to $\sim$ contains exactly $|\operatorname{val}(X, \mathbf{B})|/|\operatorname{Aut}\mathbf{B}|$ valuations.*

Corollary 2.6. *Suppose $\mathcal{V}$ is a locally finite variety, $|X| = n$, and $\mathcal{S}_n$ is a transversal with respect to $\cong$ of the subdirectly irreducible algebras in $\mathcal{V}$ that are at most n-generated. Let U_n be a transversal with respect to $\sim$ of $\operatorname{val}(X, \mathcal{S}_n)$. Then $|U_n| = \sum_{\mathbf{S} \in \mathcal{S}_n} |\operatorname{val}(X, \mathbf{S})|/|\operatorname{Aut}\mathbf{S}|$.*

Proof. If $\mathbf{S}_1$ and $\mathbf{S}_2$ are two distinct algebras in $\mathcal{S}_n$, then $\operatorname{val}(X, \mathbf{S}_1)$ and $\operatorname{val}(X, \mathbf{S}_2)$ are pairwise disjoint. ∎

Definition 2.7. A variety is *arithmetical* if it is congruence distributive and congruence permutable. A variety $\mathcal{V}$ is called *n-semisimple* if every at most n-generated subdirectly irreducible algebra in $\mathcal{V}$ is simple and $\mathcal{V}$ is *semisimple* if every subdirectly irreducible algebra in $\mathcal{V}$ is simple. An algebra $\mathbf{A}$ is *hereditarily simple* if every nontrivial subalgebra of $\mathbf{A}$ is simple. A finite algebra is called *quasiprimal* if it is hereditarily simple and generates an arithmetical variety. Every quasiprimal algebra $\mathbf{A}$ has a *discriminator term*, i.e., a ternary term t such that $t^{\mathbf{A}}(a, b, c) = a$ if $a \neq b$ and $t^{\mathbf{A}}(a, b, c) = c$ if $a = b$. A *discriminator variety* is any variety generated by a class of algebras possessing a common discriminator term. The free algebra in a variety $\mathcal{V}$ generated by a set X of free generators is denoted $\mathbf{F}_{\mathcal{V}}(X)$ while $\mathbf{F}_{\mathcal{V}}(n)$ denotes the free algebra generated by n free generators.

Theorem 2.8. *Suppose $\mathcal{V}$ is a locally finite variety, $|X| = n$, and $\mathcal{S}_n$ is a transversal with respect to $\cong$ of the subdirectly irreducible algebras in $\mathcal{V}$ that are at most n-generated. Let U_n be a transversal with respect to $\sim$ of $\operatorname{val}(X, \mathcal{S}_n)$. Then for any n-generated algebra $\mathbf{B} \in \mathcal{V}$,*

$$|B| \leq \prod_{u \in U_n} |\mathbf{Alg}(u)| = \prod_{\mathbf{S} \in \mathcal{S}_n} |S|^{\frac{|\operatorname{val}(X, \mathbf{S})|}{|\operatorname{Aut}\mathbf{S}|}}. \tag{4}$$

Moreover, if $n \geq 3$, the following are equivalent:

(i) *There is an n-generated $\mathbf{B} \in \mathcal{V}$ for which equality holds in* (4),
(ii) *$\mathcal{V}$ is arithmetical and n-semisimple.*

Proof. The inequality in (4) is the inequality (2) and the equality in (4) follows from Corollary 2.6.

Now, if $\mathcal{V}$ is arithmetical and n-semisimple, and if $\mathbf{A} \in \mathcal{V}$ is n-generated, then the congruence lattice of $\mathbf{A}$, $\mathsf{Con}\,\mathbf{A}$, is finite and every $\theta \in \mathsf{Con}\,\mathbf{A}$ is the meet of coatoms. So $\mathsf{Con}\,\mathbf{A}$ is a finite Boolean lattice of permuting congruence relations. Thus $\mathbf{A}$ is isomorphic to the product of the $\mathbf{A}/\alpha$, where α ranges over the coatoms of $\mathsf{Con}\,\mathbf{A}$. If $u \in \mathrm{val}(X, \mathcal{S}_n)$, then by the universal mapping property of free algebras, there exists a homomorphism $h : \mathbf{F}_{\mathcal{V}}(X) \to \mathbf{Alg}(u)$ such that $h(x) = u(x)$ for all $x \in X$. Then $\ker h$ is a coatom in $\mathsf{Con}\,(\mathbf{F}_{\mathcal{V}}(X))$. The number of coatoms in $\mathsf{Con}\,(\mathbf{F}_{\mathcal{V}}(X))$ is equal to $|U_n|$ by virtue of Proposition 2.3. Thus $|\mathbf{F}_{\mathcal{V}}(n)| = \prod_{u\in U_n} |\mathbf{Alg}(u)|$.

Next, suppose $\mathbf{B} \in \mathcal{V}$ is generated by X and that $|B| = \prod_{u\in U_n} |\mathbf{Alg}(u)|$. By Proposition 2.3 there is an injection δ from the set M of all meet-irreducible congruences in $\mathsf{Con}\,\mathbf{B}$ into U_n given by $\delta(\theta) = u$ for $\mathbf{B}/\theta \cong \mathbf{S} \in \mathcal{S}_n$ with x/θ corresponding to $u(x) \in S$ for all $x \in X$. Then $\mathbf{B}$ is subdirectly embedded in $\prod_{u\in U_n} \mathbf{Alg}(u)$. Thus

$$\prod_{u\in U_n} |\mathbf{Alg}(u)| \le |B| \le \prod_{u\in\delta(M)} |\mathbf{Alg}(u)|.$$

Therefore δ is a bijection between M and U_n since $\mathbf{B}$ is finite. It follows that we may write $\mathbf{B} = \prod_{u\in U_n} \mathbf{Alg}(u)$. Suppose $U_n = \{u_1, \ldots, u_m\}$. Let $\tau_1, \ldots, \tau_m$ be the congruences of $\mathbf{B}$ for which $\mathbf{B}/\tau_i$ is $\mathbf{Alg}(u_i)$, $1 \le i \le m$. That is, τ_i is the kernel of the projection map from $\mathbf{B}$ onto $\mathbf{Alg}(u_i)$. Each $\tau_i \ne 1_B$ since $\mathbf{Alg}(u)_i$ is subdirectly irreducible and thus nontrivial. Since $\mathbf{B} = \prod_{u\in U_n} \mathbf{Alg}(u)$ the set $\{\tau_1, \ldots, \tau_m\}$ forms a factorizing set, i.e.

$$0_B = \bigwedge_{j=1}^{m} \tau_j \text{ and } 1_B = \tau_i \circ \bigwedge_{j\ne i} \tau_j \text{ for each } i.$$

If $\tau_1 < \tau_2$, say, then $0_B = \tau_1 \wedge \tau_3 \wedge \cdots \wedge \tau_m$ and $1_B = \tau_1 \circ (\tau_2 \wedge \cdots \wedge \tau_m) \le \tau_1 \circ (\tau_3 \wedge \cdots \wedge \tau_m)$. So τ_1 and $\tau_3 \wedge \cdots \wedge \tau_m$ form a factorizing pair for $\mathbf{B}$. But $|\mathbf{B}| = \prod_{1\le j\le m} |\mathbf{B}/\tau_j|$ while $|\mathbf{B}| = |\mathbf{B}/\tau_1| \times |\mathbf{B}/(\tau_3 \wedge \cdots \wedge \tau_m)| \le |\mathbf{B}/\tau_1| \times |\mathbf{B}/\tau_3| \times \cdots \times |\mathbf{B}/\tau_m|$, which is a contradiction since $\mathbf{B}/\tau_2$ is nontrivial. Hence the τ_i form an antichain in $\mathsf{Con}\,\mathbf{B}$. The τ_i are also meet-irreducible in $\mathsf{Con}\,\mathbf{B}$ and every meet-irreducible element of $\mathsf{Con}\,\mathbf{B}$ is a τ_i for some i. Hence the set of all τ_i is the set of all coatoms of $\mathsf{Con}\,\mathbf{B}$ since for every congruence $\theta \ne 1_B$ there is a coatom α with $\theta \le \alpha$. Thus, every algebra in $\mathcal{S}_n$ is simple.

We next show that the lattice $\mathsf{Con}\,\mathbf{B}$ is distributive and that all the congruence relations of $\mathbf{B}$ permute. As is the case for every congruence relation θ of a finite algebra, θ is the meet of meet-irreducible congruences and thus each $\theta \in \mathsf{Con}\,\mathbf{B}$ is the meet of those τ_j for which $\theta \le \tau_j$. But by considering the kernels of projection maps of $\mathbf{B} = \prod_{1\le j\le m} \mathbf{Alg}(u_j)$ we see that every set of coatoms determines a unique congruence relation of $\mathbf{B}$. Thus, the congruence lattice of $\mathbf{B}$ is isomorphic to the lattice of subsets of $\{1, \ldots, m\}$ and is therefore distributive. That the congruence relations of $\mathbf{B}$

permute follows easily from the fact that every congruence relation of $\mathbf{B}$ is the kernel of a projection homomorphism.

Finally, if the upper bound for $|\mathbf{B}|$ is obtained, then $\mathbf{B}$ must be the free algebra for $\mathcal{V}$ with n free generators since $\mathcal{V}$ is locally finite. From the assumption that $n \geq 3$ we have that all congruences of $\mathbf{F}_{\mathcal{V}}(3)$ permute and $\mathsf{Con}\,(\mathbf{F}_{\mathcal{V}}(3))$ is distributive. Therefore the variety $\mathcal{V}$ is congruence distributive and congruence permutable, e.g., [9, p. 81]. ∎

The upper bound presented in (4) of Theorem 2.8 may be viewed as an extension of G. Birkhoff's classic theorem from 1935 that bounds the size of a finitely generated algebra:

Theorem 2.9 (Birkhoff). *Let $\mathcal{K}$ any set of algebras $\mathbf{A}_1, \ldots, \mathbf{A}_r$. The cardinality of any algebra generated by n elements and obeying the equations of $\mathcal{K}$ is at most*

$$\prod_{i=1}^{r} |A_i|^{|A_i|^n}.$$

Sioson [20] provides a companion characterization theorem: The upper bound in Birkhoff's theorem is obtained for finite algebras if and only if the $\mathbf{A}_1 \ldots, \mathbf{A}_r$ form a primal cluster. Thus, Theorem 2.8 is an extension of Birkhoff's's upper bound theorem and Sioson's characterization theorem. Further extensions of this kind are presented in [4].

Note that if the two equivalent conditions of Theorem 2.8 hold for $\mathcal{V}$, then the free algebra on n free generators for $\mathcal{V}$ is $\prod_{\mathbf{S} \in \mathcal{S}_n} \mathbf{S}^{\frac{|\mathrm{val}(X,\mathbf{S})|}{|\mathrm{Aut}\,\mathbf{S}|}}$. Thus, given the set $\mathcal{S}_n$ of n-generated subdirectly irreducible algebras in a locally finite variety $\mathcal{V}$, an upper bound for the cardinality of any n-generated algebra in $\mathcal{V}$ is given completely in terms of $|\mathrm{Aut}\,\mathbf{S}|$ and $|\mathrm{val}(X, \mathbf{S})|$ where $\mathbf{S}$ ranges over the algebras in the transversal $\mathcal{S}_n$. In the next section we consider effective methods for actually computing $|\mathrm{val}(X, \mathbf{A})|$ for $\mathbf{A}$ an arbitrary finite algebra.

3. Finding the cardinality of $\mathrm{val}(X, A)$

In this section we present general methods for determining the exact value of $|\mathrm{val}(X, \mathbf{A})|$ for $\mathbf{A}$ a finite algebra. Four methods are given. The first is based on the Möbius function for a finite partially ordered set. This method is completely general but in order to use the method to actually compute $|\mathrm{val}(X, \mathbf{A})|$ one must know the behavior of the Möbius function on the partially ordered set of subuniverses of $\mathbf{A}$. The second method is based solely on knowledge of the cardinalities of the maximal subuniverses of $\mathbf{A}$ and of the subuniverses that are the intersections of maximal subuniverses. A third method, which provides an especially simple computation for $|\mathrm{val}(X, \mathbf{A})|$, may be used when the algebra $\mathbf{A}$ is known to be uniquely generated. The fourth method, which is also completely general, aggregates isomorphic subuniverses together in the computation and makes use of matrix inversion.

Note that if $\mathbf{A}$ is an algebra and X is a set, then $\operatorname{val}(X, \mathbf{A})$ is nonempty if and only if $\mathbf{A}$ is generated by a set of cardinality at most $|X|$. If $\mathbf{B}_1 \lhd \mathbf{A}$ and $\mathbf{B}_2 \lhd \mathbf{A}$ with $\mathbf{B}_1 \neq \mathbf{B}_2$, then $\operatorname{val}(X, \mathbf{B}_1) \cap \operatorname{val}(X, \mathbf{B}_2) = \emptyset$. These comments lead to the following:

$$\operatorname{val}(X, \mathbf{A}) = A^X \setminus \Big(\bigcup_{\substack{\mathbf{B} \lhd \mathbf{A} \\ \mathbf{B} \neq \mathbf{A}}} \operatorname{val}(X, \mathbf{B}) \Big)$$

and

$$|\operatorname{val}(X, \mathbf{A})| = |A|^{|X|} - \Big(\sum_{\substack{\mathbf{B} \lhd \mathbf{A} \\ \mathbf{B} \neq \mathbf{A}}} |\operatorname{val}(X, \mathbf{B})| \Big). \tag{5}$$

Although formula (5) is completely general, it is unwieldy in application since it is recursively presented and at least on a *prima facie* level, requires knowledge of $|\operatorname{val}(X, \mathbf{B})|$ for all proper subalgebras $\mathbf{B}$ of the given algebra $\mathbf{A}$. However, we can overcome these obstacles by means of Möbius inversion.

Let $\mathbf{A}$ be a finite algebra and $\mathbf{S}_1, \dots, \mathbf{S}_r$ the subalgebras of $\mathbf{A}$, listed so that $\mathbf{S}_i \lhd \mathbf{S}_j$ implies $i \leq j$. In particular, $\mathbf{S}_r$ is $\mathbf{A}$. Consider any set Q of functions from X to A. For most of our work on determining $|\operatorname{val}(X, \mathbf{A})|$ we use $Q = A^X$, but for some applications we work with Q a proper subset of A^X. For each k, with $1 \leq k \leq r$, let

$$q_k = |Q \cap S_k^X| \quad \text{and} \quad v_k = |Q \cap \operatorname{val}(X, \mathbf{S}_k)|.$$

Proposition 3.1. *For a finite algebra* $\mathbf{A}$ *and* $\operatorname{Sub}\mathbf{A} = \{\mathbf{S}_1, \dots, \mathbf{S}_r\}$ *with* q_k *and* v_k *as above,*

$$q_k = \sum_{\mathbf{S}_p \lhd \mathbf{S}_k} v_p.$$

Proof. For each $f \in Q \cap S_k^X$ there is a unique p with $\mathbf{S}_p \lhd \mathbf{S}_k$ and a unique $g \in \operatorname{val}(X, \mathbf{S}_p)$ such that $f(X)$ generates the algebra $\mathbf{S}_p$ and $f(x) = g(x)$ for all $x \in X$. Conversely, for every $\mathbf{S}_p \lhd \mathbf{S}_k$, if $g \in Q \cap \operatorname{val}(X, \mathbf{S}_p)$, then $g \in Q \cap S_k^X$. ∎

We use Möbius inversion in Proposition 3.1 to express the v_k in terms of the q_p, for $1 \leq p \leq k$. Our terminology and notation follows that of Aigner [1].

Let ζ be the zeta-function associated with the poset $\operatorname{Sub}\mathbf{A}$. That is, $\zeta : (\operatorname{Sub}\mathbf{A})^2 \to \mathbb{R}$ is given by

$$\zeta(\mathbf{S}_i, \mathbf{S}_j) = \begin{cases} 1 & \text{if } \mathbf{S}_i \lhd \mathbf{S}_j \\ 0 & \text{otherwise.} \end{cases}$$

Then Proposition 3.1 may be expressed as

$$q_k = \sum_{\mathbf{S}_p \lhd \mathbf{S}_k} v_p = \sum_{1 \le p \le k} v_p \; \zeta(\mathbf{S}_p, \mathbf{S}_k). \tag{6}$$

If $\mu : (\mathrm{Sub}\,\mathbf{A})^2 \to \mathbb{R}$ is the Möbius function for the poset $\mathrm{Sub}\,\mathbf{A}$, then, by Möbius inversion, formula (6) yields the following formula for $|Q \cap \mathrm{val}(X, \mathbf{S}_k)|$:

$$v_k = \sum_{1 \le p \le k} q_p \; \mu(\mathbf{S}_p, \mathbf{S}_k). \tag{7}$$

We rewrite (6) and (7) in matrix notation. Let Z be the $r \times r$ matrix defined by $Z_{ij} = \zeta(\mathbf{S}_i, \mathbf{S}_j)$. By virtue of the fact that $\mathbf{S}_i \lhd \mathbf{S}_j$ implies $i \le j$, we have that Z is an upper triangular matrix. The matrix Z is therefore invertible since Z has 1's on the main diagonal. If M denotes the inverse of the matrix Z, then $M_{ij} = \mu(\mathbf{S}_i, \mathbf{S}_j)$. If $\overline{q}$ and $\overline{v}$ denote the row vectors $(q_1, \dots, q_r)$ and $(v_1, \dots, v_r)$, then we have

$$\overline{q} = \overline{v}\, Z \quad \text{and} \quad \overline{v} = \overline{q}\, M.$$

By letting $Q = A^X$ in (7) we obtain a general formula for $|\mathrm{val}(X, \mathbf{A})|$.

Theorem 3.2. *Let* $\mathbf{A}$ *be a finite algebra with* $\mathrm{Sub}\,\mathbf{A} = \{\mathbf{S}_1, \dots, \mathbf{S}_r\}$ *listed so that* $\mathbf{S}_i \lhd \mathbf{S}_j$ *implies* $i \le j$. *Let* μ *be the Möbius function for the poset* $\mathrm{Sub}\,\mathbf{A}$. *Then*

$$|\mathrm{val}(X, \mathbf{A})| = \sum_{1 \le p \le r} |S_p|^n \mu(\mathbf{S}_p, \mathbf{S}_r).$$

The only information about $\mathbf{A}$ needed to apply this formula is the cardinality of each subalgebra of $\mathbf{A}$ and the Möbius function for the poset $\mathrm{Sub}\,\mathbf{A}$.

Example 3.3. Let $\mathcal{K} = \{\mathbf{A}_1, \mathbf{A}_2, \dots\}$ be a set of nonisomorphic algebras indexed by the positive integers such that $\mathbf{A}_i \lhd \mathbf{A}_j$ if and only if i divides j. Thus, for every k the lattice $\mathrm{Sub}\,(\mathbf{A}_k)$ is isomorphic to the lattice of positive divisors of k. The set $\mathcal{K}$, having $\lhd$ as its partial order, is lattice isomorphic to the divisibility lattice of the positive integers. There are a number of examples of this phenomenon for varieties that occur as algebras of logic. One such is analyzed in Example 4.2. For each k the Möbius function on the poset $\mathrm{Sub}\,(\mathbf{A}_k)$ corresponds to the usual Möbius function $\overline{\mu}$ of number theory. Thus,

$$\mu(\mathbf{A}_i, \mathbf{A}_k) = \begin{cases} 1 & \text{if } i = k \text{ or } \frac{k}{i} \text{ is a product of an even number} \\ & \text{of distinct primes,} \\ -1 & \text{if } \frac{k}{i} \text{ is a product of an odd number} \\ & \text{of distinct primes,} \\ 0 & \text{otherwise,} \end{cases} = \overline{\mu}\left(\frac{k}{i}\right).$$

Theorem 3.2 then gives for the algebras in $\mathcal{K}$:

$$|\mathrm{val}(X,\mathbf{A}_k)| = \sum_{i|k} \overline{\mu}\left(\frac{k}{i}\right) |A_i|^n. \tag{8}$$

For every $\mathbf{A}_k \in \mathcal{K}$ the value of $|\mathrm{val}(X,\mathbf{A}_k)|$ is uniquely determined by the cardinalities of those $\mathbf{A}_i$ for $i \mid k$.

Example 4.2 provides a detailed example of a variety for which (8) applies and for which the upper bound (4) of Theorem 2.8 is obtained.

A different formula for $|\mathrm{val}(X,\mathbf{A})|$ based on intersections of maximal proper subalgebras of $\mathbf{A}$ may be obtained by use of an inclusion-exclusion argument directly on A^X and the observation that $f \in \mathrm{val}(X,\mathbf{A})$ if and only if $f(X)$ is not contained in any maximal proper subalgebra of $\mathbf{A}$. Thus,

Proposition 3.4. *If $\mathbf{A}$ is a finite algebra with $\mathbf{C}_1,\dots,\mathbf{C}_m$ the maximal proper subalgebras of $\mathbf{A}$, then*

$$\mathrm{val}(X,\mathbf{A}) = A^X \setminus \left(\bigcup_{i=1}^{m} C_i^X\right)$$

and

$$|\mathrm{val}(X,\mathbf{A})| = |A|^{|X|} - \sum_{1\le i\le m} |C_i|^{|X|} + \sum_{1\le i<j\le m} |C_i\cap C_j|^{|X|} - \sum_{1\le i<j<k\le m} |C_i\cap C_j\cap C_k|^{|X|} + \dots + (-1)^m |C_1\cap\dots\cap C_m|^{|X|}. \tag{9}$$

By virtue of Proposition 3.4 the cardinality of $\mathrm{val}(X,\mathbf{A})$ for any finite algebra $\mathbf{A}$ is completely determined by the cardinalities the universes of the maximal proper subalgebras of $\mathbf{A}$ and the cardinalities of the intersections of all of these sets. Once these maximal proper subuniverses are determined no other algebraic properties of $\mathbf{A}$ need be considered in order to compute $|\mathrm{val}(X,\mathbf{A})|$. Thus if $\mathbf{A}_1$ and $\mathbf{A}_2$ are two finite algebras sharing the same universe and with the same maximal proper subalgebras, then $|\mathrm{val}(X,\mathbf{A}_1)| = |\mathrm{val}(X,\mathbf{A}_2)|$. For example, if $\mathbf{A}$ is any finite algebra that has exactly one maximal proper subalgebra $\mathbf{C}$, then $|\mathrm{val}(X,\mathbf{A})| = |A|^{|X|} - |C|^{|X|}$.

Another immediate consequence of Proposition 3.4 is that for every finite algebra $\mathbf{A}$

$$\lim_{|X|\to\infty} \frac{|\mathrm{val}(X,\mathbf{A})|}{|A|^{|X|}} = 1.$$

Formula (9) can be simplified if the maximal subalgebras of $\mathbf{A}$ are well-behaved. For example, if there is a set $G \subseteq A$ such that an algebra $\mathbf{C} \lhd \mathbf{A}$ is maximal proper if and only if $C = A \setminus \{g\}$ for $g \in G$, then the formula becomes

$$|\mathrm{val}(X, \mathbf{A})| = \sum_{0 \le j \le |G|} (-1)^j \binom{|G|}{j} (|A| - j)^{|X|}.$$

If $\mathbf{A}$ has such a subset G, then $\mathbf{A}$ has the property that G generates $\mathbf{A}$ and every generating set for $\mathbf{A}$ must contain G. The algebra $\mathbf{A}$ is said to be *uniquely generated* by G in this case. The paper [5] contains a discussion of uniquely generated algebras and related notions. We slightly extend these ideas with an eye to applications of Proposition 3.4 later in the paper.

Proposition 3.5. *Suppose $\mathbf{A}$ is an algebra with G a nonempty subset of A, and $\equiv$ is an equivalence relation on G with blocks $D_1, D_2, \ldots, D_m$. The following are equivalent.*

1. *The maximal proper subuniverses of $\mathbf{A}$ are precisely the sets $A \setminus D_i$ for $1 \le i \le m$.*
2. *Every transversal of G with respect to $\equiv$ generates $\mathbf{A}$, and for every D_i and every fundamental operation f of $\mathbf{A}$,*

$$f(a_1, \ldots, a_r) \in D_i \text{ implies at least one } a_j \in D_i.$$

Proof. If (1) holds, then every transversal of G with respect to $\equiv$ is not in any maximal proper subalgebra of $\mathbf{A}$. Hence the transversal generates all of $\mathbf{A}$. If $f(a_1, \ldots, a_r) \in D_i$, then at least one $a_j \in D_i$ else all a_j are in the subuniverse $A \setminus D_i$.

If (2) holds, then for each i the set $A \setminus D_i$ is a proper subuniverse since if $a_1, \ldots, a_r \in A \setminus D_i$, then $f(a_1, \ldots, a_r)$ cannot be in D_i. No $A \setminus D_i$ is a subset of any other $A \setminus D_j$ since the D_i form a partition of G. If B is a subuniverse that is not a subset of any $A \setminus D_i$, then for each i there is at least one $b \in B \cap D_i$. So B contains a transversal of $\equiv$ and is therefore A. Hence each $A \setminus D_i$ is the subuniverse of a maximal proper subalgebra, and every maximal proper subalgebra must have universe $A \setminus D_i$. ∎

Definition 3.6. For an algebra $\mathbf{A}$, a nonempty set $G \subseteq A$, and an equivalence relation $\equiv$ on G, we say that $\mathbf{A}$ is *uniquely generated by $G/{\equiv}$* if the equivalent conditions of Proposition 3.5 hold. If $\mathbf{A}$ is uniquely generated by $G/0_G$, then $\mathbf{A}$ is said to be *uniquely generated by G*.

If the algebra $\mathbf{A}$ is uniquely generated by $G/\equiv$, and if the cardinalities of the blocks of $\equiv$ are well–behaved, then explicit simple formulas for $|\mathrm{val}(X, \mathbf{A})|$ based on (9) may be possible. The following proposition illustrates this and will be used later in the paper for providing explicit formulas for the cardinalities of free algebras, e.g., in Example 4.3 for certain varieties of residuated monoids and in Example 4.8 for many varieties of orthomodular lattices.

Proposition 3.7. *Suppose* $\mathbf{A}$ *is a finite algebra uniquely generated by* $G/\equiv$ *with* $\equiv$ *having* m *equivalence classes, all of the same cardinality* d. *Then*

$$|\mathrm{val}(X,\mathbf{A})| = \sum_{j=0}^{m}(-1)^j\binom{m}{j}(|A|-jd)^{|X|}$$
$$= \sum_{p=m}^{n}\binom{n}{p}S(p,m)\,m!\,d^p\,(|A|-|G|)^{n-p}.$$

Proof. The first summation is (9) in Proposition 3.4. Each summand in the second sum is a product of five factors. The first two factors are the number $\binom{n}{p}$ of ways to choose p elements from X and the Stirling number of the second kind $S(p,m)$, which is the number of ways to partition these p elements into m nonempty blocks. The factor $m!$ is the number of bijections between these m blocks and the m equivalence classes of $\equiv$. The fourth factor d^p counts the number of ways to assign each x of the p chosen members of X to one of the d elements of the equivalence class of $\equiv$ that corresponds to the block of the partition that contains x. The final factor is the number of ways to assign each of the other $n-p$ members of X to an element of $A\setminus G$. ∎

We note that in these formulas the value of $|\mathrm{val}(X,\mathbf{A})|$ depends only on the values of $|X|, |A|, |G|$ and d, with $m = |G|/d$.

We now give a fourth method for computing $|\mathrm{val}(X,\mathbf{A})|$. This method may be used to find formulas that depend only on the cardinalities of the subalgebras of $\mathbf{A}$, and for each pair $\mathbf{B}$ and $\mathbf{C}$ of subalgebras of $\mathbf{A}$, the number $\mathrm{iso}\,(\mathbf{B},\mathbf{C})$ of subalgebras of $\mathbf{C}$ that are isomorphic to $\mathbf{B}$. This method does not explicitly use the Möbius function, but it requires finding the inverse of the matrix that codes all the values of $\mathrm{iso}\,(\mathbf{B},\mathbf{C})$. The method is completely general. It is particularly effective for computing $|\mathrm{val}(X,\mathbf{A})|$ when $\mathbf{A}$ has many subalgebras but the equivalence relation of isomorphism on the set $\mathrm{Sub}\,\mathbf{A}$ has relatively few equivalence classes. By aggregating with respect to isomorphism in this manner the number of summands in an expression for $|\mathrm{val}(X,\mathbf{A})|$ can thereby be reduced.

Our presentation of this method is phrased in a wider context in order to use the method in certain applications later in the paper. Thus, for a given set D of algebraic constants in the language of the algebra $\mathbf{A}$, we are interested in those f in A^X or in $\mathrm{val}(X,\mathbf{A})$ for which $f(X)\cap D=\emptyset$. We are sometimes interested in situations in which the family of subalgebras considered extends beyond $\mathrm{Sub}\,\mathbf{A}$ for some single finite algebra $\mathbf{A}$.

Definition 3.8. For two finite algebras $\mathbf{B}$ and $\mathbf{C}$, let $\mathrm{iso}\,(\mathbf{B},\mathbf{C})$ denote the number of subalgebras of $\mathbf{C}$ that are isomorphic to $\mathbf{B}$.

Definition 3.9. Let K be either the set of all positive integers or some nonvoid initial segment of them. Suppose $\mathcal{K} = \{\mathbf{A}_k \mid k \in K\}$ is a family of pairwise nonisomorphic algebras, all of the same similarity type, indexed by K. We say $\mathcal{K}$ is *hereditarily closed* if for each $k \in K$ every subalgebra of $\mathbf{A}_k$ is isomorphic to $\mathbf{A}_p$ for some $p \in K$ with $1 \leq p \leq k$.

Note that if $\mathcal{K}$ is a hereditarily closed family and $\mathbf{A}_p \mathrel{\tilde{\triangleleft}} \mathbf{A}_k$, then $p \leq k$. If $\mathbf{A}$ is a finite algebra, then any transversal with respect to isomorphism of the subalgebras of $\mathbf{A}$ can be listed in such a way so as to form a hereditarily closed family. Likewise, if $\{\mathbf{A}_i \mid i \in I\}$ is any finite or countably infinite collection of finite algebras of the same similarity type, then any transversal with respect to isomorphism of $\bigcup_{i\in I} \mathrm{Sub}\,(\mathbf{A}_i)$ can be indexed to form a hereditarily closed family.

Suppose $\mathcal{K} = \{\mathbf{A}_k \mid k \in K\}$ is a hereditarily closed family of finite algebras and that D is a set, possibly empty, of algebraic constants in the language of $\mathcal{K}$.

We introduce the following notation:

- Q_k denotes $(A_k \setminus D)^X$.
- $q(n,k) := |Q_k| = (|A_k| - |D|)^n$.
- $v(n,k) := |Q_k \cap \mathrm{val}(X, \mathbf{A}_k)|$.

Typically in applications $D = \emptyset$ in which case $Q_k = \mathbf{A}_k^X$, $q(n,k) = |A_k|^n$ and $v(n,k) = |\mathrm{val}(X, \mathbf{A}_k)|$.

Lemma 3.10. *Suppose $\mathbf{B}_1$ and $\mathbf{B}_2$ are isomorphic finite algebras and D is a set of algebraic constants in the language of the $\mathbf{B}_i$. Then*

$$|\mathrm{val}(X, \mathbf{B}_1) \cap (B_1 \setminus D)^X| = |\mathrm{val}(X, \mathbf{B}_2) \cap (B_2 \setminus D)^X|.$$

Proof. Let $\alpha : \mathbf{B}_1 \to \mathbf{B}_2$ be an isomorphism. The lemma follows from the fact $\alpha(d) = d$ for every $d \in D$ and that a set G generates $\mathbf{B}_1$ if and only if $\alpha(G)$ generates $\mathbf{B}_2$. ∎

Since every $f \in (A \setminus D)^X$ is a valuation to some subalgebra $\mathbf{B}$ of $\mathbf{A}$ and hence f is in $\mathrm{val}(X, \mathbf{B}) \cap (B \setminus D)^X$, we have

$$(A \setminus D)^X = \bigcup_{\mathbf{B} \triangleleft \mathbf{A}} \mathrm{val}(X, \mathbf{B}) \cap (A \setminus D)^X = \bigcup_{\mathbf{B} \triangleleft \mathbf{A}} \mathrm{val}(X, \mathbf{B}) \cap (B \setminus D)^X. \tag{10}$$

Then (10), by virtue of Lemma 3.10, yields

$$q(n,k) := |A_k \setminus D|^n = \sum_{1 \leq p \leq k} v(n,p)\ \mathrm{iso}\,(\mathbf{A}_p, \mathbf{A}_k). \tag{11}$$

Theorem 3.11. *Let $\mathcal{K} = \{\mathbf{A}_k \mid k \in K\}$ be a hereditarily closed family of algebras and D any set of algebraic constants in the language of $\mathcal{K}$. For*

$k \in K$ let $\overline{q} = (q(n,1)\dots, q(n,k))$ and $\overline{v} = (v(n,1),\dots,v(n,k))$ be row vectors and let C the upper triangular $k \times k$ matrix with $C_{ij} = \operatorname{iso}(\mathbf{A}_i, \mathbf{A}_j)$. If M denotes the inverse of the matrix C, then

$$\overline{q} = \overline{v}C \text{ and hence } \overline{v} = \overline{q}M.$$

In particular, if $D = \emptyset$, then

$$|\mathrm{val}(X, \mathbf{A}_k)| = \sum_{1 \le p \le k} |A_p|^n M_{pk}.$$

Proof. The matrix C is upper triangular because $\mathcal{K}$ is hereditarily closed. C has 1's on the main diagonal. So the inverse M exists. The matrix equation $\overline{q} = \overline{v}C$ is essentially (11). If $D = \emptyset$, then $q(n,p) = |A_p|^n$ for all p. ∎

Theorem 3.11 provides a general formula for computing $|\mathrm{val}(X, \mathbf{A})|$ that depends only on the cardinalities of the members of a transversal with respect to isomorphism of the subalgebras of $\mathbf{A}$ and the inverse of the matrix whose entries are $\operatorname{iso}(\mathbf{A}_i, \mathbf{A}_j)$ where $\mathbf{A}_i$ and $\mathbf{A}_j$ range over the algebras in this transversal. For some familiar algebras the matrix C with entries $\operatorname{iso}(\mathbf{A}_i, \mathbf{A}_j)$ and its inverse M have an especially pleasant form.

One upper triangular invertible k by k matrix that we will consider is the matrix B of binomial coefficients with entries $B_{ij} = \binom{j-1}{i-1}$ for $1 \le i, j \le k$. From the standard identity

$$\sum_{t=p}^{q} \binom{t}{p}\binom{q}{t}(-1)^{t-p} = \delta_{pq}$$

it follows that the inverse M of B is given by

$$M_{ij} = (-1)^{j-i} B_{ij} = \begin{cases} (-1)^{j-i}\binom{j-1}{i-1} & \text{if } j \ge i \\ 0 & \text{if } j < i. \end{cases} \tag{12}$$

Further details on Pascal matrices such as these may be found in [10]. Example 4.1 provides an example of the matrix method applied when the entries of the matrix C in Theorem 3.11 are a collection of binomial coefficients.

Example 3.12. To illustrate this method we consider $|\mathrm{val}(X, \mathbf{A}_k)|$ for $\mathbf{A}_k$ a Boolean algebra with k atoms. The formulas we obtain will be used in Example 4.8. Every subalgebra of $\mathbf{A}_k$ is isomorphic to $\mathbf{A}_p$ for $1 \le p \le k$. If $\mathbf{A}_p$ is a subalgebra of $\mathbf{A}_k$, then each atom of $\mathbf{A}_p$ is the join of atoms of $\mathbf{A}_k$. So $\mathbf{A}_p$ is determined by a partition of the k atoms of $\mathbf{A}_k$ into p nonempty blocks. Conversely, every such partition of the atoms of $\mathbf{A}_k$ determines a unique subalgebra of $\mathbf{A}_k$. The lattice $\operatorname{Sub}\mathbf{A}_k$ is dually isomorphic to the lattice of partitions of the set $\{1, 2, \dots, k\}$. So the matrix C in Theorem 3.11 has $C_{ij} = S(j,i)$ and as observed in (1) the inverse of C is the matrix

M with $M_{ij} = s(j,i)$, with $s(j,i)$ a Stirling number of the first kind. By Theorem 3.11

$$|\mathrm{val}(X,\mathbf{A}_k)| = \sum_{1\le p\le k} (2^p)^n s(k,p) = \sum_{1\le p\le k} s(k,p)(2^n)^p \tag{13}$$
$$= 2^n(2^n-1)\dots(2^n-k+1) = \binom{2^n}{k}k!.$$

Here the third equality is based on the identity

$$\sum_{p=1}^{k} s(k,p)x^p = x(x-1)(x-2)\dots(x-k+1).$$

We present an easy application of the previous paragraph to monadic algebras. The variety $\mathcal{M}$ of monadic algebras is known to be (e.g., [18]) semisimple arithmetical and the subdirectly irreducible algebras in $\mathcal{M}$ are Boolean algebras $\mathbf{B}_k$ with a closure operator C as a fundamental operation such that $C(\bot) = \bot$ and $C(x) = \top$ for all $x \neq \bot$. If $\mathbf{M}_k$ denotes the simple monadic algebra with k atoms, then $\mathrm{val}(X,\mathbf{M}_k) = \mathrm{val}(X,\mathbf{B}_k)$ and $|\mathrm{Aut}\,\mathbf{M}_k| = k!$, since $\mathbf{M}_k$ and $\mathbf{B}_k$ have the same subuniverses and the same automorphisms. Therefore by (13)

$$\frac{|\mathrm{val}(X,\mathbf{M}_k)|}{|\mathrm{Aut}\ (\mathbf{M}_k)|} = \binom{2^n}{k}.$$

Thus, from Theorem 2.8

$$\mathbf{F}_{\mathcal{M}}(n) = \prod_{k=1}^{2^n} \mathbf{M}_k^{\binom{2^n}{k}}. \tag{14}$$

R. Quackenbush, in his paper on quasiprimal algebras [18], presents a different proof that $|\mathrm{val}(X,\mathbf{B}_k)| = k!\binom{2^n}{k}$ for $|X| = n$ and he uses this fact to derive (14). Other proofs of (14) in the literature include that of Bass [2].

In Lemmas 4.10 and 4.11 we present a similar argument in order to provide new results concerning the cardinalities of free algebras in varieties of orthomodular lattices generated by algebras that are horizontal sums of Boolean algebras.

4. Applications and examples

In this section we consider applications of the upper bound for the size of free algebras presented in Theorem 2.8 and of the methods for determining the cardinality of $\mathrm{val}(X,\mathbf{A})$ for an arbitrary finite algebra $\mathbf{A}$ as presented in Section 3. We find explicit formulas for the free spectra of some important semisimple varieties and revisit some varieties for which formulas for the free spectra have previously appeared in the literature. Indeed, our original motivation for the present paper was noticing in various published articles formulas for the free spectra of varieties that had the general format of the upper bound presented in Theorem 2.8. This theorem emerged from our attempt to unify these different results into one general framework, and the methods presented in Section 3 resulted from our attempt to find some uniform methods to replace the *ad hoc* arguments that have appeared in various sources. Examples 4.1 and 4.2 illustrate this. Having obtained Theorem 2.8 and the methods for computing $|\mathrm{val}(X, \mathbf{A})|$ in Section 3, we applied them to obtain new results for the structure and cardinality of $\mathbf{F}_{\mathcal{V}}(n)$ for specific $\mathcal{V}$ or families of varieties, or we extended previous results on $\mathbf{F}_{\mathcal{V}}(n)$ in the literature to larger n or to a wider family of varieties. Examples 4.3 and 4.8 are representative of this type of result.

Example 4.1. An algebra $\mathbf{A} = \langle A, \wedge, \vee, \Rightarrow, \sim, 0, 1\rangle$ is called a *symmetric Heyting algebra* if $\langle A, \wedge, \vee, \Rightarrow, 0, 1\rangle$ is a Heyting algebra and $\langle A, \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra. The variety and many subvarieties of symmetric Heyting algebras have been extensively studied. A long paper [17] by A. Monteiro contains a wealth of results about these algebras, many of which he obtained in the 1960's and 1970's. In this paper Monteiro considers for each positive integer m a variety $\mathcal{I}_m\mathcal{K}$. He shows that the subdirectly irreducible algebras in $\mathcal{I}_m\mathcal{K}$ are the symmetric Heyting algebras $\mathbf{C}_2, \ldots \mathbf{C}_m$, where for any integer $k \geq 2$ the universe of $\mathbf{C}_k$ is

$$C_k = \left\{0 = \frac{0}{k-1}, \frac{1}{k-1}, \ldots \frac{k-2}{k-1}, \frac{k-1}{k-1} = 1\right\}$$

with the order structure a chain and with $\sim \frac{i}{k-1} = \frac{k-1-i}{k-1}$. If $\mathbf{F}_{\mathcal{I}_m\mathcal{K}}(n)$ is the free algebra for $\mathcal{I}_m\mathcal{K}$ on n free generators, then he argues that

$$\mathbf{F}_{\mathcal{I}_m\mathcal{K}}(n) = \mathbf{C}_2^{p_2} \times \cdots \times \mathbf{C}_m^{p_m}$$

where the p_k are non-negative integers. He then shows the value of each p_k is the number of functions f from the set X of n free generators to C_k for which $f(X)$ generates $\mathbf{C}_k$. That is, in our terminology, $p_k = |\mathrm{val}(X, \mathbf{C}_k)|$. In pages 139–149 he finds explicit formulas for the p_k. He shows that if $k = 2t$, then

$$p_k = 2^n \sum_{i=0}^{t-1} (-1)^i \binom{t-1}{i} (t-i)^n, \tag{15}$$

and if $k = 2t + 1$, then

$$p_k = \sum_{i=0}^{t-1} (-1)^i \binom{t-1}{i} (2t-2i+1)^n - 2^n \sum_{i=0}^{t-1} (-1)^i \binom{t-1}{i} (t-i)^n. \tag{16}$$

The facts that the free algebra $\mathbf{F}_{\mathcal{I}_m\mathcal{K}}(n)$ is the direct product of subdirectly irreducibles $\mathbf{C}_k$ and that the number of copies of each C_k is $|\mathrm{val}(X, \mathbf{C}_k)|$

suggest that Theorem 2.8 and the techniques for computing the cardinality of $\mathrm{val}(X, \mathbf{A})$ of Section 3 might apply to give a proof of the results of Monteiro [17] for $\mathcal{I}_m\mathcal{K}$ described above.

To this end consider the variety $\mathcal{I}_m\mathcal{K}$ with $(\mathcal{I}_m\mathcal{K})_{SI} = \{\mathbf{C}_2, \ldots, \mathbf{C}_m\}$. Each $\mathbf{C}_k$ is easily seen to be simple. The subuniverses of $\mathbf{C}_m$ consist precisely of those sets S that contain 0 and 1 and for which $\frac{i}{m-1} \in S$ if and only if $\frac{m-1-i}{m-1} \in S$. Thus, every subalgebra of $\mathbf{C}_m$ is isomorphic to some $\mathbf{C}_k$ for $2 \leq k \leq m$. Hence each $\mathbf{C}_m$ is semisimple and so is the variety $\mathcal{I}_m\mathcal{K}$. Heyting algebras are known to be congruence distributive and congruence permutable so, in particular, $\mathcal{I}_m\mathcal{K}$ is an arithmetical variety. Therefore Theorem 2.8 applies. Since each $\mathbf{C}_m$ has a linear order, $|\mathrm{Aut}\,(\mathbf{C}_k)| = 1$ for all k. Thus

$$\mathbf{F}_{\mathcal{I}_m\mathcal{K}}(n) = \prod_{k=2}^{m} \mathbf{C}_k^{|\mathrm{val}(X,\mathbf{C}_k)|}.$$

Now, we use the methods of Section 3 to determine $|\mathrm{val}(X, \mathbf{C}_k)|$ for arbitrary k. Since $(\mathcal{I}_m\mathcal{K})_{SI} = \{\mathbf{C}_2, \ldots, \mathbf{C}_m\}$ is a hereditarily closed family and the values of $\mathrm{iso}\,(\mathbf{C}_p, \mathbf{C}_k)$ are readily found, we use the matrix methods of Theorem 3.11. Note if k is even and p is odd, then $\mathrm{iso}\,(\mathbf{C}_p, \mathbf{C}_k) = 0$. From this and from the characterization of subalgebras of $\mathbf{C}_k$ given above we have

$$\mathrm{iso}\,(\mathbf{C}_{2s+1}, \mathbf{C}_{2t}) = 0$$

and

$$\mathrm{iso}\,(\mathbf{C}_{2s}, \mathbf{C}_{2t}) = \mathrm{iso}\,(\mathbf{C}_{2s+1}, \mathbf{C}_{2t+1}) = \mathrm{iso}\,(\mathbf{C}_{2s}, \mathbf{C}_{2t+1}) = \binom{t-1}{s-1}.$$

If D is the m by m matrix with $D_{ij} = \mathrm{iso}\,(\mathbf{C}_i, \mathbf{C}_j)$, and if M denotes the inverse of D, then arguing as in (12) for Pascal matrices, it can be seen that

$$M_{ij} = D_{ij}(-1)^{\gamma} = \mathrm{iso}\,(\mathbf{C}_i, \mathbf{C}_j)(-1)^{\gamma},$$

where

$$\gamma = \begin{cases} t-s-1, & \text{if } i = 2s,\ j = 2t+1; \\ t-s, & \text{otherwise.} \end{cases}$$

Then letting $\overline{q} = (1^n, 2^n, \ldots, k^n)$ in Theorem 3.11 we get

$$|\mathrm{val}(X, \mathbf{C}_k)| = \sum_{p=1}^{k} p^n M_{pk} =$$

$$\begin{cases} \displaystyle\sum_{j=1}^{t} (2j)^n \binom{t-1}{j-1} (-1)^{t-j}, & \text{if } k = 2t; \\ \displaystyle\sum_{j=1}^{t} (2j)^n \binom{t-1}{j-1} (-1)^{t-j+1} + (2j+1)^n \binom{t-1}{j-1} (-1)^{t-j}, & \text{if } k = 2t+1. \end{cases}$$

Substituting $t-i$ for j and factoring 2^n outside the summation sign, Monteiro's formulas (15) and (16) are obtained.

Example 4.2. We present a specific example of the situation described in Example 3.3 for algebras $\mathbf{Ł} = \langle A, \to, \wedge, 1\rangle$ that are the algebraic counterpart of the $\{\to, \wedge\}$-fragment of Łukasiewicz's many-valued logic.

For each nonnegative integer k let $Ł_{k+1}$ be the set $\{e^0, e^1, \dots, e^k\}$. The algebra $\mathbf{Ł}_{k+1} = \langle Ł_{k+1}, \to, \wedge, 1\rangle$ is a $(k+1)$-element algebra with the meet operation $\wedge$ corresponding to the order $1 = e^0 > e^1 > \cdots > e^k$ and with the residuation operator $\to$ given by

$$e^i \to e^j = \begin{cases} 1 & \text{if } i \geq j \\ e^{j-i} & \text{otherwise.} \end{cases}$$

It is easily checked that the subuniverses of $\mathbf{Ł}_{k+1}$ consist of all sets of the form $\{e^0, e^\ell, e^{2\ell}, \dots, e^{q\ell}\}$ with $0 \leq \ell \leq k$ and positive integers q for which $q\ell \leq k$.

The paper [11] by Figallo, Figallo, Figallo and Zilani provides a formula for the cardinality of the free algebra on n free generators for the variety generated by $\mathbf{Ł}_{k+1}$. They do this by observing that each $\mathbf{Ł}_{k+1}$ is quasi-primal with its nontrivial subalgebras consisting of algebras isomorphic to $\mathbf{Ł}_{i+1}$ for $1 \leq i \leq k$. Therefore the free algebra in this variety is

$$\prod_{i=1}^{k} \mathbf{Ł}_{i+1}^{|\mathrm{val}(X, \mathbf{Ł}_{i+1})|}.$$

They then argue that the cardinality of $\mathrm{val}(X, \mathbf{Ł}_{i+1})$ for $|X| = n$ is given recursively by

$$|\mathrm{val}(X, \mathbf{Ł}_{i+1})| = (i+1)^n - i^n - \sum_{\substack{j|i \\ j \neq i}} |\mathrm{val}(X, \mathbf{Ł}_{j+1})|.$$

We now use the matrix method of Theorem 3.11 to give a description of this free algebra by providing an explicit formula for the cardinality of $\mathrm{val}(X, \mathbf{Ł}_{i+1})$.

Every subalgebra of $\mathbf{Ł}_{k+1}$ is isomorphic to either the trivial algebra $\mathbf{Ł}_1$ or to $\mathbf{Ł}_p$ for some $2 \leq p \leq k+1$. Thus the subalgebras of $\mathbf{Ł}_{k+1}$ form a hereditarily closed family $\mathcal{K} = \{\mathbf{Ł}_1, \dots, \mathbf{Ł}_{k+1}\}$. With this scheme we have that $\mathrm{iso}\,(\mathbf{Ł}_p, \mathbf{Ł}_r) = 1$ if $p = 1$ and $\mathrm{iso}\,(\mathbf{Ł}_p, \mathbf{Ł}_r) = \lfloor(\frac{r-1}{p-1})\rfloor$ if $2 \leq p \leq r \leq k+1$. If, say, we let Q_p be all p^n functions from X to the p-element set $Ł_p$, then in order to apply the matrix method to find a formula for $|\mathrm{val}(X, \mathbf{Ł}_{k+1})|$ we are faced with the problem of finding the inverse of the $k+1$ by $k+1$ matrix $C = (c_{pr})$ in which $c_{pr} = 1$ if $p = 1$ and $c_{pr} = \lfloor(\frac{r-1}{p-1})\rfloor$ if $p > 1$. This appears to be difficult.

A more workable approach for finding $|\mathrm{val}(X, \mathbf{Ł}_{k+1})|$ is to first observe that any valuation $f \in \mathrm{val}(X, \mathbf{Ł}_{k+1})$ must have $e^k \in f(X)$. For each i, with $1 \leq i \leq k$, let $\mathbf{A}_i$ denote the algebra obtained from $\mathbf{Ł}_{i+1}$ by adding the element e^i, the least element in the order relation, to the similarity type as a constant. The family $\mathcal{K} = \{\mathbf{A}_1, \ldots, \mathbf{A}_k\}$ forms a hereditarily closed family with $|A_i| = i+1$ for $1 \leq i \leq k$. For $1 \leq i \leq j \leq k$, if $\mathrm{iso}\,(\mathbf{A}_i, \mathbf{A}_j)$ denotes the number of subalgebras of $\mathbf{A}_j$ isomorphic to $\mathbf{A}_i$, then we have

$$\mathrm{iso}\,(\mathbf{A}_i, \mathbf{A}_j) = \begin{cases} 1 & \text{if } i|j \\ 0 & \text{otherwise.} \end{cases}$$

Let Q_p denote the set of all $f : X \to A_p$ for which $e^p \in f(X)$. Then $q(n,p) := |Q_p| = (p+1)^n - p^n$. As observed in Example 3.3 the inverse of the matrix C formed from the $\mathrm{iso}\,(\mathbf{A}_i, \mathbf{A}_j)$ is the matrix M with $M_{ij} = \overline{\mu}(\frac{j}{i})$ if i divides j, with $\overline{\mu}$ the usual Möbius function of number theory, and $M_{ij} = 0$ otherwise.

We therefore have by Theorem 3.11

$$|\mathrm{val}(X, \mathbf{Ł}_{k+1})| = \sum_{t|k} \overline{\mu}\left(\frac{k}{t}\right)((t+1)^n - t^n). \tag{17}$$

If $k = p_1^{e_1} \cdots p_m^{e_m}$ for m distinct primes $p_1, \ldots, p_m$ and positive exponents e_i, and if

$$h(t) := (t+1)^n - t^n,$$

then by the definition of the Möbius function we have from (17)

$$|\mathrm{val}(X, \mathbf{Ł}_{k+1})| = \sum_{I \subseteq \{1,\ldots,m\}} (-1)^{|I|} h\left(\frac{k}{\prod_{i \in I} p_i}\right), \tag{18}$$

where $\prod_{i \in I} p_i = 1$ for $I = \emptyset$. Note that the values of the expressions that appear on the right sides of (17) and (18) depend only on the values of k and $|X|$ and the primes that appear in the prime factorization of k.

Let $\mathcal{L}_{k+1}$ denote the variety generated by $\mathbf{Ł}_{k+1}$ for $k \geq 1$. As observed in [11] this algebra is quasiprimal, its nontrivial subalgebras are isomorphic to $\mathbf{Ł}_{i+1}$ for $1 \leq i \leq k$, and each subalgebra is rigid since the underlying order is a finite chain. Therefore by Theorem 2.8

$$\mathbf{F}_{\mathcal{L}_{k+1}}(n) = \prod_{i=1}^{k} \mathbf{Ł}_{i+1}^{|\mathrm{val}(X, \mathbf{L}_{i+1})|} = \prod_{i=1}^{k} \mathbf{Ł}_{i+1}^{\sum_{t|i} \overline{\mu}\left(\frac{i}{t}\right)((t+1)^n - t^n)}.$$

Example 4.3. Algebras that lend themselves well to the analysis of Section 3 are certain varieties of pocrims considered by Blok and Raftery in [8]. I thank James Raftery for suggesting and outlining this example. A *pocrim* (partially ordered commutative residuated integral monoid) is an algebra

$\mathbf{A} = \langle A; \cdot, \to, 1\rangle$ in which $\cdot$ is a commutative monoid operation with identity 1 that is obtained from a partially ordered monoid on A having 1 as greatest element and $\to$ is a residuation operation with

$$z \le x \to y \text{ if and only if } x \cdot z \le y. \tag{19}$$

(I work with the order dual of the definition of a pocrim given in [8].) It follows from (19) that the order $\le$ and the residuation $\to$ are linked by $x \to y = 1$ if and only if $x \le y$.

For every positive integer n, let $\mathcal{M}_n^s$ denote the variety generated by all simple pocrims that satisfy the monoid identity $x^n = x^{n+1}$. As shown in [8], the varieties $\mathcal{M}_n^s$ are semisimple and they are discriminator varieties.

The variety $\mathcal{M}_1^s$ is the variety of generalized Boolean algebras and is generated by the 2-element algebra $\mathbf{B} = \langle \{0,1\}; \cdot, \to, 1\rangle$. Then $\mathrm{val}(X, \mathbf{B})$ consists of all of $\{0,1\}^X$ except for the constant function 1. The automorphism group of $\mathbf{B}$ is trivial, so by Theorem 2.8 we have the well-known fact that

$$\mathbf{F}_{\mathcal{M}_1^s}(X) = \mathbf{B}^{2^{|X|}-1}.$$

As shown in [8] the varieties $\mathcal{M}_n^s$ for $n \ge 3$ are not locally finite but the variety $\mathcal{M}_2^s$ is. Moreover, the subdirectly irreducible algebras of this variety are explicitly described. We use this description to investigate the free algebras for the variety $\mathcal{M}_2^s$ and for some of its subvarieties.

In addition to the generalized Boolean algebra $\mathbf{B}$ described above, the other subdirectly irreducible algebras in $\mathcal{M}_2^s$ have the partial order

$$a^2 < P < a < 1,$$

where P denotes an arbitrary partially ordered set, 1 is the monoid identity, and for any $x, y \le a$, we have $x \cdot y = a^2$. If P is the empty set, then we get the usual 3-element Łukasiewicz chain, which we denote by $\mathbf{Ł}$. For a poset P let $\mathbf{A}_P$ denote this subdirectly irreducible (and hence simple) member of $\mathcal{M}_2^s$.

Note that for any $x, y \in P$, either $x \to y = 1$ or $x \to y = a$ in $\mathbf{A}_P$. Also, if $x \in P$, then $x \to a^2 = a$. Thus if $P \ne \emptyset$, then condition (2) of Proposition 3.5 holds for P with $\equiv$ being the identity relation 1_P. So if P is nonempty, then P uniquely generates the algebra $\mathbf{A}_P$ as in Definition 3.6. If $P = \emptyset$, then the Łukasiewicz 3-chain $\mathbf{Ł}$ is uniquely generated by the set $\{a\}$.

It is easily checked that if Q is a subposet of P, then $\mathbf{A}_Q$ is a subalgebra of $\mathbf{A}_P$. The subuniverses of $\mathbf{A}_P$ are $\{1\}, \{1, a^2\}, \{1, a, a^2\}$ and $\{1, a, a^2\} \cup Q$ where Q ranges over all nonvoid subposets of P. There is an obvious bijection between the automorphisms of $\mathbf{A}_P$ and the order automorphisms of the partially ordered set P since the partial order on $\mathbf{A}_P$ is definable in terms of the fundamental operations of the algebra.

These observations and Proposition 3.7 yield the following.

Lemma 4.4. *Let P be a finite partially ordered set with $|P| = p \geq 1$ and let $\mathbf{A}_P \in \mathcal{M}_2^s$. Suppose $|X| = n$.*

$$|\mathrm{val}(X, \mathbf{A}_P)| = \sum_{j=0}^{p} (-1)^j \binom{p}{j} (p + 3 - j)^n. \tag{20}$$

Definition 4.5. For n and p positive integers let $e(n, p)$ denote the value of the function of n and p given by (20) in Lemma 4.4. Note that $e(n, p) = 0$ if $n < p$.

Lemma 4.6. *Let $\mathcal{V}$ be any subvariety of $\mathcal{M}_2^s$ that contains the 3-element Łukasiewicz chain* **Ł**. *The free algebra $\mathbf{F}_{\mathcal{V}}(n)$ is a direct product of the simple algebras* **B**, **Ł**, *and those $\mathbf{A}_P \in \mathcal{V}$ with $1 \leq |P| = p \leq n$. The algebra* **B** *appears as a direct factor $2^n - 1$ times,* **Ł** *appears as a direct factor $3^n - 2^n$ times, and each $\mathbf{A}_P$ appears as a direct factor $\frac{e(n,p)}{|\mathrm{Aut}\,(P)|}$ times.*

Proof. The variety $\mathcal{V}$ is a discriminator variety since $\mathcal{M}_2^s$ is. So it is locally finite, n-semisimple and arithmetical. Thus Theorem 2.8 applies. ∎

Corollary 4.7. *Let n be a positive integer and let $\mathcal{P}$ be a transversal with respect to order isomorphism of all nonvoid partially ordered sets with at most n elements. Then*

$$\mathbf{F}_{\mathcal{M}_2^s}(n) = \mathbf{B}^{2^n - 1} \text{\L}^{3^n - 2^n} \prod_{P \in \mathcal{P}} \mathbf{A}_P^{\frac{e(n,|P|)}{|\mathrm{Aut}\ P|}}$$

and

$$|\mathrm{F}_{\mathcal{M}_2^s}(n)| = 2^{2^n - 1} 3^{3^n - 2^n} \prod_{k=1}^{n} (k + 3)^{\frac{e(n,k)}{a(k)}},$$

with

$$a(k) = \sum_{P \in \mathcal{P},\, |P| = k} \frac{1}{|\mathrm{Aut}\ P|}.$$

We apply Corollary 4.7 for small values of n. The number of partially ordered sets with 1, 2 and 3 elements is 1, 2 and 5 respectively.

$$|\mathrm{F}_{\mathcal{M}_2^s}(1)| = 2^{2-1} 3^{3-2} 4^{4-3} = 24,$$

$$|\mathrm{F}_{\mathcal{M}_2^s}(2)| = 2^{2^2-1} 3^{3^2-2^2} 4^{4^2-3^2} 5^{\frac{3}{2}(5^2 - 2\cdot 4^2 + 3^2)} = 2^3 3^5 4^7 5^3 = 3981312000,$$

$$\begin{aligned} |\mathrm{F}_{\mathcal{M}_2^s}(3)| &= 2^{2^3-1} 3^{3^3-2^3} 4^{4^3-3^3} 5^{\frac{3}{2}(5^3 - 2\cdot 4^3 + 3^3)} 6^{\frac{19}{6}(6^3 - 3\cdot 5^3 + 3\cdot 4^3 - 3^3)} \\ &= 2^7 3^{19} 4^{37} 5^{36} 6^{19}. \end{aligned}$$

Every finite simple member of $\mathcal{M}_2^s$ is quasiprimal. For a finite partially ordered set P let $\mathcal{V}$ be the variety generated by the quasiprimal algebra $\mathbf{A}_P$. Let $\mathcal{T}$ be a transversal with respect to order isomorphism of the nonvoid subposets of P. Then we have

$$\mathbf{F}_{\mathcal{V}}(n) = \mathbf{B}^{2^n-1}\mathbf{Ł}^{3^n-2^n} \prod_{Q\in\mathcal{T}} \mathbf{A}_Q^{\frac{e(n,|Q|)}{|\mathrm{Aut}\ Q|}}. \tag{21}$$

Although $e(n,|Q|)$ depends only on the values of n and $|Q|$, the sets Q and values of $|Q|$ depend on the order structure of P, and thus evaluation of the formula (21) may require detailed knowledge of the order structure of P. However, for certain familiar and important classes of partially ordered sets we can find relatively simple expressions for (21) that depend only on n and $|P|$.

Let P be a finite linearly ordered set with $|P| = p$ and let $\mathcal{V}$ be the variety generated by $\mathbf{A}_P$. The transversal $\mathcal{T}$ in (21) consists of one linearly ordered set Q_k of size k, for every $1 \le k \le p$. Each $\mathbf{A}_{Q_k}$ is rigid. Hence we have

$$\mathbf{F}_{\mathcal{V}}(n) = \mathbf{B}^{2^n-1}\mathbf{Ł}^{3^n-2^n} \prod_{k=1}^{p} \mathbf{A}_{Q_k}^{e(n,k)},$$

and

$$|\mathrm{F}_{\mathcal{V}}(n)| = 2^{2^n-1}3^{3^n-2^n} \prod_{k=1}^{p} (k+3)^{e(n,k)}.$$

At the other extreme, let P be an antichain of cardinality p. The transversal $\mathcal{T}$ consists of one antichain Q_k of size k for $1 \le k \le p$. Each antichain Q_k has $k!$ automorphisms. This gives

$$\mathbf{F}_{\mathcal{V}}(n) = \mathbf{B}^{2^n-1}\mathbf{Ł}^{3^n-2^n} \prod_{k=1}^{p} \mathbf{A}_{Q_k}^{\frac{e(n,k)}{k!}},$$

and

$$|\mathrm{F}_{\mathcal{V}}(n)| = 2^{2^n-1}3^{3^n-2^n} \prod_{k=1}^{p} (k+3)^{\frac{e(n,k)}{k!}}.$$

Example 4.8. The variety of modular ortholattices, $\mathcal{MO}$, is known to be congruence distributive, congruence permutable, and finitely semisimple. This variety and its subvarieties therefore lend themselves well to our analysis. An *ortholattice* is an algebra $\mathbf{A} = \langle A; \wedge, \vee, ', 0, 1\rangle$, that is a bounded lattice with a complement operation $'$ that satisfies the De Morgan laws and $x'' = x$. An ortholattice is *modular* if the underlying lattice is modular. The finite subdirectly irreducible algebras in $\mathcal{MO}$ are the 2-element Boolean algebra $\mathbf{B}$, and for each integer $k \geq 2$ the ortholattice $\mathbf{MO}_k$ with universe $\{0, 1, a_1, \ldots, a_k, b_1, \ldots, b_k\}$ that is a lattice of height 2 having $a_i' = b_i$ and $b_i' = a_i$ for all i. We note that $\mathbf{MO}_k$ is not hereditarily simple since for each i the 4-element subalgebra of $\mathbf{MO}_k$ with universe $\{0, 1, a_i, b_i\}$ is not simple.

Let $\mathcal{MO}_k$ denote the variety generated by $\mathbf{MO}_k$. The subdirectly irreducible algebras in $\mathcal{MO}_k$ consist of $\mathbf{B}$ and $\mathbf{MO}_\ell$ for $2 \leq \ell \leq k$. Each $\mathbf{MO}_k$ is k-generated and is not $(k-1)$-generated. Hence for every $n \geq 2$ we have $\mathbf{F}_{\mathcal{MO}}(n) = \mathbf{F}_{\mathcal{MO}_n}(n)$. However, it is known that the variety $\mathcal{MO}$ is not generated by its finite members.

The structure and cardinality of the finitely generated free algebras in $\mathcal{MO}$ and the $\mathcal{MO}_k$ are determined in the papers [12] and [13] using methods of natural duality. We consider these free algebras using Theorem 2.8. (A similar approach to determining the structure of free algebras in $\mathcal{MO}_k$ is given in Chapter 6 of [6].) Let $|X| = n$. A transversal with respect to isomorphism of the n-generated subdirectly irreducible algebras in $\mathcal{MO}$ is $\{\mathbf{B}, \mathbf{MO}_2, \ldots, \mathbf{MO}_n\}$. Since $\mathcal{MO}$ is arithmetical and finitely semisimple, Theorem 2.8 gives us

$$\mathbf{F}_{\mathcal{MO}}(n) = \mathbf{B}^{e_0} \times \mathbf{MO}_2^{e_2} \times \cdots \times \mathbf{MO}_n^{e_n},$$

where

$$e_0 = \frac{|\mathrm{val}(X, \mathbf{B})|}{|\mathrm{Aut}\,\mathbf{B}|} \text{ and } e_k = \frac{|\mathrm{val}(X, \mathbf{MO}_k)|}{|\mathrm{Aut}\,\mathbf{MO}_k|}, \text{ for } 2 \leq k \leq n.$$

As already observed, $e_0 = 2^n$, the number of functions from X to $\{0, 1\}$.

It is easily seen that $|\mathrm{Aut}\,\mathbf{MO}_k| = 2^k k!$ since every automorphism is determined by a permutation π of $\{1, \ldots, k\}$ and for each i, whether a_i is mapped to $a_{\pi(i)}$ or to $b_{\pi(i)}$.

To determine the cardinality of $\mathrm{val}(X, \mathbf{MO}_k)$ for $2 \leq k \leq n$, we first note that if R is the equivalence relation on $G = \{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ consisting of k blocks $\{a_i, b_i\}$, then $\mathbf{MO}_k$ is uniquely generated by G/R since condition (2) of Proposition 3.5 is satisfied. By Proposition 3.7 we have

$$|\mathrm{val}(X, \mathbf{MO}_k)| = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (2k + 2 - 2j)^n$$

$$= 2^n \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j + 1)^n = \frac{2^n}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k+1}{j} (k - j + 1)^{n+1}$$

$$= 2^n k! S(n + 1, k + 1),$$

where $S(n, k)$ denotes a Stirling number of the second kind.

Thus,

$$\mathbf{F}_{\mathcal{MO}}(n) = \mathbf{B}^{2^n} \times \prod_{k=2}^{n} \mathbf{MO}_k^{2^{n-k}S(n+1,k+1)}.$$

$$\begin{aligned}
|\mathbf{F}_{\mathcal{MO}}(1)| &= 2^1 &&= 2\\
|\mathbf{F}_{\mathcal{MO}}(2)| &= 2^4 6^1 &&= 96\\
|\mathbf{F}_{\mathcal{MO}}(3)| &= 2^8 6^{12} 8^1 &&= 4458050224128\\
|\mathbf{F}_{\mathcal{MO}}(4)| &= 2^{16} 6^{100} 8^{20} 10^1 &&\approx .4936335952 \times 10^{102}.
\end{aligned}$$

We use the next lemma to determine the structure of $\mathbf{F}_{\mathcal{V}}(n)$ for a wide collection of orthomodular lattice varieties.

Lemma 4.9. *Suppose $|X| = n$ and $\mathbf{B}$ is the Boolean algebra with k atoms. If $w(n, k)$ denotes the number of valuations from X to $\mathbf{B}$ having neither have $\perp$ nor $\top$ in their range, then*

$$w(n, k) = \sum_{j=1}^{k} s(k, j)(2^j - 2)^n = 2^n \sum_{j=1}^{k} s(k, j)(2^{j-1} - 1)^n.$$

Proof. We use hereditarily closed families and the matrix method of Theorem 3.11. As observed in Example 3.12, the finite Boolean algebras $\mathbf{B}_k$ for $k \geq 1$ form a hereditarily closed family with $C_{ij} = \operatorname{iso}(\mathbf{B}_i, \mathbf{B}_j) = S(j, i)$. We let Q_k denote all $f : X \to \mathbf{B}_k$ for which $f(X)$ contains neither $\perp$ nor $\top$. Thus $q(n, k) = (2^k - 2)^n$. So, as in Example 3.12, the inverse M of matrix C has $M_{kp} = s(k, p)$ for $s(k, p)$ a Stirling number of the first kind. An application of Theorem 3.11 completes the proof. ∎

For positive integers r and $k_1 \geq \cdots \geq k_r \geq 2$ let $\mathbf{A}(k_1, \ldots, k_r)$ denote the orthomodular lattice that is the horizontal sum of the r Boolean algebras $\mathbf{B}_1, \ldots, \mathbf{B}_r$. The top and bottom elements of $\mathbf{A}(k_1, \ldots, k_r)$ are denoted by $\top$ and $\perp$ and this lattice has $k_1 + \cdots + k_r$ atoms and $k_1 + \cdots + k_r$ coatoms. If $r = 1$, then $\mathbf{A}(k_1)$ is the Boolean algebra $\mathbf{B}_{k_1}$ and if $r > 1$ and all $k_i = 2$, then we have the simple modular ortholattice $\mathbf{MO}_k$. If $r \geq 2$, then every $\mathbf{A}(k_1, \ldots, k_r)$ is a simple.

Lemma 4.10. *For $|X| = n$ and positive integers $k_1 \geq \cdots \geq k_r \geq 2$*

$$|\operatorname{val}(X, \mathbf{A}(k_1, \ldots, k_r))| = \sum_{\substack{n_0 + \cdots + n_r = n\\ n_i \geq 1 \text{ for } i > 0}} \left(\binom{n}{n_0\ n_1 \ldots n_r} 2^{n_0} \prod_{i=1}^{r} w(n_i, k_i) \right)$$

where the $w(n, j)$ are as defined in Lemma 4.9.

Proof. Let $f : X \to \mathbf{A}(k_1, \ldots, k_r)$ be an arbitrary function. Let n_0 be the number of elements of X sent into the set $\{\perp, \top\}$ and let n_j be the number of elements of X sent into $B_j \setminus \{\bot, \top\}$. Then f is a valuation if and only if for each $1 \le j \le r$, the image of the set of n_j elements of X sent to $B_j \setminus \{\bot, \top\}$ generates $\mathbf{B}_j$. Hence, for a given choice of positive integers $n_1, \ldots, n_r$ whose sum is at most n, and for $n_0 = n - (n_1 + \cdots + n_r)$, there are

$$\binom{n}{n_0\ n_1 \ldots n_r} 2^{n_0} \times w(n_1, k_1) \times \cdots \times w(n_r, k_r)$$

functions f that are valuations. Note that if $1 \le n_j$ for $1 \le j \le r$ and no set of n_j elements generates $\mathbf{B}_j$, then $w(n_j, k_j) = 0$. So summing over all choices of nonnegative n_0 and positive $n_1, \ldots, n_r$ whose sum is n gives the desired formula for the number of valuations. ∎

Lemma 4.11. *For positive integers $k_1 \ge \cdots \ge k_r \ge 2$*

$$|\mathrm{Aut}\ \mathbf{A}(k_1, \ldots, k_r)| = k_1! \times \cdots \times k_r! \times m_1! \times \cdots \times m_t!$$

where there are t distinct values among the k_j and $m_1, \ldots, m_t$ are the multiplicities of these t values.

Let $\mathcal{V}$ denote the variety of orthomodular lattices generated by all algebras $\mathbf{A}(k_1, \ldots, k_r)$ where r and the k_i are integers, $r \ge 2$ and $k_1 \ge \cdots \ge k_r \ge 2$. The subvariety generated by all the algebras $\mathbf{A}(2, 2, \ldots, 2)$ is the variety generated by the modular ortholattices $\mathbf{MO}_k$ that we have just considered. The structure and cardinality of the free algebras for those subvarieties of $\mathcal{V}$ generated by $\mathbf{A}(3, 2, \ldots, 2)$ is the subject of [14] and the varieties generated by $\mathbf{A}(3, \ldots, 3)$ is the subject of [15]. The arguments presented there, as in [12] and [13], make use of duality theory. The formulas of Lemmas 4.10 and 4.11 may be used to determine the structure of $\mathbf{F}_{\mathcal{W}}(n)$ for any subvariety $\mathcal{W}$ of $\mathcal{V}$. We now compute the actual cardinalities of these free algebras for $n \le 3$. This work is summarized in the following table. The entries in a cell in a row indexed by a value of n and a column headed by an algebra is the number of valuations into the algebra from a set X of size n. The rightmost column gives the cardinalities of the $\mathbf{F}_{\mathcal{V}}(n)$. Note that the subdirectly irreducible algebras in $\mathcal{V}$ generated by 3 or fewer elements are $\mathbf{B}_1, \mathbf{A}(2,2), \mathbf{A}(2,2,2), \mathbf{A}(3,2)$ and $\mathbf{A}(4,2)$.

	$\mathbf{B}_1$	$\mathbf{A}(2,2)$	$\mathbf{A}(2,2,2)$	$\mathbf{A}(3,2)$	$\mathbf{A}(4,2)$	$\lvert\mathbf{F}_{\mathcal{V}}(n)\rvert$
$\lvert\mathbf{A}\rvert$	2	6	8	10	18	
$\lvert\mathrm{Aut}\ \mathbf{A}\rvert$	1	8	48	12	48	
$n = 1$	2					$2^2 = 4$
$n = 2$	4	8				$2^4 6^1 = 96$
$n = 3$	8	96	48	144	144	$2^8 6^{12} 8^1 10^{12} 18^3$

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DEPARTMENT OF MATHEMATICS
STATISTICS, AND COMPUTER SCIENCE
UNIVERSITY OF ILLINOIS AT CHICAGO
851 S. MORGAN STREET
CHICAGO, IL, USA
E-mail: jberman@uic.edu

Received September 27, 2010.