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ON MONOIDS OF INJECTIVE PARTIAL SELFMAPS ALMOST EVERYWHERE THE IDENTITY

Abstract. In this paper we study the semigroup $\mathscr{I}_\lambda^\infty$ of injective partial selfmaps almost everywhere the identity of a set of infinite cardinality λ. We describe the Green relations on $\mathscr{I}_\lambda^\infty$, all (two-sided) ideals and all congruences of the semigroup $\mathscr{I}_\lambda^\infty$. We prove that every Hausdorff hereditary Baire topology τ on $\mathscr{I}_\omega^\infty$ such that $(\mathscr{I}_\omega^\infty, \tau)$ is a semitopological semigroup is discrete and describe the closure of the discrete semigroup $\mathscr{I}_\lambda^\infty$ in a topological semigroup. Also we show that for an infinite cardinal λ the discrete semigroup $\mathscr{I}_\lambda^\infty$ does not embed into a compact topological semigroup and construct two non-discrete Hausdorff topologies turning $\mathscr{I}_\lambda^\infty$ into a topological inverse semigroup.

1. Introduction and preliminaries

In this paper all spaces are assumed to be Hausdorff. Furthermore we shall follow the terminology of [3, 5, 7, 9, 23]. By ω we shall denote the first infinite cardinal and by $|A|$ the cardinality of the set A. If Y is a subspace of a topological space X and $A \subseteq Y$, then by $\operatorname{cl}_Y(A)$ and $\operatorname{Int}_Y(A)$ we shall denote the topological closure and the interior of A in Y, respectively.

For a semigroup S we denote the semigroup S with the adjoined unit by S^1 (see [5]).

An algebraic semigroup S is called *inverse* if for any element $x \in S$ there exists a unique element $x^{-1} \in S$ (called the *inverse* of x) such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. If S is an inverse semigroup, then the function $\operatorname{inv}\colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called *inversion*.

If S is an inverse semigroup, then by $E(S)$ we shall denote the *band* (i.e., the subsemigroup of idempotents) of S. If the band $E(S)$ is a non-empty

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subset of S, then the semigroup operation on S determines a partial order $\leqslant$ on $E(S)$: $e \leqslant f$ if and only if $ef = fe = e$. This order is called *natural.* A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or a *chain* if the semilattice operation induces a linear natural order on E. A *maximal chain* of a semilattice E is a chain which is properly contained in no other chain of E. The Axiom of Choice implies the existence of maximal chains in any partially ordered set. According to [21, Definition II.5.12] a chain L is called an ω-chain if L is isomorphic to $\{0, -1, -2, -3, \ldots\}$ with the usual order $\leqslant$. Let E be a semilattice and $e \in E$. We denote ${\downarrow}e = \{f \in E \mid f \leqslant e\}$ and ${\uparrow}e = \{f \in E \mid e \leqslant f\}$. By $(\mathscr{P}_{<\omega}(\lambda), \cup)$ we shall denote the free semilattice with identity over a cardinal $\lambda \geqslant \omega$, i.e., $\mathscr{P}_{<\omega}(\lambda)$ is the set of all finite subsets of λ with the binary operation $a \cdot b = a \cup b$, for $a, b \in \mathscr{P}_{<\omega}(\lambda)$.

If S is a semigroup, then we shall denote by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ the Green relations on S (see [5]):

$$\begin{aligned}
&a\mathscr{R}b \text{ if and only if } aS^1 = bS^1;\\
&a\mathscr{L}b \text{ if and only if } S^1a = S^1b;\\
&a\mathscr{J}b \text{ if and only if } S^1aS^1 = S^1bS^1;\\
&\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L};\\
&\mathscr{H} = \mathscr{L} \cap \mathscr{R}.
\end{aligned}$$

The relation $\mathscr{J}$ induced a quasi-order $\leqslant_{\mathscr{J}}$ on S as follows:

$$a \leqslant_{\mathscr{J}} b \qquad \text{if and only if} \qquad S^1aS^1 \subseteq S^1bS^1,$$

for $a, b \in S$. This implies that the inclusion order among two-sided ideals of S induces a partial order among the $\mathscr{J}$-equivalence classes:

$$J_a \preccurlyeq J_b \qquad \text{if and only if} \qquad S^1aS^1 \subseteq S^1bS^1,$$

for $a, b \in S$, where by J_a we denote the $\mathscr{J}$-class in S which contains an element $a \in S$ (see [17, Section 2.1]). Then we may thus regard $S/\mathscr{J}$ with the relation $\leqslant$ as a partially ordered set.

A semigroup S is called *simple* if S does not contain proper two-sided ideals.

A *semitopological* (resp. *topological*) *semigroup* is a topological space together with a separately (resp. jointly) continuous semigroup operation. An inverse topological semigroup with the continuous inversion is called a *topological inverse semigroup.*

In the remainder of the paper λ denotes an infinite cardinal.

Let $\mathscr{I}_\lambda$ denote the set of all partial one-to-one transformations of an infinite cardinal λ together with the following semigroup operation: $x(\alpha\beta) = (x\alpha)\beta$ if $x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom}\alpha \mid y\alpha \in \operatorname{dom}\beta\}$, for $\alpha, \beta \in \mathscr{I}_\lambda$. The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the cardinal λ (see [5]). The symmetric inverse semigroup was introduced by Wagner [25] and it plays a major role in the theory of semigroups.

A partial map $\alpha \in \mathscr{I}_\lambda$ is called *almost everywhere the identity* if the set $\lambda \setminus \operatorname{dom} \alpha$ is finite and $(x)\alpha \neq x$ only for finitely many $x \in \lambda$. We denote

$$\mathscr{I}_\lambda^\infty = \{\alpha \in \mathscr{I}_\lambda \mid \alpha \text{ is almost everywhere the identity}\}.$$

Obviously, $\mathscr{I}_\lambda^\infty$ is an inverse subsemigroup of the semigroup $\mathscr{I}_\omega$. The semigroup $\mathscr{I}_\lambda^\infty$ is called *the semigroup of injective partial selfmaps almost everywhere the identity* of λ. We shall denote every element α of the semigroup $\mathscr{I}_\lambda^\infty$ by

$$\left(\begin{array}{ccc|c} x_1 & \cdots & x_n & \multirow{2}{*}{A} \\ y_1 & \cdots & y_n & \end{array}\right)$$

and this means that the following conditions hold:

(i) A is the maximal subset of λ with the finite complement such that $\alpha|_A \colon A \to A$ is an identity map;

(ii) $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are finite (not necessary non-empty) subsets of $\lambda \setminus A$; and

(iii) α maps x_i into y_i for all $i = 1, \ldots, n$.

We denote the identity of the semigroup $\mathscr{I}_\lambda^\infty$ by $\mathbb{I}$.

Many semigroup theorists have considered topological semigroups of (continuous) transformations of topological spaces. Beĭda [2], Orlov [19, 20], and Subbiah [24] have considered semigroup and inverse semigroup topologies on semigroups of partial homeomorphisms of some classes of topological spaces.

Gutik and Pavlyk [12] considered the special case of the semigroup $\mathscr{I}_\lambda^n$: an infinite topological semigroup of $\lambda \times \lambda$-matrix units B_λ. They showed that an infinite topological semigroup of $\lambda \times \lambda$-matrix units B_λ does not embed into a compact topological semigroup and that B_λ is algebraically h-closed in the class of topological inverse semigroups. They also described the Bohr compactification of B_λ, minimal semigroup and minimal semigroup inverse topologies on B_λ.

Gutik, Lawson and Repovš [11] introduced the notion of a semigroup with a tight ideal series and investigated their closures in semitopological semigroups, in particular, in inverse semigroups with continuous inversion. As a corollary they showed that the symmetric inverse semigroup of finite transformations $\mathscr{I}_\lambda^n$ of infinite cardinal λ is algebraically closed in the class of (semi) topological inverse semigroups with continuous inversion. They also derived related results about the nonexistence of (partial) compactifications of semigroups with a tight ideal series.

Gutik and Reiter [14] showed that the topological inverse semigroup $\mathscr{I}_{\lambda}^{n}$ is algebraically h-closed in the class of topological inverse semigroups. They also proved that a topological semigroup S with countably compact square $S \times S$ does not contain the semigroup $\mathscr{I}_{\lambda}^{n}$ for infinite cardinals λ and showed that the Bohr compactification of an infinite topological semigroup $\mathscr{I}_{\lambda}^{n}$ is the trivial semigroup.

In [15] Gutik and Reiter showed that the symmetric inverse semigroup of finite transformations $\mathscr{I}_{\lambda}^{n}$ of infinite cardinal λ is algebraically closed in the class of semitopological inverse semigroups with continuous inversion. Also there they described all congruences on the semigroup $\mathscr{I}_{\lambda}^{n}$ and all compact and countably compact topologies τ on $\mathscr{I}_{\lambda}^{n}$ such that $(\mathscr{I}_{\lambda}^{n}, \tau)$ is a semitopological semigroup.

Gutik, Pavlyk and Reiter [13] showed that a topological semigroup of finite partial bijections $\mathscr{I}_{\lambda}^{n}$ of an infinite cardinal with a compact subsemigroup of idempotents is absolutely H-closed. They proved that no Hausdorff countably compact topological semigroup and no Tychonoff topological semigroup with pseudocompact square contain $\mathscr{I}_{\lambda}^{n}$ as a subsemigroup. They proved that every continuous homomorphism from a topological semigroup $\mathscr{I}_{\lambda}^{n}$ into a Hausdorff countably compact topological semigroup or Tychonoff topological semigroup with pseudocompact square is annihilating. They also gave sufficient conditions for a topological semigroup $\mathscr{I}_{\lambda}^{1}$ to be non-H-closed and showed that the topological inverse semigroup $\mathscr{I}_{\lambda}^{1}$ is absolutely H-closed if and only if the band $E(\mathscr{I}_{\lambda}^{1})$ is compact [13].

In [16] Gutik and Repovš studied the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ of partial cofinite monotone bijective transformations of the set of positive integers $\mathbb{N}$. They showed that the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ has algebraic properties similar to the bicyclic semigroup: it is bisimple and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. They proved that every locally compact topology τ on $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ such that $(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}), \tau)$ is a topological inverse semigroup, is discrete and described the closure of $(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}), \tau)$ in a topological semigroup.

In [4] Gutik and Chuchman studied the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ of partial co-finite almost monotone bijective transformations of the set of positive integers $\mathbb{N}$. They showed that the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ has algebraic properties similar to the bicyclic semigroup: it is bisimple and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. Also they proved that every Baire topology τ on $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ such that $(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}), \tau)$ is a semitopological semigroup is discrete, described the closure of $(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}), \tau)$ in a topological semigroup and constructed non-discrete Hausdorff semigroup topologies on the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$.

In this paper we study the semigroup $\mathscr{I}_{\lambda}^{\infty}$ of injective partial selfmaps almost everywhere the identity of a set of infinite cardinality λ. We describe the Green relations on $\mathscr{I}_{\lambda}^{\infty}$, all (two-sided) ideals and all congruences of the semigroup $\mathscr{I}_{\lambda}^{\infty}$. We prove that every Hausdorff hereditary Baire topology τ on $\mathscr{I}_{\omega}^{\infty}$ such that $(\mathscr{I}_{\omega}^{\infty},\tau)$ is a semitopological semigroup is discrete and describe the closure of the discrete semigroup $\mathscr{I}_{\lambda}^{\infty}$ in a topological semigroup. Also we show that for an infinite cardinal λ the discrete semigroup $\mathscr{I}_{\lambda}^{\infty}$ does not embed into a compact topological semigroup and construct two non-discrete Hausdorff topologies turning $\mathscr{I}_{\lambda}^{\infty}$ into a topological inverse semigroup.

2. Algebraic properties of the semigroup $\mathscr{I}_{\lambda}^{\infty}$

The definition of the semigroup $\mathscr{I}_{\lambda}^{\infty}$ implies the following proposition:

Proposition 2.1. *A partial map $\alpha\in\mathscr{I}_{\lambda}$ is an element of the semigroup $\mathscr{I}_{\lambda}^{\infty}$ if and only if the following assertions hold:*

(*i*) $|\lambda\setminus\operatorname{dom}\alpha|=|\lambda\setminus\operatorname{ran}\alpha|$; *and*
(*ii*) *there exists a subset $A\subseteq\operatorname{dom}\alpha\cap\operatorname{ran}\alpha$ such that $\lambda\setminus A$ is a finite subset of λ and the restriction $\alpha|_A\colon A\to A$ is the identity map.*

Proposition 2.2.

(*i*) *An element α of the semigroup $\mathscr{I}_{\lambda}^{\infty}$ is an idempotent if and only if $(x)\alpha=x$ for every $x\in\operatorname{dom}\alpha$.*
(*ii*) *If $\varepsilon,\iota\in E(\mathscr{I}_{\lambda}^{\infty})$, then $\varepsilon\leqslant\iota$ if and only if* $\operatorname{dom}\varepsilon\subseteq\operatorname{dom}\iota$.
(*iii*) *The semilattice $E(\mathscr{I}_{\lambda}^{\infty})$ is isomorphic to $(\mathscr{P}_{<\omega}(\lambda),\cup)$ under the mapping $(\varepsilon)h=\lambda\setminus\operatorname{dom}\varepsilon$.*
(*iv*) *Every maximal chain in $E(\mathscr{I}_{\lambda}^{\infty})$ is an ω-chain.*
(*v*) *$\alpha\mathscr{R}\beta$ in $\mathscr{I}_{\lambda}^{\infty}$ if and only if* $\operatorname{dom}\alpha=\operatorname{dom}\beta$.
(*vi*) *$\alpha\mathscr{L}\beta$ in $\mathscr{I}_{\lambda}^{\infty}$ if and only if* $\operatorname{ran}\alpha=\operatorname{ran}\beta$.
(*vii*) *$\alpha\mathscr{H}\beta$ in $\mathscr{I}_{\lambda}^{\infty}$ if and only if* $\operatorname{dom}\alpha=\operatorname{dom}\beta$ *and* $\operatorname{ran}\alpha=\operatorname{ran}\beta$.
(*viii*) *$\alpha\mathscr{D}\beta$ in $\mathscr{I}_{\lambda}^{\infty}$ if and only if* $|\lambda\setminus\operatorname{dom}\alpha|=|\lambda\setminus\operatorname{dom}\beta|$.
(*ix*) *If n is a non-negative integer, then for every $\alpha,\beta\in\mathscr{I}_{\lambda}^{\infty}$ such that $|\lambda\setminus\operatorname{dom}\alpha|=|\lambda\setminus\operatorname{dom}\beta|=n$ there exist $\gamma,\delta\in\mathscr{I}_{\lambda}^{\infty}$ such that $\alpha=\gamma\cdot\beta\cdot\delta$ and $|\lambda\setminus\operatorname{dom}\gamma|=|\lambda\setminus\operatorname{dom}\delta|=n$.*
(*x*) *For every non-negative integer n the set $I_n=\{\alpha\in\mathscr{I}_{\lambda}^{\infty}\mid|\lambda\setminus\operatorname{dom}\alpha|\geqslant n\}$ is an ideal in $\mathscr{I}_{\lambda}^{\infty}$. Moreover, for every ideal I in $\mathscr{I}_{\lambda}^{\infty}$ there exists an integer $n\geqslant 0$ such that I is equal to I_n.*
(*xi*) *$\mathscr{D}=\mathscr{J}$ in $\mathscr{I}_{\lambda}^{\infty}$.*
(*xii*) *If λ_1 and λ_2 are infinite cardinals such that $\lambda_1\leqslant\lambda_2$ then $\mathscr{I}_{\lambda_1}^{\infty}$ is a subsemigroup of the semigroup $\mathscr{I}_{\lambda_2}^{\infty}$.*
(*xiii*) *$(\mathscr{I}_{\lambda}^{\infty}/\mathscr{J},\preccurlyeq)$ is an ω-chain for any infinite cardinal λ.*

Proof. Statements $(i)-(iv)$ are trivial and they follow from the definition of the semigroup $\mathscr{I}_\lambda^\infty$.

(v) Let be $\alpha,\beta\in\mathscr{I}_\lambda^\infty$ such that $\alpha\mathscr{R}\beta$. Since $\alpha\mathscr{I}_\lambda^\infty=\beta\mathscr{I}_\lambda^\infty$ and $\mathscr{I}_\lambda^\infty$ is an inverse semigroup, Theorem 1.17 [5] implies that $\alpha\mathscr{I}_\lambda^\infty=\alpha\alpha^{-1}\mathscr{I}_\lambda^\infty$, $\beta\mathscr{I}_\lambda^\infty=\beta\beta^{-1}\mathscr{I}_\lambda^\infty$ and hence $\alpha\alpha^{-1}=\beta\beta^{-1}$. Therefore we get that $\operatorname{dom}\alpha=\operatorname{dom}\beta$.

Conversely, let be $\alpha,\beta\in\mathscr{I}_\lambda^\infty$ such that $\operatorname{dom}\alpha=\operatorname{dom}\beta$. Then $\alpha\alpha^{-1}=\beta\beta^{-1}$. Since $\mathscr{I}_\lambda^\infty$ is an inverse semigroup, Theorem 1.17 [5] implies that $\alpha\mathscr{I}_\lambda^\infty=\alpha\alpha^{-1}\mathscr{I}_\lambda^\infty=\beta\mathscr{I}_\lambda^\infty$ and hence $\alpha\mathscr{I}_\lambda^\infty=\beta\mathscr{I}_\lambda^\infty$.

The proof of statement (vi) is similar to (v).

Statement (vii) follows from (v) and (vi).

$(viii)$ Let $\alpha,\beta\in\mathscr{I}_\lambda^\infty$ be such that $\alpha\mathscr{D}\beta$. Then there exists $\gamma\in\mathscr{I}_\lambda^\infty$ such that $\alpha\mathscr{L}\gamma$ and $\gamma\mathscr{R}\beta$. Therefore by statements (v) and (vi) we have that $\operatorname{ran}\alpha=\operatorname{ran}\gamma$ and $\operatorname{dom}\gamma=\operatorname{dom}\beta$. Then Proposition 2.1 implies that $|\lambda\setminus\operatorname{ran}\gamma|=|\lambda\setminus\operatorname{dom}\gamma|$ and $|\lambda\setminus\operatorname{ran}\beta|=|\lambda\setminus\operatorname{dom}\beta|$, and hence we get that $|\lambda\setminus\operatorname{dom}\alpha|=|\lambda\setminus\operatorname{dom}\beta|$.

Let α and β are elements of the semigroup $\mathscr{I}_\lambda^\infty$ such that $|\lambda\setminus\operatorname{dom}\alpha|=|\lambda\setminus\operatorname{dom}\beta|$. Then Proposition 2.1 implies that $|\lambda\setminus\operatorname{ran}\alpha|=|\lambda\setminus\operatorname{dom}\alpha|$ and $|\lambda\setminus\operatorname{ran}\beta|=|\lambda\setminus\operatorname{dom}\beta|$. Let A_α and A_β be maximal subsets of λ such that the sets $\lambda\setminus A_\alpha$ and $\lambda\setminus A_\beta$ are finite and the restrictions $\alpha|_{A_\alpha}\colon A_\alpha\to A_\alpha$ and $\beta|_{A_\beta}\colon A_\beta\to A_\beta$ are identity maps. We put $A=A_\alpha\cap A_\beta$. Since $\lambda\setminus A_\alpha$ and $\lambda\setminus A_\beta$ are finite subsets of λ we conclude that $\lambda\setminus A$ is a finite subset of λ too. Since $|\lambda\setminus\operatorname{dom}\alpha|=|\lambda\setminus\operatorname{dom}\beta|<\omega$ Proposition 2.1 implies that

$$|\operatorname{dom}\alpha\setminus A|=|\operatorname{ran}\alpha\setminus A|=|\operatorname{dom}\beta\setminus A|=|\operatorname{ran}\beta\setminus A|=n$$

for some non-negative integer n. If $n=0$, then $\alpha=\beta$. Suppose that $n\geqslant 1$. Let $\{x_1,\ldots,x_n\}=\operatorname{ran}\alpha\setminus A$ and $\{y_1,\ldots,y_n\}=\operatorname{dom}\beta\setminus A$. We define

$$\gamma=\left(\begin{array}{ccc|c} y_1 & \cdots & y_n & \multirow{2}{*}{A}\\ x_1 & \cdots & x_n & \end{array}\right).$$

Then by statements (v) and (vi) we have that $\alpha\mathscr{L}\gamma$ and $\gamma\mathscr{R}\beta$ in $\mathscr{I}_\lambda^\infty$. Hence $\alpha\mathscr{D}\beta$ in $\mathscr{I}_\lambda^\infty$.

(ix) Let α and β be arbitrary elements of the semigroup $\mathscr{I}_\lambda^\infty$ such that $|\lambda\setminus\operatorname{dom}\alpha|=|\lambda\setminus\operatorname{dom}\beta|=n$ for some non-negative integer n. Let A_α and A_β be maximal subsets of λ such that the sets $\lambda\setminus A_\alpha$ and $\lambda\setminus A_\beta$ are finite and the restrictions $\alpha|_{A_\alpha}\colon A_\alpha\to A_\alpha$ and $\beta|_{A_\beta}\colon A_\beta\to A_\beta$ are identity maps. We put $A=A_\alpha\cap A_\beta$. Since $\lambda\setminus A_\alpha$ and $\lambda\setminus A_\beta$ are finite subsets of λ we conclude that $\lambda\setminus A$ is a finite subset of λ too. Since $|\lambda\setminus\operatorname{dom}\alpha|=|\lambda\setminus\operatorname{dom}\beta|$ the definition of the semigroup $\mathscr{I}_\lambda^\infty$ implies that $|\operatorname{dom}\alpha\setminus A|=|\operatorname{dom}\beta\setminus A|<\omega$. If $\operatorname{dom}\alpha\setminus A=\operatorname{dom}\beta\setminus A=\varnothing$ then $\alpha=\beta$ and hence $\alpha=\gamma\cdot\beta\cdot\delta$ for $\gamma=\delta=\mathbb{I}$. Otherwise we put $\{x_1,\ldots,x_k\}=\operatorname{dom}\alpha\setminus A$, $\{y_1,\ldots,y_k\}=\operatorname{dom}\beta\setminus A$,

$b_1=(y_1)\beta,\dots,b_k=(y_k)\beta$ and $a_1=(x_1)\alpha,\dots,a_k=(x_k)\alpha$, for some positive integer k. We define

$$\gamma=\left(\begin{array}{ccc|c} x_1 & \cdots & x_k & \multirow{2}{*}{A}\\ y_1 & \cdots & y_k & \end{array}\right) \qquad\text{and}\qquad \delta=\left(\begin{array}{ccc|c} b_1 & \cdots & b_k & \multirow{2}{*}{A}\\ a_1 & \cdots & a_k & \end{array}\right).$$

Then $\gamma,\delta\in\mathscr{I}_\lambda^\infty$, $|\lambda\setminus\operatorname{dom}\gamma|=|\lambda\setminus\operatorname{dom}\delta|=n$ and $\alpha=\gamma\cdot\beta\cdot\delta$.

(x) Let α and β be arbitrary elements of the semigroup $\mathscr{I}_\lambda^\infty$. Since α and β are injective partial selfmaps almost everywhere the identity of the cardinal λ we conclude that

$$|\lambda\setminus\operatorname{dom}(\alpha\cdot\beta)|\geqslant\max\{|\lambda\setminus\operatorname{dom}\alpha|,|\lambda\setminus\operatorname{dom}\beta|\}.$$

This implies the first assertion of statement (x).

Let I be an ideal in $\mathscr{I}_\lambda^\infty$. Then the definition of the semigroup $\mathscr{I}_\lambda^\infty$ implies that there exists $\alpha\in I$ such that

$$|\lambda\setminus\operatorname{dom}\alpha|=\min\{|\lambda\setminus\operatorname{dom}\gamma|\mid\gamma\in I\}.$$

Then $|\lambda\setminus\operatorname{dom}\alpha|=n$ for some integer $n\geqslant 0$. Hence $I\subseteq I_n$ and by statement (ix) we get that $I_n\subseteq I$. This implies the second assertion of the statement.

Statement (xi) follows from statement (ix).

(xii) Let $\alpha=\left(\begin{array}{ccc|c} x_1 & \cdots & x_n & \multirow{2}{*}{A}\\ y_1 & \cdots & y_n & \end{array}\right)$ be an arbitrary element of the semigroup $\mathscr{I}_{\lambda_1}^\infty$ and $B=\lambda_2\setminus\lambda_1$. We put

$$\widetilde{\alpha}=\left(\begin{array}{ccc|c} x_1 & \cdots & x_n & \multirow{2}{*}{$A\cup B$}\\ y_1 & \cdots & y_n & \end{array}\right).$$

Obviously that $\widetilde{\alpha}\in\mathscr{I}_{\lambda_2}^\infty$. Simple verifications show that the map $h\colon\mathscr{I}_{\lambda_1}^\infty\to\mathscr{I}_{\lambda_2}^\infty$ defined by the formula $(\alpha)h=\widetilde{\alpha}$ is an isomorphic embedding of the semigroup $\mathscr{I}_{\lambda_1}^\infty$ into $\mathscr{I}_{\lambda_2}^\infty$.

Statement $(xiii)$ follows from items $(viii)$ and (xi). ■

Later we shall need the following proposition:

Proposition 2.3. *Let λ be an arbitrary infinite cardinal. Then for every finite subset $\{x_1,\dots,x_n\}$ of λ the semigroups $\mathscr{I}_\lambda^\infty$ and $\mathscr{I}_\eta^\infty$ are isomorphic for $\eta=\lambda\setminus\{x_1,\dots,x_n\}$.*

Proof. Since λ is infinite we conclude that there exists a bijective map $f\colon\lambda\to\eta$. Then the bijection f generates a map $h\colon\mathscr{I}_\lambda^\infty\to\mathscr{I}_\eta^\infty$ such that the following condition holds:

$$(\alpha_\lambda)h=\alpha_\eta\quad\text{if and only if}\quad((x)f)\alpha_\eta=((x)\alpha_\lambda)f\ \text{ for every }\ x\in\lambda,$$

where $\alpha_\lambda\in\mathscr{I}_\lambda^\infty$ and $\alpha_\eta\in\mathscr{I}_\eta^\infty$.

Now we shall show that so defined map h is injective. Suppose to the contrary that there exist distinct elements $\alpha_\lambda, \beta_\lambda \in \mathscr{I}_\lambda^\infty$ such that $(\alpha_\lambda)h = (\beta_\lambda)h$. We denote $\alpha_\eta = (\alpha_\lambda)h$ and $\beta_\eta = (\beta_\lambda)h$. Then $\operatorname{dom}\alpha_\eta = \operatorname{dom}\beta_\eta$ and $\operatorname{ran}\alpha_\eta = \operatorname{ran}\beta_\eta$ and since $f\colon \lambda \to \eta$ is a bijective map we conclude that $\operatorname{dom}\alpha_\lambda = \operatorname{dom}\beta_\lambda$ and $\operatorname{ran}\alpha_\lambda = \operatorname{ran}\beta_\lambda$. Therefore there exists $x \in \operatorname{ran}\alpha_\lambda$ such that $(x)\alpha_\lambda \neq (x)\beta_\lambda$. Since $(\alpha_\lambda)h = (\beta_\lambda)h$ we have that $((x)f)\alpha_\eta = ((x)f)\beta_\eta$. But $((x)f)\alpha_\eta = ((x)\alpha_\lambda)f$ and $((x)f)\beta_\eta = ((x)\beta_\lambda)f$ and since the map $f\colon \lambda \to \eta$ is bijective we conclude that $(x)\alpha_\lambda = (x)\beta_\lambda$, a contradiction. The obtained contradiction implies that the map $h\colon \mathscr{I}_\lambda^\infty \to \mathscr{I}_\eta^\infty$ is injective.

Let

$$\alpha_\eta = \left(\begin{array}{ccc|c} x_1 & \cdots & x_n & \multirow{2}{*}{A} \\ y_1 & \cdots & y_n & \end{array}\right)$$

be an arbitrary element of the semigroup $\mathscr{I}_\eta^\infty$, where $A \subseteq \eta$ and $x_1, \ldots, x_n, y_1, \ldots, y_n \in \eta$. Since the map $f\colon \lambda \to \eta$ is bijective we conclude that

$$\alpha_\lambda = \left(\begin{array}{ccc|c} (x_1)f^{-1} & \cdots & (x_n)f^{-1} & \multirow{2}{*}{$(A)f^{-1}$} \\ (y_1)f^{-1} & \cdots & (y_n)f^{-1} & \end{array}\right)$$

is a partial bijective map from λ into λ such that the sets $\lambda \setminus \operatorname{dom}\alpha_\lambda$ and $\lambda \setminus \operatorname{ran}\alpha_\lambda$ are finite. Therefore $\alpha_\lambda \in \mathscr{I}_\lambda^\infty$ and hence the map $h\colon \mathscr{I}_\lambda^\infty \to \mathscr{I}_\eta^\infty$ is bijective.

Now we prove that the map $h\colon \mathscr{I}_\lambda^\infty \to \mathscr{I}_\eta^\infty$ is a homomorphism. We fix arbitrary elements $\alpha_\lambda, \beta_\lambda \in \mathscr{I}_\lambda^\infty$ and denote $\alpha_\eta = (\alpha_\lambda)h$ and $\beta_\eta = (\beta_\lambda)h$. Then for every $x \in \operatorname{ran}\alpha_\lambda$ we have that

$$\begin{aligned}\big((x)f\big)(\alpha_\eta \cdot \beta_\eta) &= \Big(\big((x)f\big)\alpha_\eta\Big)\beta_\eta = \Big(\big((x)\alpha_\lambda\big)f\Big)\beta_\eta = \Big(\big((x)\alpha_\lambda\big)\beta_\lambda\Big)f \\ &= \big((x)(\alpha_\lambda \cdot \beta_\lambda)\big)f,\end{aligned}$$

and hence $(\alpha_\lambda \cdot \beta_\lambda)h = \alpha_\eta \cdot \beta_\eta = (\alpha_\lambda)h \cdot (\beta_\lambda)h$.

Therefore h is an isomorphism from the semigroup $\mathscr{I}_\lambda^\infty$ onto $\mathscr{I}_\eta^\infty$. ■

Proposition 2.4. *Let λ be an arbitrary infinite cardinal. Then for every idempotent ε of the semigroup $\mathscr{I}_\lambda^\infty$ the semigroups $\mathscr{I}_\lambda^\infty(\varepsilon) = \varepsilon \cdot \mathscr{I}_\lambda^\infty \cdot \varepsilon$ and $\mathscr{I}_\lambda^\infty$ are isomorphic.*

Proof. Since

$$\begin{aligned}\mathscr{I}_\lambda^\infty(\varepsilon) &= \varepsilon \cdot \mathscr{I}_\lambda^\infty \cdot \varepsilon = \varepsilon \cdot \mathscr{I}_\lambda^\infty \cap \mathscr{I}_\lambda^\infty \cdot \varepsilon = \\ &= \{\alpha \in \mathscr{I}_\lambda^\infty \mid \operatorname{dom}\alpha \subseteq \operatorname{dom}\varepsilon\} \cap \{\alpha \in \mathscr{I}_\lambda^\infty \mid \operatorname{ran}\alpha \subseteq \operatorname{ran}\varepsilon\} = \\ &= \{\alpha \in \mathscr{I}_\lambda^\infty \mid \operatorname{dom}\alpha \subseteq \operatorname{dom}\varepsilon \text{ and } \operatorname{ran}\alpha \subseteq \operatorname{ran}\varepsilon\},\end{aligned}$$

Proposition 2.3 implies the assertion of the proposition. ■

Proposition 2.5. *For every $\alpha, \beta \in \mathscr{I}_\lambda^\infty$, both sets $\{\chi \in \mathscr{I}_\lambda^\infty \mid \alpha \cdot \chi = \beta\}$ and $\{\chi \in \mathscr{I}_\lambda^\infty \mid \chi \cdot \alpha = \beta\}$ are finite. Consequently, every right translation* *and every left translation by an element of the semigroup $\mathscr{I}_\lambda^\infty$ is a finite-to-one map.*

Proof. We denote $S = \{\chi \in \mathscr{I}_\lambda^\infty \mid \alpha \cdot \chi = \beta\}$ and $T = \{\chi \in \mathscr{I}_\lambda^\infty \mid \alpha^{-1} \cdot \alpha \cdot \chi = \alpha^{-1} \cdot \beta\}$. Then $S \subseteq T$ and the restriction of any partial map $\chi \in T$ to $\operatorname{dom}(\alpha^{-1} \cdot \alpha)$ coincides with the partial map $\alpha^{-1} \cdot \beta$. Since every partial map from the semigroup $\mathscr{I}_\lambda^\infty$ is an injective partial selfmap almost everywhere the identity we have that there exist maximal subsets $A_{\alpha^{-1}\alpha}$ and $A_{\alpha^{-1}\beta}$ in λ such that the sets $\lambda \setminus A_{\alpha^{-1}\alpha}$ and $\lambda \setminus A_{\alpha^{-1}\beta}$ are finite and the restrictions $(\alpha^{-1} \cdot \alpha)|_{A_{\alpha^{-1}\alpha}} \colon A_{\alpha^{-1}\alpha} \to A_{\alpha^{-1}\alpha}$ and $(\alpha^{-1} \cdot \beta)|_{A_{\alpha^{-1}\beta}} \colon A_{\alpha^{-1}\beta} \to A_{\alpha^{-1}\beta}$ are identity maps. We put $A = A_{\alpha^{-1}\alpha} \cap A_{\alpha^{-1}\beta}$. Then the definition of the semigroup $\mathscr{I}_\lambda^\infty$ implies that the restrictions $(\alpha^{-1} \cdot \alpha)|_A \colon A \to A$ and $(\alpha^{-1} \cdot \beta)|_A \colon A \to A$ are identity maps and the set $\lambda \setminus A$ is finite. This implies that the set T is finite and hence the set S is finite too. ■

For an arbitrary non-zero cardinal λ we denote by $\mathcal{S}_\infty(\lambda)$ the group of all bijective transformations of λ with finite supports (i.e., $\alpha \in \mathcal{S}_\infty(\lambda)$ if and only if the set $\{x \in \lambda \mid (x)\alpha \neq x\}$ is finite).

The definition of the semigroup $\mathscr{I}_\lambda^\infty$ and Proposition 2.4 imply the following proposition:

Proposition 2.6. *Every maximal subgroup of the semigroup $\mathscr{I}_\lambda^\infty$ is isomorphic to $\mathcal{S}_\infty(\lambda)$.*

3. On congruences on the semigroup $\mathscr{I}_\lambda^\infty$

If $\mathfrak{R}$ is an arbitrary congruence on a semigroup S, then we denote by $\Phi_{\mathfrak{R}} \colon S \to S/\mathfrak{R}$ the natural homomorphisms from S onto $S/\mathfrak{R}$. Also we denote by Ω_S and Δ_S the *universal* and the *identity* congruences, respectively, on the semigroup S, i. e., $\Omega(S) = S \times S$ and $\Delta(S) = \{(s, s) \mid s \in S\}$.

The following lemma follows from the definition of a congruence on a semilattice:

Lemma 3.1. *Let $\mathfrak{R}$ is an arbitrary congruence on a semilattice E. Let a and b be elements of the semilattice E such that $a\mathfrak{R}b$. Then*

(i) $a\mathfrak{R}(ab)$; and
(ii) if $a \leqslant b$ then $a\mathfrak{R}c$ for all $c \in E$ such that $a \leqslant c \leqslant b$.

Proposition 3.2. *Let $\mathfrak{R}$ be an arbitrary congruence on the semigroup $\mathscr{I}_\lambda^\infty$. Let ε and φ be idempotents of $\mathscr{I}_\lambda^\infty$ such that $\varepsilon\mathfrak{R}\varphi$ and $\varepsilon \leqslant \varphi$. If $|\operatorname{dom}\varphi \setminus \operatorname{dom}\varepsilon| = 1$ then the following conditions hold:*

(i) $\varphi\mathfrak{R}\iota$ for all idempotents $\iota \in {\downarrow}\varphi$; and
(ii) $\varphi\mathfrak{R}\chi$ for all idempotents $\chi \in \mathscr{I}_\lambda^\infty$ such that $|\lambda \setminus \operatorname{dom}\varphi| = |\lambda \setminus \operatorname{dom}\chi|$.

Proof. (i) First we shall show that $\varphi\mathfrak{R}\psi$ for all idempotents $\psi \in {\downarrow}\varepsilon$. By Proposition 2.2 (iv) there exists a maximal (not necessary unique) ω-chain L in $E(\mathscr{I}_\lambda^\infty)$ which contains ε and ψ. Let $L_0 = \{\varepsilon_1, \ldots, \varepsilon_n\}$ be a maximal subchain in L such that $\psi = \varepsilon_n < \ldots < \varepsilon_1 = \varepsilon$, where n is some positive integer. The existence of the subchain L follows from Proposition 2.2 (iv) too. Let

$$x_n = \operatorname{dom}\varepsilon_{n-1} \setminus \operatorname{dom}\varepsilon_n,\ x_{n-1} = \operatorname{dom}\varepsilon_{n-2} \setminus \operatorname{dom}\varepsilon_{n-1},\ \ldots,\\ x_2 = \operatorname{dom}\varepsilon_1 \setminus \operatorname{dom}\varepsilon_2,\ x_1 = \operatorname{dom}\varphi \setminus \operatorname{dom}\varepsilon_1.$$

We put

$$\alpha_1 = \left(\begin{array}{c|c} x_1 & \\ & \operatorname{dom}\varepsilon_2 \\ x_2 & \end{array}\right), \quad \alpha_2 = \left(\begin{array}{c|c} x_2 & \\ & \operatorname{dom}\varepsilon_3 \\ x_3 & \end{array}\right), \quad \ldots,\\ \alpha_{n-1} = \left(\begin{array}{c|c} x_{n-1} & \\ & \operatorname{dom}\varepsilon_n \\ x_n & \end{array}\right).$$

Then we have that

$$\begin{array}{ccc} \alpha_1^{-1}\cdot\varphi\cdot\alpha_1 = \varepsilon_1 & \text{and} & \alpha_1^{-1}\cdot\varepsilon_1\cdot\alpha_1 = \varepsilon_2; \\ \alpha_2^{-1}\cdot\varepsilon_1\cdot\alpha_2 = \varepsilon_2 & \text{and} & \alpha_2^{-1}\cdot\varepsilon_2\cdot\alpha_2 = \varepsilon_3; \\ \cdots & \cdots & \cdots \\ \alpha_{n-1}^{-1}\cdot\varepsilon_{n-2}\cdot\alpha_{n-1} = \varepsilon_{n-1} & \text{and} & \alpha_{n-1}^{-1}\cdot\varepsilon_{n-1}\cdot\alpha_{n-1} = \varepsilon_n, \end{array}$$

and hence $\varepsilon_1\mathfrak{R}\varepsilon_2, \varepsilon_2\mathfrak{R}\varepsilon_3, \ldots, \varepsilon_{n-1}\mathfrak{R}\varepsilon_n$. Since $\varphi\mathfrak{R}\varepsilon$ we have that $\varphi\mathfrak{R}\varepsilon_n$. This completes the proof of the statement.

Let ι be an arbitrary idempotent of the semigroup $\mathscr{I}_\lambda^\infty$ such that $\iota \in {\downarrow}\varphi$. We put $\iota_0 = \varepsilon\cdot\iota$. Then by previous part of the proof we have that $\iota_0\mathfrak{R}\varphi$ and hence by Lemma 3.1 we get $\iota\mathfrak{R}\varphi$.

(ii) Let χ be an arbitrary idempotent of the semigroup $\mathscr{I}_\lambda^\infty$ such that $\varphi \neq \chi$ and $|\lambda\setminus\operatorname{dom}\varphi| = |\lambda\setminus\operatorname{dom}\chi|$. Then $\varepsilon\cdot\chi \leqslant \varphi$ and hence by statement (i) we get that $(\varepsilon\cdot\chi)\mathfrak{R}\varphi$. Since $|\lambda\setminus\operatorname{dom}\varphi| = |\lambda\setminus\operatorname{dom}\chi|$ we conclude that $|\operatorname{dom}\varphi\setminus\operatorname{dom}(\varepsilon\cdot\chi)| = |\operatorname{dom}\chi\setminus\operatorname{dom}(\varepsilon\cdot\chi)|$. Let be $\{x_1, \ldots, x_k\} = \operatorname{dom}\varphi\setminus\operatorname{dom}(\varepsilon\cdot\chi)$ and $\{y_1, \ldots, y_k\} = \operatorname{dom}\chi\setminus\operatorname{dom}(\varepsilon\cdot\chi)$. We put

$$\alpha = \left(\begin{array}{ccc|c} x_1 & \cdots & x_k & \\ & & & \operatorname{dom}(\varepsilon\cdot\chi) \\ y_1 & \cdots & y_k & \end{array}\right).$$

Then $\alpha^{-1}\cdot\varphi\cdot\alpha = \chi$ and $\alpha^{-1}\cdot(\varepsilon\cdot\chi)\cdot\alpha = \varepsilon\cdot\chi$. Therefore we get that $(\varepsilon\cdot\chi)\mathfrak{R}\chi$ and hence $\varphi\mathfrak{R}\chi$. This completes the proof of our statement. ■

Theorem 3.3. *Let $\mathfrak{R}$ be an arbitrary congruence on the semigroup $\mathscr{I}_\lambda^\infty$ and ε and φ be distinct $\mathfrak{R}$-equivalent idempotents of $\mathscr{I}_\lambda^\infty$. Then $\alpha\mathfrak{R}\varepsilon$ for every $\alpha \in \mathscr{I}_\lambda^\infty$ such that*

$$|\lambda\setminus\operatorname{dom}\alpha| \geqslant \min\{|\lambda\setminus\operatorname{dom}\varphi|, |\lambda\setminus\operatorname{dom}\varepsilon|\}.$$

Proof. In the case when α is an idempotent of the semigroup $\mathscr{I}_{\lambda}^{\infty}$ the statement of the theorem follows from Lemma 3.1 and Proposition 3.2.

Suppose that α is an arbitrary non-idempotent element of the semigroup $\mathscr{I}_{\lambda}^{\infty}$ such that $|\lambda \setminus \operatorname{dom} \alpha| \geqslant \max\{|\lambda \setminus \operatorname{dom} \varphi|, |\lambda \setminus \operatorname{dom} \varepsilon|\}$. Since $\mathscr{I}_{\lambda}^{\infty}$ is an inverse semigroup we have that $\alpha \cdot \alpha^{-1} \cdot \alpha = \alpha$ and Propositions 2.1 and 2.2 imply that

$$\begin{aligned} |\lambda \setminus \operatorname{dom} \alpha| &= |\lambda \setminus \operatorname{dom} \alpha^{-1}| = |\lambda \setminus \operatorname{dom}(\alpha \cdot \alpha^{-1})| = |\lambda \setminus \operatorname{dom}(\alpha^{-1} \cdot \alpha)| \\ &\geqslant \min\{|\lambda \setminus \operatorname{dom} \varphi|, |\lambda \setminus \operatorname{dom} \varepsilon|\}. \end{aligned}$$

Hence $(\alpha \cdot \alpha^{-1})\mathfrak{R}\varepsilon$ and by Proposition 3.2 we have that $(\alpha \cdot \alpha^{-1})\mathfrak{R}\iota$ for every idempotent ι of the semigroup $\mathscr{I}_{\lambda}^{\infty}$ such that $\iota \in {\downarrow}\varepsilon$. Definition of the semigroup $\mathscr{I}_{\lambda}^{\infty}$ implies that for every $\alpha \in \mathscr{I}_{\lambda}^{\infty}$ there exists an idempotent $\varsigma_{\alpha} \in \mathscr{I}_{\lambda}^{\infty}$ such that $\alpha \cdot \varsigma = \varsigma \cdot \alpha = \varsigma \cdot (\alpha \cdot \alpha^{-1}) = \varsigma$ for all idempotents $\varsigma \in \mathscr{I}_{\lambda}^{\infty}$ such that $\varsigma \in {\downarrow}\varsigma_{\alpha}$. Let $\nu = \varsigma_{\alpha} \cdot \varepsilon$. Then $(\alpha \cdot \alpha^{-1})\mathfrak{R}\nu$ and $\alpha \cdot \nu = \nu \cdot \alpha = \nu \cdot (\alpha \cdot \alpha^{-1}) = \nu$. Therefore we get

$$\begin{aligned} (\alpha)\Phi_{\mathfrak{R}} &= (\alpha \cdot \alpha^{-1} \cdot \alpha)\Phi_{\mathfrak{R}} = (\alpha \cdot \alpha^{-1})\Phi_{\mathfrak{R}} \cdot (\alpha)\Phi_{\mathfrak{R}} = (\nu)\Phi_{\mathfrak{R}} \cdot (\alpha)\Phi_{\mathfrak{R}} \\ &= (\nu \cdot \alpha)\Phi_{\mathfrak{R}} = (\nu)\Phi_{\mathfrak{R}} \end{aligned}$$

and $\alpha\mathfrak{R}\nu$. Hence we have that $\alpha\mathfrak{R}\varepsilon$. ■

Proposition 3.4. *Let $\mathfrak{R}$ be an arbitrary congruence on the semigroup $\mathscr{I}_{\lambda}^{\infty}$. Let ε be an idempotent of $\mathscr{I}_{\lambda}^{\infty}$ such that $|\lambda \setminus \operatorname{dom} \varepsilon| \geqslant 1$ and the following conditions hold:*

(i) there exists an idempotent $\varphi \in \mathscr{I}_{\lambda}^{\infty}$ such that $\varepsilon\mathfrak{R}\varphi$ and $|\lambda \setminus \operatorname{dom} \varphi| \geqslant |\lambda \setminus \operatorname{dom} \varepsilon|$; and
(ii) does not exist an idempotent $\psi \in \mathscr{I}_{\lambda}^{\infty}$ such that $\varepsilon\mathfrak{R}\psi$ and $|\lambda \setminus \operatorname{dom} \psi| < |\lambda \setminus \operatorname{dom} \varepsilon|$.

Then there exists no element α of the semigroup $\mathscr{I}_{\lambda}^{\infty}$ such that $\varepsilon\mathfrak{R}\alpha$ and $|\lambda \setminus \operatorname{dom} \alpha| < |\lambda \setminus \operatorname{dom} \varepsilon|$.

Proof. Suppose to the contrary that there exists $\alpha \in \mathscr{I}_{\lambda}^{\infty}$ such that $\varepsilon\mathfrak{R}\alpha$ and $|\lambda \setminus \operatorname{dom} \alpha| < |\lambda \setminus \operatorname{dom} \varepsilon|$. Since $\mathscr{I}_{\lambda}^{\infty}$ is an inverse semigroup Lemma III.1.1 [21] implies that $\varepsilon\mathfrak{R}\alpha^{-1}$ and hence $\varepsilon\mathfrak{R}(\alpha \cdot \alpha^{-1})$. But $|\lambda \setminus \operatorname{dom}(\alpha \cdot \alpha^{-1})| = |\lambda \setminus \operatorname{dom} \alpha| < |\lambda \setminus \operatorname{dom} \varepsilon|$, a contradiction. An obtained contradiction implies the statement of the proposition. ■

Proposition 3.5. *Let $\mathfrak{R}$ be an arbitrary congruence on the semigroup $\mathscr{I}_{\lambda}^{\infty}$. Let α and β be non-$\mathscr{H}$-equivalent elements of $\mathscr{I}_{\lambda}^{\infty}$ such that $\alpha\mathfrak{R}\beta$. Then $\gamma\mathfrak{R}\alpha$ for all $\gamma \in \mathscr{I}_{\lambda}^{\infty}$ such that*

$$|\lambda \setminus \operatorname{dom} \gamma| \geqslant \min\{|\lambda \setminus \operatorname{dom} \alpha|, |\lambda \setminus \operatorname{dom} \beta|\}.$$

Proof. Since α and β are non-$\mathscr{H}$-equivalent elements of the inverse semigroup $\mathscr{I}_\lambda^\infty$ we conclude that at least one of the following conditions holds:

(*i*) $\alpha \cdot \alpha^{-1} \neq \beta \cdot \beta^{-1}$;
(*ii*) $\alpha^{-1} \cdot \alpha \neq \beta^{-1} \cdot \beta$.

Suppose that the case $\alpha \cdot \alpha^{-1} \neq \beta \cdot \beta^{-1}$ holds. In the other case the proof is similar. Since $\mathscr{I}_\lambda^\infty$ is an inverse semigroup Lemma III.1.1 [21] implies that $\beta^{-1}\mathfrak{R}\alpha^{-1}$ and hence $(\beta \cdot \beta^{-1})\mathfrak{R}(\alpha \cdot \alpha^{-1})$. Then we have that

$$|\lambda \setminus \operatorname{dom} \alpha| = |\lambda \setminus \operatorname{dom}(\alpha \cdot \alpha^{-1})| \qquad \text{and} \qquad |\lambda \setminus \operatorname{dom} \beta| = |\lambda \setminus \operatorname{dom}(\beta \cdot \beta^{-1})|$$

and hence the assumptions of the Theorem 3.3 hold. This completes the proof of the proposition. ■

Proposition 3.6. *Let $\mathfrak{R}$ be an arbitrary congruence on the semigroup $\mathscr{I}_\lambda^\infty$. If α and β are distinct $\mathscr{H}$-equivalent elements of $\mathscr{I}_\lambda^\infty$ such that $\alpha\mathfrak{R}\beta$, then $\gamma\mathfrak{R}\alpha$ for all $\gamma \in \mathscr{I}_\lambda^\infty$ such that*

$$|\lambda \setminus \operatorname{dom} \gamma| > |\lambda \setminus \operatorname{dom} \alpha|.$$

Proof. Since $\mathscr{I}_\lambda^\infty$ is an inverse semigroup Theorem 2.20 [5] and Proposition 2.2 (*viii*) imply that without loss of generality we can assume that α and β are elements of a maximal subgroup $H(\varepsilon)$ of $\mathscr{I}_\lambda^\infty$ with unity ε. Since $(\alpha\cdot\alpha^{-1})\mathfrak{R}(\beta\cdot\alpha^{-1})$ we can assume that α is an identity of the subgroup $H(\varepsilon)$. Let $x \in \operatorname{dom} \alpha$ such that $(x)\beta \neq x$. We put $\varepsilon_1\colon \operatorname{dom} \alpha \setminus \{x\} \to \operatorname{dom} \alpha \setminus \{x\}$ be an identity map. Then $\varepsilon_1 \cdot \alpha = \varepsilon_1$ and $\operatorname{ran}(\varepsilon_1 \cdot \beta) \neq \operatorname{ran}(\varepsilon_1)$. Therefore by Proposition 2.2 (*vii*) we get that the elements ε_1 and $\varepsilon_1 \cdot \beta$ are not $\mathscr{H}$-equivalent. Since $|\lambda\setminus\operatorname{dom} \varepsilon_1| = |\lambda\setminus\operatorname{dom}(\varepsilon_1\cdot\beta)|$ we have that the assumptions of Proposition 3.5 hold. This completes the proof of the proposition. ■

Theorem 3.3 and Propositions 3.4, 3.5 and 3.6 imply the following proposition:

Proposition 3.7. *Let $\mathfrak{R}$ be an arbitrary congruence on the semigroup $\mathscr{I}_\lambda^\infty$. Let α and β be distinct $\mathscr{H}$-equivalent elements of $\mathscr{I}_\lambda^\infty$ such that $\alpha\mathfrak{R}\beta$ and suppose that there does not exist $\gamma \in \mathscr{I}_\lambda^\infty$ such that $\alpha\mathfrak{R}\gamma$ and $|\lambda \setminus \operatorname{dom} \gamma| < |\lambda\setminus\operatorname{dom} \alpha|$. Then elements $\mu, \nu \in \mathscr{I}_\lambda^\infty$ with $|\lambda\setminus\operatorname{dom} \mu| < |\lambda\setminus\operatorname{dom} \alpha|$ and $|\lambda \setminus \operatorname{dom} \nu| < |\lambda \setminus \operatorname{dom} \alpha|$ are $\mathfrak{R}$-equivalent if and only if $\mu = \nu$.*

Definition 3.8. For every non-negative integer n we denote by $\mathfrak{K}_n(I)$ the congruence on the semigroup $\mathscr{I}_\lambda^\infty$ generated by the ideal I_n, i. e., $\mathfrak{K}_n(I) = (I_n \times I_n) \cup \Delta(\mathscr{I}_\lambda^\infty)$. We observe that $\mathfrak{K}_0(I) = \Omega(\mathscr{I}_\lambda^\infty)$.

Remark 3.9. The group $\mathsf{S}_\infty(\lambda)$ has only one non-trivial normal subgroup: that is a group $\mathsf{A}_\infty(\lambda)$ of all even permutations of the set λ (see [10, pp. 313–314, Example] or [18]). Therefore every non-trivial homomorphism of $\mathsf{S}_\infty(\lambda)$ is either an isomorphism or its image is a two-elements cyclic group.

Definition 3.10. Fix an arbitrary non-negative integer n. We shall say that elements α and β of the semigroup $\mathscr{I}_\lambda^\infty$ are n_{S_∞}-*equivalent* if the following conditions hold:

(i) $\alpha\mathscr{H}\beta$; and
(ii) $|\lambda \setminus \operatorname{dom}\alpha| = |\lambda \setminus \operatorname{dom}\beta| = n$.

We define a relation $\mathfrak{K}_n(\mathsf{S}_\infty)$ on the semigroup $\mathscr{I}_\lambda^\infty$ as follows:

$$\mathfrak{K}_n(\mathsf{S}_\infty) = \{(\alpha,\beta) \mid (\alpha,\beta) \in n_{\mathsf{S}_\infty}\} \cup (I_{n+1} \times I_{n+1}) \cup \Delta(\mathscr{I}_\lambda^\infty).$$

Simple verifications show that so defined relation $\mathfrak{K}_n(\mathsf{S}_\infty)$ on $\mathscr{I}_\lambda^\infty$ is an equivalence relation for every non-negative integer n.

Proposition 3.11. *The relation $\mathfrak{K}_n(S_\infty)$ is a congruence on the semigroup $\mathscr{I}_\lambda^\infty$.*

Proof. First we consider the case when $n = 0$. If α and β are distinct elements of the semigroup $\mathscr{I}_\lambda^\infty$ such that $\alpha\mathfrak{K}_0(\mathsf{S}_\infty)\beta$, then either $\alpha, \beta \in H(\mathbb{I})$ or $\alpha, \beta \in I_1$. Suppose that $\alpha, \beta \in H(\mathbb{I})$. Then for every $\gamma \in \mathscr{I}_\lambda^\infty$ we have that either $\alpha \cdot \gamma, \beta \cdot \gamma \in H(\mathbb{I})$ or $\alpha \cdot \gamma, \beta \cdot \gamma \in I_1$, and similarly we get that either $\gamma \cdot \alpha, \gamma \cdot \beta \in H(\mathbb{I})$ or $\gamma \cdot \alpha, \gamma \cdot \beta \in I_1$. If $\alpha, \beta \in I_1$ then for every $\gamma \in \mathscr{I}_\lambda^\infty$ we have that $\alpha \cdot \gamma, \beta \cdot \gamma, \alpha \cdot \gamma, \beta \cdot \gamma \in I_1$. Therefore $\mathfrak{K}_0(\mathsf{S}_\infty)$ is a congruence on the semigroup $\mathscr{I}_\lambda^\infty$.

Suppose that n is an arbitrary positive integer. Let α and β be distinct elements of the semigroup $\mathscr{I}_\lambda^\infty$ such that $\alpha\mathfrak{K}_n(\mathsf{S}_\infty)\beta$. The definition of the relation $\mathfrak{K}_n(\mathsf{S}_\infty)$ implies that only one of the following conditions holds:

(i) $|\lambda \setminus \operatorname{dom}\alpha| = |\lambda \setminus \operatorname{dom}\beta| = n$; or
(ii) $|\lambda \setminus \operatorname{dom}\alpha| > n$ and $|\lambda \setminus \operatorname{dom}\beta| > n$.

First we suppose that $|\lambda\setminus\operatorname{dom}\alpha| = |\lambda\setminus\operatorname{dom}\beta| = n$. Let γ be an arbitrary element of the semigroup $\mathscr{I}_\lambda^\infty$. We consider two cases:

$a)$ $\operatorname{dom}\alpha \subseteq \operatorname{ran}\gamma$; and
$b)$ $\operatorname{dom}\alpha \nsubseteq \operatorname{ran}\gamma$.

Since the elements α and β are $\mathscr{H}$-equivalent in $\mathscr{I}_\lambda^\infty$ Proposition 2.2 (vii) implies that in case $a)$ we have that $\operatorname{dom}(\gamma\cdot\alpha) = \operatorname{dom}(\gamma\cdot\beta)$ and $\operatorname{ran}(\gamma\cdot\alpha) = \operatorname{ran}(\gamma\cdot\beta)$. Then again by Proposition 2.2 (vii) the elements $\gamma\cdot\alpha$ and $\gamma\cdot\beta$ are $\mathscr{H}$-equivalent in $\mathscr{I}_\lambda^\infty$. Since $\operatorname{dom}\alpha \subseteq \operatorname{ran}\gamma$ we get that $|\lambda\setminus\operatorname{dom}(\gamma\cdot\alpha)| = |\lambda \setminus \operatorname{dom}(\gamma\cdot\beta)| = n$. Hence we obtain that $(\gamma\cdot\alpha)\mathfrak{K}_n(\mathsf{S}_\infty)(\gamma\cdot\beta)$. In case $b)$ we have that $\gamma\cdot\alpha, \gamma\cdot\beta \in I_{n+1}$ and hence $(\gamma\cdot\alpha)\mathfrak{K}_n(\mathsf{S}_\infty)(\gamma\cdot\beta)$.

The proof that the assertion $\alpha\mathfrak{K}_n(\mathsf{S}_\infty)\beta$ implies $(\alpha\cdot\delta)\mathfrak{K}_n(\mathsf{S}_\infty)(\beta\cdot\delta)$ for every $\delta \in \mathscr{I}_\lambda^\infty$ is similar.

Suppose that $|\lambda \setminus \operatorname{dom}\alpha| > n$ and $|\lambda \setminus \operatorname{dom}\beta| > n$. Then $\alpha, \beta \in I_{n+1}$. By Proposition 2.2 (x) we have that $\gamma\cdot\alpha, \gamma\cdot\beta, \alpha\cdot\delta, \beta\cdot\delta \in I_{n+1}$ and hence

$(\gamma\cdot\alpha)\mathfrak{K}_n(\mathsf{S}_\infty)(\gamma\cdot\beta)$ and $(\alpha\cdot\delta)\mathfrak{K}_n(\mathsf{S}_\infty)(\beta\cdot\delta)$ for all $\gamma,\delta\in\mathscr{I}_\lambda^\infty$. This completes the proof of the proposition. ■

Definition 3.12. Fix an arbitrary non-negative integer n. We shall say that elements α and β of the semigroup $\mathscr{I}_\lambda^\infty$ are n_{A_∞}-*equivalent* if the following conditions hold:

(i) $\alpha\mathscr{H}\beta$;
(ii) $\alpha\cdot\beta^{-1}$ is an even permutation of the set $\operatorname{dom}\alpha$; and
(iii) $|\lambda\setminus\operatorname{dom}\alpha| = |\lambda\setminus\operatorname{dom}\beta| = n$.

We define a relation $\mathfrak{K}_n(\mathsf{A}_\infty)$ on the semigroup $\mathscr{I}_\lambda^\infty$ as follows:

$$\mathfrak{K}_n(\mathsf{A}_\infty)=\{(\alpha,\beta)\mid(\alpha,\beta)\in n_{\mathsf{A}_\infty}\}\cup(I_{n+1}\times I_{n+1})\cup\Delta(\mathscr{I}_\lambda^\infty).$$

Simple verifications show that so defined relation $\mathfrak{K}_n(\mathsf{A}_\infty)$ on $\mathscr{I}_\lambda^\infty$ is an equivalence relation for every non-negative integer n.

Proposition 3.13. *The relation $\mathfrak{K}_n(A_\infty)$ is a congruence on the semigroup $\mathscr{I}_\lambda^\infty$.*

Proof. First we consider the case when $n=0$. If α and β are distinct elements of the semigroup $\mathscr{I}_\lambda^\infty$ such that $\alpha\mathfrak{K}_0(\mathsf{S}_\infty)\beta$, then either $\alpha,\beta\in H(\mathbb{I})$ or $\alpha,\beta\in I_1$. Suppose that $\alpha,\beta\in H(\mathbb{I})$. Then for every $\gamma\in H(\mathbb{I})$ we have that $\alpha\cdot\gamma,\beta\cdot\gamma,\gamma\cdot\alpha,\gamma\cdot\beta\in H(\mathbb{I})$. Then $(\alpha\cdot\gamma)\cdot(\beta\cdot\gamma)^{-1}=\alpha\cdot\gamma\cdot\gamma^{-1}\cdot\beta^{-1}=\alpha\cdot\beta^{-1}$ is an even permutation of the set λ. Also, since $\alpha\cdot\beta^{-1}$ is an even permutation of the set λ we get that $(\gamma\cdot\alpha)\cdot(\gamma\cdot\beta)^{-1}=\gamma\cdot\alpha\cdot\beta^{-1}\cdot\gamma^{-1}$ is an even permutation of the set λ too. For every $\gamma\in I_1$ we have that $\alpha\cdot\gamma,\beta\cdot\gamma,\gamma\cdot\alpha,\gamma\cdot\beta\in I_1$. If $\alpha,\beta\in I_1$ then for every $\gamma\in\mathscr{I}_\lambda^\infty$ we have that $\alpha\cdot\gamma,\beta\cdot\gamma,\alpha\cdot\gamma,\beta\cdot\gamma\in I_1$. Therefore $\mathfrak{K}_0(\mathsf{A}_\infty)$ is a congruence on the semigroup $\mathscr{I}_\lambda^\infty$.

Suppose that n is an arbitrary positive integer. Let α and β be distinct elements of the semigroup $\mathscr{I}_\lambda^\infty$ such that $\alpha\mathfrak{K}_n(\mathsf{A}_\infty)\beta$. The definition of the relation $\mathfrak{K}_n(\mathsf{A}_\infty)$ implies that only one of the following conditions holds:

(i) $|\lambda\setminus\operatorname{dom}\alpha|=|\lambda\setminus\operatorname{dom}\beta|=n$; or
(ii) $|\lambda\setminus\operatorname{dom}\alpha|>n$ and $|\lambda\setminus\operatorname{dom}\beta|>n$.

First we suppose that $|\lambda\setminus\operatorname{dom}\alpha|=|\lambda\setminus\operatorname{dom}\beta|=n$. Let γ be an arbitrary element of the semigroup $\mathscr{I}_\lambda^\infty$. We consider two cases:

a) $\operatorname{dom}\alpha\subseteq\operatorname{ran}\gamma$; and
b) $\operatorname{dom}\alpha\nsubseteq\operatorname{ran}\gamma$.

Suppose case a) holds. Since the elements α and β are $\mathscr{H}$-equivalent in $\mathscr{I}_\lambda^\infty$ we have that Proposition 2.2 (vii) implies that $\operatorname{dom}(\gamma\cdot\alpha)=\operatorname{dom}(\gamma\cdot\beta)$ and $\operatorname{ran}(\gamma\cdot\alpha)=\operatorname{ran}(\gamma\cdot\beta)$. Then again by Proposition 2.2 (vii) the elements $\gamma\cdot\alpha$ and $\gamma\cdot\beta$ are $\mathscr{H}$-equivalent in $\mathscr{I}_\lambda^\infty$. Since $\operatorname{dom}\alpha\subseteq\operatorname{ran}\gamma$ we get that $|\lambda\setminus\operatorname{dom}(\gamma\cdot\alpha)|=|\lambda\setminus\operatorname{dom}(\gamma\cdot\beta)|=n$. We define a partial map

$\gamma_1 \colon \lambda \rightharpoonup \lambda$ as follows $\gamma_1 = \gamma|_{(\operatorname{dom}\alpha)\gamma^{-1}} \colon (\operatorname{dom}\alpha)\gamma^{-1} \to \operatorname{dom}\alpha$. Then we get that $|\lambda \setminus \operatorname{dom}\gamma_1| = |\lambda \setminus \operatorname{dom}\alpha| = |\lambda \setminus \operatorname{dom}\beta| = n$, $\gamma\cdot\alpha = \gamma_1\cdot\alpha$, $\gamma\cdot\beta = \gamma_1\cdot\beta$ and hence $(\gamma\cdot\alpha)\cdot(\gamma\cdot\beta)^{-1} = (\gamma_1\cdot\alpha)\cdot(\gamma_1\cdot\beta)^{-1} = \gamma_1\cdot\alpha\cdot\beta^{-1}\cdot\gamma_1^{-1}$. Since $\alpha\cdot\beta^{-1}$ is an even permutation of the set $\operatorname{dom}\alpha$ we conclude that $\gamma_1\cdot\alpha\cdot\beta^{-1}\cdot\gamma_1^{-1}$ is an even permutation of the set $\operatorname{dom}\gamma_1 = (\operatorname{dom}\alpha)\gamma^{-1}$. Hence we obtain that $(\gamma\cdot\alpha)\mathfrak{K}_n(\mathsf{A}_\infty)(\gamma\cdot\beta)$. In case b) we have that $\gamma\cdot\alpha, \gamma\cdot\beta \in I_{n+1}$ and hence $(\gamma\cdot\alpha)\mathfrak{K}_n(\mathsf{A}_\infty)(\gamma\cdot\beta)$.

The proof the assertion that $\alpha\mathfrak{K}_n(\mathsf{A}_\infty)\beta$ implies $(\alpha\cdot\delta)\mathfrak{K}_n(\mathsf{A}_\infty)(\beta\cdot\delta)$ for every $\delta \in \mathscr{I}_\lambda^\infty$ is similar.

Suppose that $|\lambda \setminus \operatorname{dom}\alpha| > n$ and $|\lambda \setminus \operatorname{dom}\beta| > n$. Then $\alpha, \beta \in I_{n+1}$. By Proposition 2.2 ($x$) we have that $\gamma\cdot\alpha, \gamma\cdot\beta, \alpha\cdot\delta, \beta\cdot\delta \in I_{n+1}$ and hence $(\gamma\cdot\alpha)\mathfrak{K}_n(\mathsf{A}_\infty)(\gamma\cdot\beta)$ and $(\alpha\cdot\delta)\mathfrak{K}_n(\mathsf{A}_\infty)(\beta\cdot\delta)$, for all $\gamma, \delta \in \mathscr{I}_\lambda^\infty$. This completes the proof of the proposition. ∎

Theorem 3.14. *The family*

$$\begin{aligned} \mathit{Cong}(\mathscr{I}_\lambda^\infty), = \{\Delta(\mathscr{I}_\lambda^\infty), \Omega(\mathscr{I}_\lambda^\infty)\} &\cup \{\mathfrak{K}_n(S_\infty) \mid n = 0,1,2,\ldots\} \\ \cup \{\mathfrak{K}_n(A_\infty) \mid n = 0,1,2,\ldots\} &\cup \{\mathfrak{K}_n(I_n) \mid n = 1,2,\ldots\} \end{aligned}$$

determines all congruences on the semigroup $\mathscr{I}_\lambda^\infty$.

Proof. Let $\mathfrak{R}$ be non-identity congruence on the semigroup $\mathscr{I}_\lambda^\infty$. Since the set of all non-negative integers with respect to the usual order $\leqslant$ is well ordered there exists a minimal non-negative integer n such that there are two distinct elements α and β in $\mathscr{I}_\lambda^\infty$ such that $\alpha\mathfrak{R}\beta$ and

$$\min\{|\lambda \setminus \operatorname{dom}\alpha|, |\lambda \setminus \operatorname{dom}\beta|\} = n,$$

i.e., for any non-negative integer $m < n$ if for α and β in $\mathscr{I}_\lambda^\infty$ such that $\alpha\mathfrak{R}\beta$ and

$$\min\{|\lambda \setminus \operatorname{dom}\alpha|, |\lambda \setminus \operatorname{dom}\beta|\} = m$$

then $\alpha = \beta$.

We consider two cases:

(i) $|\lambda \setminus \operatorname{dom}\alpha| \neq |\lambda \setminus \operatorname{dom}\beta|$; and
(ii) $|\lambda \setminus \operatorname{dom}\alpha| = |\lambda \setminus \operatorname{dom}\beta|$.

Suppose case (i) holds and $|\lambda \setminus \operatorname{dom}\alpha| = n < |\lambda \setminus \operatorname{dom}\beta|$. Then α and β are not $\mathscr{H}$-equivalent elements in $\mathscr{I}_\lambda^\infty$ and hence by Proposition 3.5 we obtain that $\alpha\mathfrak{R}\gamma$ for all $\gamma \in \mathscr{I}_\lambda^\infty$ with $|\lambda \setminus \operatorname{dom}\gamma| \geqslant n$. Then Proposition 3.7 implies that $\mu\mathfrak{R}\nu$ if and only if $\mu = \nu$ for all elements $\mu, \nu \in \mathscr{I}_\lambda^\infty$ such that $|\lambda \setminus \operatorname{dom}\mu| < n$ and $|\lambda \setminus \operatorname{dom}\nu| < n$. Hence we get that $\mathfrak{R} = \mathfrak{K}_n(I)$. We observe if $n = 0$ then $\mathfrak{R} = \Omega(\mathscr{I}_\lambda^\infty)$.

We henceforth assume that case (ii) holds.

If α and β are not $\mathscr{H}$-equivalent elements in $\mathscr{I}_\lambda^\infty$ and then by Proposition 3.5 we have that $\alpha\mathfrak{R}\gamma$ for all $\gamma \in \mathscr{I}_\lambda^\infty$ such that $|\lambda \setminus \operatorname{dom}\gamma| \geqslant n$. Then Proposition 3.7 implies that $\mu\mathfrak{R}\nu$ if and only if $\mu = \nu$ for all elements $\mu, \nu \in \mathscr{I}_\lambda^\infty$ such that $|\lambda \setminus \operatorname{dom}\mu| < n$ and $|\lambda \setminus \operatorname{dom}\nu| < n$, and hence we have that $\mathfrak{R} = \mathfrak{K}_n(I)$. Also in this case if $n = 0$ then $\mathfrak{R} = \Omega(\mathscr{I}_\lambda^\infty)$.

Suppose that α and β are $\mathscr{H}$-equivalent elements in $\mathscr{I}_\lambda^\infty$ and there exists no non-$\mathscr{H}$-equivalent element δ of the semigroup $\mathscr{I}_\lambda^\infty$ such that $\alpha\mathfrak{R}\delta$. Otherwise by the previous part of the proof we have that $\mathfrak{R} = \mathfrak{K}_n(I)$. Since $(\alpha\cdot\alpha^{-1})\mathfrak{R}(\beta\cdot\alpha^{-1})$ we conclude that without loss of generality we can assume that α is an identity element of $\mathscr{H}$-class $H(\alpha)$ which contains α and $\beta \neq \alpha$. Since α is an idempotent of the semigroup $\mathscr{I}_\lambda^\infty$ we have that $\operatorname{dom}\alpha = \operatorname{ran}\alpha$ and the restriction $\alpha|_{\operatorname{dom}\alpha}\colon \operatorname{dom}\alpha \to \operatorname{dom}\alpha$ is an identity map. Also we observe that the restriction of the partial map $\beta|_{\operatorname{dom}\alpha}\colon \operatorname{dom}\alpha \to \operatorname{dom}\alpha$ is a permutation of the set $\operatorname{dom}\alpha$. Therefore without loss of generality we can consider β as a permutation of the set $\operatorname{dom}\alpha$.

We consider two cases:

(1) β is an odd permutation of the set $\operatorname{dom}\alpha$; and
(2) β is an even permutation of the set $\operatorname{dom}\alpha$.

Suppose that β is an odd permutation of the set $\operatorname{dom}\alpha$. Since $H(\alpha)$ is a subgroup of the semigroup $\mathscr{I}_\lambda^\infty$ we conclude that the image $(H(\alpha))\Phi_{\mathfrak{R}}$ of $H(\alpha)$ is a subgroup in $\mathscr{I}_\lambda^\infty/\mathfrak{R}$. Since the subgroup $H(\alpha)$ is isomorphic to the group $\mathsf{S}_\infty(\lambda)$ and the group of all even permutations $\mathsf{A}_\infty(\lambda)$ of the set λ is a unique normal subgroup in $\mathsf{S}_\infty(\lambda)$ (see [10, pp. 313–314, Example] or [18]) we conclude that the image $(H(\alpha))\Phi_{\mathfrak{R}}$ is singleton. Then by Theorem 2.20 [5] and Proposition 2.2 $(viii)$ for every $\gamma \in \mathscr{I}_\lambda^\infty$ with $|\lambda\setminus\operatorname{dom}\gamma| = |\lambda\setminus\operatorname{dom}\alpha|$ the image $(H_\gamma)\Phi_{\mathfrak{R}}$ of the $\mathscr{H}$-class H_γ which contains the element γ is singleton and hence by Propositions 3.5, 3.6 and 3.7, we get that $\mathfrak{R} = \mathfrak{K}_n(\mathsf{S}_\infty)$.

Suppose that β is an even permutation of the set $\operatorname{dom}\alpha$. If the subgroup $H(\alpha)$ contains an odd permutation δ of the set $\operatorname{dom}\alpha$, then by previous proof we get that $\mathfrak{R} = \mathfrak{K}_n(\mathsf{S}_\infty)$. Suppose the subgroup $H(\alpha)$ does not contain an odd permutation δ of the set $\operatorname{dom}\alpha$. Since the subgroup $H(\alpha)$ is isomorphic to the group $\mathsf{S}_\infty(\lambda)$ and the group of all even permutations $\mathsf{A}_\infty(\lambda)$ of the set λ is a unique normal subgroup in $\mathsf{S}_\infty(\lambda)$ we conclude that the image $(H(\alpha))\Phi_{\mathfrak{R}}$ is a two-element subgroup in $\mathscr{I}_\lambda^\infty/\mathfrak{R}$. Then by Theorem 2.20 [5] and Proposition 2.2 $(viii)$ for every $\gamma \in \mathscr{I}_\lambda^\infty$ with $|\lambda \setminus \operatorname{dom}\gamma| = |\lambda \setminus \operatorname{dom}\alpha|$ the image $(H_\gamma)\Phi_{\mathfrak{R}}$ of the $\mathscr{H}$-class H_γ which contains the element γ is a two-element subset in $\mathscr{I}_\lambda^\infty/\mathfrak{R}$ and hence by Propositions 3.5, 3.6 and 3.7, we get that $\mathfrak{R} = \mathfrak{K}_n(\mathsf{A}_\infty)$. ∎

4. On topologizations of the free semilattice $(\mathscr{P}_{<\omega}(\lambda), \cup)$

Definition 4.1. ([4]) We shall say that a semigroup S has the F-*property* if for every $a, b, c, d \in S^1$ the sets $\{x \in S \mid a \cdot x = b\}$ and $\{x \in S \mid x \cdot c = d\}$ are finite or empty.

Recall [9] an element x of a semitopological semilattice S is a *local minimum* if there exists an open neighbourhood $U(x)$ of x such that $U(x) \cap {\downarrow}x = \{x\}$. This is equivalent to statement that ${\uparrow}x$ is an open subset in S.

A topological space X is called *Baire* if for each sequence $A_1, A_2, \ldots, A_i, \ldots$ of nowhere dense subsets of X the union $\bigcup_{i=1}^{\infty} A_i$ is a co-dense subset of X [7]. A Tychonoff space X is called *Čech complete* if for every compactification cX of X the remainder $cX \setminus c(X)$ is an F_σ-set in cX [7].

A topological space X is called *hereditary Baire* if every closed subset of X is a Baire space [7]. Every Čech complete (and hence locally compact) space is hereditary Baire (see [7, Theorem 3.9.6]). We shall say that a Hausdorff semitopological semigroup S is an *I-Baire space* if for every $s \in S$ either sS or Ss is a Baire space [4].

Remark 4.2. We observe that every left ideal Ss and every right ideal sS of a regular semigroup S is generated by its idempotents. Therefore every principal left (right) ideal of a regular Hausdorff semitopological semigroup S is a closed subset of S. Hence every regular Hausdorff hereditary Baire semitopological semigroup is a I-Baire space.

Theorem 4.3. *Let S be a semilattice with the F-property. Then every I-Baire topology τ on S such that (S, τ) is a Hausdorff semitopological semilattice is discrete.*

Proof. Let x be an arbitrary element of the semilattice S. We need to show that x is an isolated point in (S, τ).

Since τ is an I-Baire topology on S we conclude that the subspace ${\downarrow}x$ is Baire. We denote $S_x = {\downarrow}x$. For every positive integer n we put

$$F_n = \{y \in S_x \mid |{\uparrow}y| = n\}.$$

Then we have that $S_x = \bigcup_{i=1}^{\infty} F_n$. Since the topological space S_x is Baire we conclude that there exists $F_n \in \mathscr{F}$ such that $\operatorname{Int}_{S_x}(F_n) \neq \varnothing$. We fix an arbitrary $y_0 \in \operatorname{Int}_{S_x}(F_n)$. We observe that the definition of the family $\{F_n \mid n \in \mathbb{N}\}$ implies that for every non-empty subset F_n and for any $s \in F_n$ the sets ${\uparrow}s \cap F_n$ and ${\downarrow}s \cap F_n$ are singleton. This implies that y_0 is a local minimum in S_x, i.e., ${\uparrow}y_0$ is an open subset of S. Since the semilattice S_x has the F-property we conclude that the Hausdorffness of S implies that x is an isolated point in S_x. Then x is a local minimum in S and hence ${\uparrow}x$ is an open subset in S. Since the semilattice S has the F-property we conclude that the Hausdorffness of S implies that x is an isolated point in S. ∎

Remark 4.4. We observe that the statement of Theorem 4.3 is true for a T_1-semitopological I-Baire semilattice with the F-property.

Since every Čech complete (and hence locally compact) space is hereditary Baire, Theorem 4.3 implies the following corollary:

Corollary 4.5. *Let S be a semilattice with the F-property. Then every Čech complete (locally compact) topology τ on S such that (S,τ) is a semitopological semilattice is discrete.*

Since the free semilattice $(\mathscr{P}_{<\omega}(\lambda),\cup)$ has F-property, Theorem 4.3 implies the following corollary:

Corollary 4.6. *Every Hausdorff I-Baire (Čech complete, locally compact) topology τ on the free semilattice $\mathscr{P}_{<\omega}(\lambda)$ such that $(\mathscr{P}_{<\omega}(\lambda),\tau)$ is a semitopological semilattice is discrete.*

5. On a topological semigroup $\mathscr{I}_\lambda^\infty$

Theorem 5.1. *Every hereditary Baire topology τ on the semigroup $\mathscr{I}_\omega^\infty$ such that $(\mathscr{I}_\omega^\infty,\tau)$ is a Hausdorff semitopological semigroup is discrete.*

Proof. Let α be an arbitrary element of the semigroup $\mathscr{I}_\omega^\infty$. We need to show that α is an isolated point in $(\mathscr{I}_\omega^\infty,\tau)$.

For every non-negative integer n we denote $C_n = \mathscr{I}_\omega^\infty \setminus I_{n+1}$.

By induction we shall prove that for every non-negative integer n the following statement holds: *every $\alpha \in C_n$ is an isolated point in $(\mathscr{I}_\omega^\infty,\tau)$.*

First we shall show that our statement is true for $n=0$. We define a family $\mathscr{C} = \{\{\beta\} \mid \beta \in \mathscr{I}_\omega^\infty\}$. Since the topological space $(\mathscr{I}_\omega^\infty,\tau)$ is Baire we have that the family $\mathscr{C}$ has an element with non-empty interior and hence the topological space $(\mathscr{I}_\omega^\infty,\tau)$ has an isolated point γ in $(\mathscr{I}_\omega^\infty,\tau)$. Then $|\omega\setminus \operatorname{dom}\alpha| = 0$ and hence statements $(viii)-(xi)$ of Proposition 2.2 imply that there exist $\mu,\nu\in\mathscr{I}_\omega^\infty$ such that $\mu\cdot\alpha\cdot\nu=\gamma$. Since translations in $(\mathscr{I}_\omega^\infty,\tau)$ are continuous we conclude that Hausdorffness of the space $(\mathscr{I}_\omega^\infty,\tau)$ and Proposition 2.5 imply that α is an isolated point in $(\mathscr{I}_\omega^\infty,\tau)$.

Suppose our statement is true for all $n<k$, $k\in\mathbb{N}$. We shall show that it is true for $n=k$. Our assumption implies that I_k is a closed subset of $(\mathscr{I}_\omega^\infty,\tau)$. Later we shall denote by τ_k the topology induced from $(\mathscr{I}_\omega^\infty,\tau)$ onto I_k. Then (I_k,τ_k) is a Baire space. We define a family $\mathscr{C}_k = \{\{\beta\} \mid \beta\in I_k\}$. Since the topological space (I_k,τ_k) is Baire we have that the family $\mathscr{C}_k$ has an element with non-empty interior and hence the topological space (I_k,τ_k) has an isolated point γ in (I_k,τ_k). Let $U(\gamma)$ be an open neighbourhood $U(\gamma)$ of γ in $(\mathscr{I}_\omega^\infty,\tau)$ such that $U(\gamma)\cap I_k=\{\gamma\}$. Since $(\mathscr{I}_\omega^\infty,\tau)$ is a

semitopological semigroup we have that there exists an open neighbourhood $V(\gamma)$ of γ in $(\mathscr{I}_\omega^\infty, \tau)$ such that $V(\gamma) \subseteq U(\gamma)$ and $\gamma \cdot \gamma^{-1} \cdot V(\gamma) \subseteq U(\gamma)$. We remark that $\gamma \cdot \gamma^{-1} \cdot V(\gamma) \subseteq \{\gamma\}$. Hence by Proposition 2.5 the neighbourhood $V(\gamma)$ is finite and Hausdorffness of the space $(\mathscr{I}_\omega^\infty, \tau)$ implies that γ is an isolated point in $(\mathscr{I}_\omega^\infty, \tau)$. Let α be an arbitrary element of the set $I_k \setminus I_{k+1}$. Then $|\omega \setminus \operatorname{dom} \alpha| = k$ and hence statements $(viii) - (xi)$ of Proposition 2.2 imply that there exist $\mu, \nu \in \mathscr{I}_\omega^\infty$ such that $\mu \cdot \alpha \cdot \nu = \gamma$. Since translations in $(\mathscr{I}_\omega^\infty, \tau)$ are continuous we conclude that Hausdorffness of the space $(\mathscr{I}_\omega^\infty, \tau)$ and Proposition 2.5 imply that α is an isolated point in $(\mathscr{I}_\omega^\infty, \tau)$. This completes the proof of our theorem. ■

Remark 5.2. We observe that the statement of Theorem 5.1 holds for every topology τ on the semigroup $\mathscr{I}_\omega^\infty$ such that $(\mathscr{I}_\omega^\infty, \tau)$ is a Hausdorff semitopological semigroup and every (two-sided) ideal in $(\mathscr{I}_\omega^\infty, \tau)$ is a Baire space.

Theorem 5.1 implies the following corollary:

Corollary 5.3. *Every Čech complete (locally compact) topology τ on the semigroup $\mathscr{I}_\omega^\infty$ such that $(\mathscr{I}_\omega^\infty, \tau)$ is a Hausdorff semitopological semigroup is discrete.*

Theorem 5.4. *Let λ be an infinite cardinal and S be a topological semigroup which contains a dense discrete subsemigroup $\mathscr{I}_\lambda^\infty$. If $I = S \setminus \mathscr{I}_\lambda^\infty \neq \varnothing$ then I is an ideal of S.*

Proof. Suppose that I is not an ideal of S. Then at least one of the following conditions holds:

$$1)\ I \cdot \mathscr{I}_\lambda^\infty \nsubseteq I, \qquad 2)\ \mathscr{I}_\lambda^\infty \cdot I \nsubseteq I, \qquad \text{or} \qquad 3)\ I \cdot I \nsubseteq I.$$

Since $\mathscr{I}_\lambda^\infty$ is a discrete dense subspace of S, Theorem 3.5.8 [7] implies that $\mathscr{I}_\lambda^\infty$ is an open subspace of S. Suppose there exist $a \in \mathscr{I}_\lambda^\infty$ and $b \in I$ such that $b \cdot a = c \notin I$. Since $\mathscr{I}_\lambda^\infty$ is a dense open discrete subspace of S the continuity of the semigroup operation in S implies that there exists an open neighbourhood $U(b)$ of b in S such that $U(b) \cdot \{a\} = \{c\}$. But by Proposition 2.5 the equation $x \cdot a = c$ has finitely many solutions in $\mathscr{I}_\lambda^\infty$. This contradicts the assumption that $b \in S \setminus \mathscr{I}_\lambda^\infty$. Therefore $b \cdot a = c \in I$ and hence $I \cdot \mathscr{I}_\lambda^\infty \subseteq I$. The proof of the inclusion $\mathscr{I}_\lambda^\infty \cdot I \subseteq I$ is similar.

Suppose there exist $a, b \in I$ such that $a \cdot b = c \notin I$. Since $\mathscr{I}_\lambda^\infty$ is a dense open discrete subspace of S, the continuity of the semigroup operation in S implies that there exist open neighbourhoods $U(a)$ and $U(b)$ of a and b in S, respectively, such that $U(a) \cdot U(b) = \{c\}$. But by Proposition 2.5 the equations $x \cdot b_0 = c$ and $a_0 \cdot y = c$ have finitely many solutions in $\mathscr{I}_\lambda^\infty$. This contradicts the assumption that $a, b \in S \setminus \mathscr{I}_\lambda^\infty$. Therefore $a \cdot b = c \in I$ and hence $I \cdot I \subseteq I$. ∎

Proposition 5.5. *Let S be a topological semigroup which contains a dense discrete subsemigroup $\mathscr{I}_\lambda^\infty$. Then for every $c \in \mathscr{I}_\lambda^\infty$ the set*

$$D_c(\mathscr{I}_\lambda^\infty) = \{(x, y) \in \mathscr{I}_\lambda^\infty \times \mathscr{I}_\lambda^\infty \mid x \cdot y = c\}$$

is a closed-and-open subset of $S \times S$.

Proof. Since $\mathscr{I}_\lambda^\infty$ is a discrete subspace of S, we have that $D_c(\mathscr{I}_\lambda^\infty)$ is an open subset of $S \times S$.

Suppose that there exists $c \in \mathscr{I}_\lambda^\infty$ such that $D_c(\mathscr{I}_\lambda^\infty)$ is a non-closed subset of $S \times S$. Then there exists an accumulation point $(a, b) \in S \times S$ of the set $D_c(\mathscr{I}_\lambda^\infty)$. The continuity of the semigroup operation in S implies that $a \cdot b = c$. But $\mathscr{I}_\lambda^\infty \times \mathscr{I}_\lambda^\infty$ is a discrete subspace of $S \times S$ and hence by Theorem 5.4 the points a and b belong to the ideal $I = S \setminus \mathscr{I}_\lambda^\infty$ and hence $a \cdot b \in S \setminus \mathscr{I}_\lambda^\infty$ cannot be equal to c. ∎

A topological space X is defined to be *pseudocompact* if each locally finite open cover of X is finite. According to [7, Theorem 3.10.22] a Tychonoff topological space X is pseudocompact if and only if each continuous real-valued function on X is bounded.

Theorem 5.6. *If a topological semigroup S contains $\mathscr{I}_\lambda^\infty$ as a dense discrete subsemigroup then the square $S \times S$ is not pseudocompact.*

Proof. Since the square $S \times S$ contains an infinite closed-and-open discrete subspace $D_c(\mathscr{I}_\lambda^\infty)$, we conclude that $S \times S$ fails to be pseudocompact (see [7, Ex. 3.10.F(d)] or [6]). ∎

A topological space X is called *countably compact* if any countable open cover of X contains a finite subcover [7]. We observe that every Hausdorff countably compact space is pseudocompact.

Since the closure of an arbitrary subspace of a countably compact space is countably compact (see [7, Theorem 3.10.4]) Theorem 5.6 implies the following corollary:

Corollary 5.7. *For every infinite cardinal λ the discrete semigroup $\mathscr{I}_\lambda^\infty$ does not embed into a topological semigroup S with the countably compact square $S \times S$.*

Since every compact topological space is countably compact Theorem 3.24 [7] and Corollary 5.7 imply

Corollary 5.8. *For every infinite cardinal λ the discrete semigroup $\mathscr{I}_\lambda^\infty$ does not embed into a compact topological semigroup.*

We recall that the Stone-Čech compactification of a Tychonoff space X is a compact Hausdorff space βX containing X as a dense subspace so that each continuous map $f\colon X \to Y$ to a compact Hausdorff space Y extends to a continuous map $\overline{f}\colon \beta X \to Y$ [7].

Theorem 5.9. *For every infinite cardinal λ the discrete semigroup $\mathscr{I}_\lambda^\infty$ does not embed into a Tychonoff topological semigroup S with the pseudocompact square $S \times S$.*

Proof. By Theorem 1.3 [1] for any topological semigroup S with the pseudocompact square $S\times S$ the semigroup operation $\mu\colon S\times S \to S$ extends to a continuous semigroup operation $\beta\mu\colon \beta S\times \beta S \to \beta S$, so S is a subsemigroup of the compact topological semigroup βS. Then Corollary 5.8 implies the statement of the theorem. ■

The following example shows, that there exists a non-discrete topology τ_F on the semigroup $\mathscr{I}_\lambda^\infty$ such that $(\mathscr{I}_\lambda^\infty, \tau_F)$ is a Tychonoff topological inverse semigroup.

Example 5.10. We define a topology τ_F on the semigroup $\mathscr{I}_\lambda^\infty$ as follows. For every $\alpha \in \mathscr{I}_\lambda^\infty$ we define a family

$$\mathscr{B}_F(\alpha) = \{U_\alpha(F) \mid F \text{ is a finite subset of } \operatorname{dom}\alpha\},$$

where

$$U_\alpha(F) = \{\beta \in \mathscr{I}_\lambda^\infty \mid \operatorname{dom}\alpha = \operatorname{dom}\beta, \operatorname{ran}\alpha = \operatorname{ran}\beta \\ \text{and } (x)\beta = (x)\alpha \text{ for all } x \in F\}.$$

Since conditions (BP1)–(BP3) [7] hold for the family $\{\mathscr{B}_F(\alpha)\}_{\alpha\in\mathscr{I}_\lambda^\infty}$ we conclude that the family $\{\mathscr{B}_F(\alpha)\}_{\alpha\in\mathscr{I}_\lambda^\infty}$ is the base of the topology τ_F on the semigroup $\mathscr{I}_\lambda^\infty$.

Proposition 5.11. *$(\mathscr{I}_\lambda^\infty, \tau_F)$ is a Tychonoff topological inverse semigroup.*

Proof. Let α and β be arbitrary elements of the semigroup $\mathscr{I}_\lambda^\infty$. We put $\gamma = \alpha\cdot\beta$ and let $F = \{n_1, \dots, n_i\}$ be a finite subset of $\operatorname{dom}\gamma$. We denote $m_1 = (n_1)\alpha, \dots, m_i = (n_i)\alpha$ and $k_1 = (n_1)\gamma, \dots, k_i = (n_i)\gamma$. Then we get that $(m_1)\beta = k_1, \dots, (m_i)\beta = k_i$. Hence we have that

$$U_\alpha(\{n_1,\dots,n_i\})\cdot U_\beta(\{m_1,\dots,m_i\}) \subseteq U_\gamma(\{n_1,\dots,n_i\})$$

and

$$\big(U_\gamma(\{n_1,\dots,n_i\})\big)^{-1} \subseteq U_{\gamma^{-1}}(\{k_1,\dots,k_i\}).$$

Therefore the semigroup operation and the inversion are continuous in $(\mathscr{I}_\infty^{\nearrow}(\mathbb{N}), \tau_F)$.

We observe that the group of units $H(\mathbb{I})$ of the semigroup $\mathscr{I}_\lambda^\infty$ with the induced topology $\tau_F(H(\mathbb{I}))$ from $(\mathscr{I}_\lambda^\infty, \tau_F)$ is a topological group (see [10, pp. 313–314, Example] or [18]) and the definition of the topology τ_F implies that every $\mathscr{H}$-class of the semigroup $\mathscr{I}_\lambda^\infty$ is an open-and-closed subset of the topological space $(\mathscr{I}_\lambda^\infty, \tau_F)$. Therefore Theorem 2.20 [5] implies that the topological space $(\mathscr{I}_\lambda^\infty, \tau_F)$ is homeomorphic to a countable topological sum of topological copies of $\big(H(\mathbb{I}), \tau_F(H(\mathbb{I}))\big)$. Since every T_0-topological group is a Tychonoff topological space (see [22, Theorem 3.10] or [8, Theorem 8.4]) we conclude that the topological space $(\mathscr{I}_\lambda^\infty, \tau_F)$ is Tychonoff too. This completes the proof of the proposition. ∎

Remark 5.12. We observe that the topology τ_F on $\mathscr{I}_\lambda^\infty$ induces the discrete topology on the band $E(\mathscr{I}_\lambda^\infty)$.

Example 5.13. We define a topology τ_{WF} on the semigroup $\mathscr{I}_\lambda^\infty$ as follows. For every $\alpha \in \mathscr{I}_\lambda^\infty$ we define a family

$$\mathscr{B}_{WF}(\alpha) = \{U_\alpha(F) \mid F \text{ is a finite subset of } \operatorname{dom}\alpha\},$$

where

$$U_\alpha(F) = \{\beta \in \mathscr{I}_\lambda^\infty \mid \operatorname{dom}\beta \subseteq \operatorname{dom}\alpha \text{ and } (x)\beta = (x)\alpha \text{ for all } x \in F\}.$$

Since conditions (BP1)–(BP3) [7] hold for the family $\{\mathscr{B}_{WF}(\alpha)\}_{\alpha \in \mathscr{I}_\lambda^\infty}$ we conclude that the family $\{\mathscr{B}_{WF}(\alpha)\}_{\alpha \in \mathscr{I}_\lambda^\infty}$ is the base of the topology τ_{WF} on the semigroup $\mathscr{I}_\lambda^\infty$.

Proposition 5.14. *$(\mathscr{I}_\lambda^\infty, \tau_{WF})$ is a Hausdorff topological inverse semigroup.*

Proof. Let α and β be arbitrary elements of the semigroup $\mathscr{I}_\lambda^\infty$. We put $\gamma = \alpha \cdot \beta$ and let $F = \{n_1, \ldots, n_i\}$ be a finite subset of $\operatorname{dom}\gamma$. We denote $m_1 = (n_1)\alpha, \ldots, m_i = (n_i)\alpha$ and $k_1 = (n_1)\gamma, \ldots, k_i = (n_i)\gamma$. Then we get that $(m_1)\beta = k_1, \ldots, (m_i)\beta = k_i$. Hence we have that

$$U_\alpha(\{n_1, \ldots, n_i\}) \cdot U_\beta(\{m_1, \ldots, m_i\}) \subseteq U_\gamma(\{n_1, \ldots, n_i\})$$

and

$$\big(U_\gamma(\{n_1, \ldots, n_i\})\big)^{-1} \subseteq U_{\gamma^{-1}}(\{k_1, \ldots, k_i\}).$$

Therefore the semigroup operation and the inversion are continuous in $(\mathscr{I}_\lambda^\infty, \tau_{WF})$.

Later we shall show that the topology τ_{WF} is Hausdorff. Let α and β be arbitrary distinct points of the space $(\mathscr{I}_\lambda^\infty, \tau_{WF})$. Then only one of the following conditions holds:

(*i*) $\operatorname{dom}\alpha = \operatorname{dom}\beta$;
(*ii*) $\operatorname{dom}\alpha \neq \operatorname{dom}\beta$.

In case $\operatorname{dom}\alpha = \operatorname{dom}\beta$ we have that there exists $x \in \operatorname{dom}\alpha$ such that $(x)\alpha \neq (x)\beta$. The definition of the topology τ_{WF} implies that $U_\alpha(\{x\}) \cap U_\beta(\{x\}) = \varnothing$.

If $\operatorname{dom}\alpha \neq \operatorname{dom}\beta$, then only one of the following conditions holds:

(a) $\operatorname{dom}\alpha \subsetneqq \operatorname{dom}\beta$;
(b) $\operatorname{dom}\beta \subsetneqq \operatorname{dom}\alpha$;
(c) $\operatorname{dom}\alpha \setminus \operatorname{dom}\beta \neq \varnothing$ and $\operatorname{dom}\beta \setminus \operatorname{dom}\alpha \neq \varnothing$.

Suppose that case (a) holds. Let $x \in \operatorname{dom}\beta \setminus \operatorname{dom}\alpha$ and $y \in \operatorname{dom}\alpha$. The definition of the topology τ_{WF} implies that $U_\alpha(\{y\}) \cap U_\beta(\{x\}) = \varnothing$.

Case (b) is similar to (a).

Suppose that case (c) holds. Let $x \in \operatorname{dom}\beta \setminus \operatorname{dom}\alpha$ and $y \in \operatorname{dom}\alpha \setminus \operatorname{dom}\beta$. The definition of the topology τ_{WF} implies that $U_\alpha(\{y\}) \cap U_\beta(\{x\}) = \varnothing$.

This completes the proof of the proposition. ■

Remark 5.15. We observe that the topology τ_{WF} on $\mathscr{I}_\lambda^\infty$ induces a non-discrete topology (and hence a non-hereditary Baire topology) on the band $E(\mathscr{I}_\lambda^\infty)$. Moreover, $\mathscr{H}$-classes in $(\mathscr{I}_\lambda^\infty, \tau_{WF})$ and $(\mathscr{I}_\lambda^\infty, \tau_F)$ are homeomorphic subspaces.

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