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FIXED POINT THEOREMS FOR NONCOMPATIBLE, DISCONTINUOUS HYBRID PAIRS OF MAPPINGS ON 2-METRIC SPACES

Abstract. In this paper, we prove some fixed point theorems for two hybrid pairs of mappings in 2-metric spaces by using some weaker conditions.

1. Introduction

During sixties, the notion of 2-metric space was introduced by Gähler ([10], [11]) as a generalization of the usual notion of metric space (X, d). It has been developed extensively by Gähler and many other mathematicians ([5], [19], [20]). The topology induced by 2-metric space is called 2-metric topology which is generated by the set of all open spheres with two centres. Many authors used this topology in many applications for example EI. Naschie [6] used this sort of topology in physical applications. Many authors studied fixed point theorems in 2-metric spaces (Hsiao [12], Iseki [13]).

It has been shown by Gähler [9] that the 2-metric d is a continuous function in any of its three arguments. It need not be continuous in two arguments. A 2-metric which is continuous in all of its arguments is said to be continuous.

In 1992, Dhage [2] introduced a new class of generalized metric spaces called D-metric spaces. Dhage attempted to develop topological structures in such spaces ([3], [4], [5]). But in 2003, Mustafa and Sims [19] proved that most claims concerning the fundamental topological structures of D-metric spaces are incorrect.

Sessa [33] introduced the concept of weakly commuting maps. Jungck [14] defined the notion of compatible maps in order to generalize the concept

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of weak commutativity, and showed that weakly commuting mappings are compatible but the converse is not true.

Jungck and Rhoades ([15], [16]) defined the concepts of $\delta-$compatibility and weak compatibility between a set valued mapping and a single-valued mapping and generalized the weak commutativity defined in [8]. Several authors used these concepts to prove some fixed point theorems ([7], [28]–[32]).

Monsef et al. [1] generalized some concepts in 2-metric spaces for set valued mappings. They also proved some common fixed point theorems in 2-metric spaces.

Fixed point theorems for set valued and single valued mappings provide technique for solving variety of applied problems in mathematical sciences and engineering (e.g. Krzyska and Kubiaczyk [17], Sessa and Khan [34]).

Pant ([22] – [25]) initiated the study of noncompatible maps and introduced pointwise R-weak commutativity of mappings in [22]. He also showed that pointwise R-weak commutativity is a necessary, hence minimal, condition for the existence of a common fixed point of contractive type maps [23].

Pathak et al. [26] introduced the concept of R-weakly commuting maps of type (A), and showed that they are not compatible.

Recently, Kubiaczyk and Deshpande [18] extended the concept of R-weakly commutativity of type (A) for single valued mappings to set valued mappings and introduced weak commutativity of type (KB) which is a weaker condition than δ-compatibility. In fact, δ-compatible maps are weakly commuting of type (KB) but converse is not true. For example we can see [18], [35] and [36].

Recently, Sharma and Deshpande [35] proved a common fixed point theorem for two pairs of hybrid mappings by using weak commutativity of type (KB) on a noncomplete metric space without assuming continuity of any mapping.

In this paper, we improve, extend and generalize the results of Tas et al. [37], Fisher [7], Rashwan and Ahmed [28], Sharma and Deshpande [35].

2. Preliminaries

Definition 1. [9] Let X denotes a nonempty set and R, the set of all nonnegative numbers. Then X together with a function $d : X \times X \times X \to R$, is called a 2-metric space if it satisfies the following properties:

(1) For distinct points $x, y \in X$, there exists a point $c \in X$ such that $d(x, y, c) \neq 0$ and $d(x, y, c) = 0$ if at least two of x, y and c are equal,

(2) $d(x, y, c) = d(x, c, y) = d(y, x, c) = d(y, c, x) = d(c, x, y) = d(c, y, x)$ (Symmetry),

(3) $d(x, y, c) \leq d(x, y, z) + d(x, z, c) + d(z, y, c)$ for $x, y, c, z \in X$. (Rectangle inequality).

The function d is called a 2-metric for the space X and the pair (X, d) denotes 2-metric space. It has been shown by Gähler in [9] that 2-metric d is non-negative and although d is a continuous function in any of its three arguments, it need not be continuous in two arguments. A 2-metric d which is continuous in all of its arguments is said to be continuous.

Geometrically, the value of a 2-metric $d(x, y, c)$ represents the area of a triangle with vertices x, y and c.

Throughout this paper, let (X, d) be a 2-metric space unless mentioned otherwise, and let $B(X)$ be the class of all nonempty bounded subsets of X.

Definition 2. [27] A sequence $\{x_n\}$ in (X, d) is said to be convergent to a point x in X, denoted by $\lim\limits_{n\to\infty} x_n = x$ if

$$\lim_{n\to\infty} d(x_n, x, c) = 0 \text{ for all } c \text{ in } X.$$

The point x is called the limit of the sequence $\{x_n\}$ in X.

Definition 3. [27] A sequence $\{x_n\}$ in (X, d) is said to be a Cauchy sequence if

$$\lim_{n\to\infty} d(x_m, x_n, c) = 0 \text{ for all } c \text{ in } X.$$

Definition 4. [27] The space (X, d) is said to be complete if every Cauchy sequence in X converges to a point of X.

Remark 1. We note that, in a metric space a convergent sequence is a Cauchy sequence and in a 2-metric space a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the 2-metric d is continuous on X [21].

For all $A, B, C \in B(X)$, let $\delta(A, B, C)$ and $D(A, B, C)$ be the functions defined by

$$\delta(A, B, C) = \sup\{d(a, b, c) : a \in A, b \in B, c \in C\},$$
$$D(A, B, C) = \inf\{d(a, b, c) : a \in A, b \in B, c \in C\}.$$

If A consists of a single point a we write $\delta(A, B, C) = \delta(a, B, C)$. If B and C also consists of single points b and c, respectively, we write

$$\delta(A, B, C) = D(A, B, C) = d(a, b, c).$$

It follows immediately from the definition that:

$$\begin{aligned}\delta(A, B, C) &= \delta(A, C, B) = \delta(C, B, A) = \delta(C, A, B) = \delta(B, C, A)\\ &= \delta(B, A, C) \geq 0,\end{aligned}$$

$$\delta(A,B,C) \le \delta(A,B,E) + \delta(A,E,C) + \delta(E,B,C) \text{ for all } A,B,C,E \in B(X),$$
$$\delta(A,B,C) = 0 \text{ if at least two of } A, B \text{ and } C \text{ are singleton.}$$

Definition 5. [1] A sequence $\{A_n\}$ of subsets of a 2-metric space (X,d) is said to be convergent to a subset A of X if:

(i) given $a \in A$, there is a sequence $\{a_n\}$ in X such that $a_n \in A_n$ for $n = 1, 2, 3\ldots$ and $\lim_{n\to\infty} d(a_n, a, c) = 0$ for all c in X,

(ii) given $\varepsilon > 0$, there exists a positive integer N such that $A_n \subseteq A_\varepsilon$ for $n > N$ where A_ε is the union of all open spheres with centers in A and radius ε.

Definition 6. [1] The mappings $F : X \to B(X)$ and $f : X \to X$ are said to be weakly commuting on X if $fFx \in B(X)$ and for all C in $B(X)$,

$$\delta(Ffx, fFx, C) \le \max\{\delta(fx, Fx, C), \delta(fFx, fFx, C)\}.$$

Note that if F is a single valued mapping, then the set fFx consists of a single point. Therefore, $\delta(fFx, fFx, C) = D(fFx, fFx, C) = 0$ for all C in $B(X)$ and the above inequality reduces to the condition given by Khan (16), that is $D(Ffx, fFx, C) \le D(fx, Fx, C)$.

Definition 7. [1] The mappings $F : X \to B(X)$ and $f : X \to X$ are said to be compatible if $\lim_{n\to\infty} d(Ffx_n, fFx_n, C) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Fx_n$ for some $t \in X$ and $A \in B(X)$.

Definition 8. [1] The mappings $F : X \to B(X)$, and $f : X \to X$ are said to be δ-compatible if $\lim_{n\to\infty} \delta(Ffx_n, fFx_n, C) = 0$, whenever $\{x_n\}$ is a sequence in X such that $fFx \in B(X)$, $Fx_n \to \{t\}$ and $fx_n \to t$ for some t in X.

Definition 9. [14] The mappings $F : X \to B(X)$ and $f : X \to X$ are said to be weakly compatible if they commute at a coincidence point u in X such that $Fu = \{fu\}$ we have $Ffu = fFu$.

Note that the equation $Fu = \{fu\}$ implies that Fu is a singleton.

It can be easily shown that any δ-compatible pair $\{F, f\}$ is weakly compatible but the converse is false.

Definition 10. [18] The mappings $F : X \to B(X)$ and $f : X \to X$ are said to be weakly commuting of type (KB) at x if there exists some positive real number R such that

$$\delta(ffx, Ffx, C) \le R\delta(fx, Fx, C) \text{ for all } C \text{ in } B(X).$$

Here F and f are weakly commuting of type (KB) on X if the above inequality holds for all $x \in X$.

Every δ-compatible pair of hybrid maps is weakly commuting of type (KB) but the converse is not necessarily true. For example we can see [18], [35] and [36].

Lemma 1. [1] *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$, respectively, then the sequence $\{\delta(A_n, B_n, C)\}$ converges to $\delta(A, B, C)$.*

3. Main results

Theorem 1. *Let (X, d) be a 2-metric space and $f, g : X \to X$ be self mappings, and $F, G : X \to B(X)$ be set valued mappings such that*

(1) *$G(X) \subseteq f(X)$ and $F(X) \subseteq g(X)$,*
(2) $$\delta^2(Fx, Gy, C) \le c_1 \max\left\{D^2(fx, gy, C), \delta^2(fx, Fx, C), \delta^2(gy, Gy, C)\right\}$$ $$+ c_2 \max\{\delta(fx, Fx, C).D(fx, Gy, C), D(gy, Fx, C).\delta(gy, Gy, C)\}$$ $$+ c_3 D(fx, Gy, C).D(gy, Fx, C)$$
for all $x, y \in X$ and $C \in B(X)$ where $c_1 + 2c_2 < 1$, $c_1 + c_3 < 1$, $c_1, c_2, c_3 \ge 0$,
(3) *one of $f(X)$ or $g(X)$ is complete,*
(4) *the pairs $\{F, f\}$ and $\{G, g\}$ are weakly commuting of type (KB) at coincidence points in X.*

Then there exists a unique fixed point z in X such that

$$\{z\} = \{fz\} = \{gz\} = Fz = Gz.$$

Proof. Let $x_0 \in X$ be an arbitrary point in X. By (1), there exists a point x_1 in X such that $gx_1 \in Fx_0 = Z_0$ and for this point x_1 there exists a point x_2 in X such that $fx_2 \in Gx_1 = Z_1$ and so on. Continuing in this manner, we can define a sequence $\{x_n\}$ as follows:

$$gx_{2n+1} \in Fx_{2n} = Z_{2n}, fx_{2n+2} \in Gx_{2n+1} = Z_{2n+1}, \text{ for } n = 0, 1, 2, 3 \ldots$$

Let $V_n = \delta(Z_n, Z_{n+1}, C)$ for $n = 0, 1, 2, 3 \ldots$. By (2), we have

$$V_{2n}^2 = \delta^2(Z_{2n}, Z_{2n+1}, C) = \delta^2(Fx_{2n}, Gx_{2n+1}, C)$$
$$\le c_1 \max\left\{D^2(fx_{2n}, gx_{2n+1}, C), \delta^2(fx_{2n}, Fx_{2n}, C), \delta^2(gx_{2n+1}, Gx_{2n+1}, C)\right\}$$
$$+ c_2 \max\left\{\begin{array}{c}\delta(fx_{2n}, Fx_{2n}, C).D(fx_{2n}, Gx_{2n+1}, C), \\ D(gx_{2n+1}, Fx_{2n}, C).\delta(gx_{2n+1}, Gx_{2n+1}, C)\end{array}\right\}$$
$$+ c_3 D(fx_{2n}, Gx_{2n+1}, C).D(gx_{2n+1}, Fx_{2n}, C)$$

$$
\begin{aligned}
&\leq c_1 \max\left\{D^2(Gx_{2n-1}, Fx_{2n}, C), \delta^2(Gx_{2n-1}, Fx_{2n}, C), \delta^2(Fx_{2n}, Gx_{2n+1}, C)\right\} \\
&\quad + c_2 \max\left\{\begin{array}{c} \delta(Gx_{2n-1}, Fx_{2n}, C).D(Gx_{2n-1}, Gx_{2n+1}, C), \\ D(Fx_{2n}, Fx_{2n}, C).\delta(Fx_{2n}, Gx_{2n+1}, C) \end{array}\right\} \\
&\quad + c_3 \delta(Gx_{2n-1}, Gx_{2n+1}, C) \\
&\leq c_1 \max\left\{\delta^2(Z_{2n-1}, Z_{2n}, C), \delta^2(Z_{2n}, Z_{2n+1}, C)\right\} \\
&\quad + c_2 \max\{\delta(Z_{2n-1}, Z_{2n}, C).\delta(Z_{2n-1}, Z_{2n+1}, C), , \delta(Z_{2n}, Z_{2n+1}, C)\} \\
&\leq c_1 \max\left\{V^2_{2n-1}, V^2_{2n}\right\} + c_2 \max\{V_{2n-1}(V_{2n-1} + V_{2n}), V_{2n}\} \\
&= c_1 \max\left\{V^2_{2n-1}, V^2_{2n}\right\} + c_2 V_{2n-1}(V_{2n-1} + V_{2n}).
\end{aligned}
$$

If $V_{2n} \geq V_{2n-1}$, then we have

$$V^2_{2n} \leq (c_1 + 2c_2)V^2_{2n} < V^2_{2n}.$$

Since $c_1 + 2c_2 < 1$, this is a contradiction. Thus

$$V_{2n} < hV_{2n-1} \text{ where } h = \sqrt{(c_1 + 2c_2)} < 1.$$

Similarly we have $V_{2n+1} < hV_{2n}$ and so

$$V_{2n} = \delta(Z_{2n}, Z_{2n+1}, C) = \delta(Fx_{2n}, Gx_{2n+1}, C) \leq \ldots . \leq h^{2n}\delta(Fx_0, Gx_1, C)$$

for $n = 1, 2, 3 \ldots.$ Let z_n be an arbitrary point in Z_n for $n = 0, 1, 2, 3 \ldots$ Thus we have

$$D(z_n, z_{n+1}, C) \leq \delta(Z_n, Z_{n+1}, C) \leq \ldots . \leq h^n \delta(Fx_0, Gx_1, C).$$

Since $h < 1$, therefore the sequence $\{z_n\}$ is a Cauchy sequence in X and hence any subsequence thereof is a Cauchy sequence in X.

Suppose that $g(X)$ is complete. Since

$$gx_{2n+1} \in Fx_{2n} = Z_{2n} \text{ for } n = 0, 1, 2, 3 \ldots,$$

then

$$D(gx_{2m+1}, gx_{2n+1}, C) \leq \delta(Z_{2m}, Z_{2n}, C) < \varepsilon \text{ for } m, n \geq n_0,\ n_0 = 1, 2, 3.$$

Therefore $\{gx_{2n+1}\}$ is a Cauchy sequence and hence $gx_{2n+1} \to z = gv \in g(X)$ for $v \in X$. But $fx_{2n} \in Gx_{2n-1} = Z_{2n-1}$, so we have

$$D(fx_{2n}, gx_{2n+1}, C) \leq \delta(Z_{2n-1}, Z_{2n}, C) = V_{2n-1} \to 0.$$

Consequently, $fx_{2n} \to z$. Moreover we have for $n = 1, 2, 3 \ldots$

$$\delta(Fx_{2n}, z, C) \leq \delta(Fx_{2n}, fx_{2n}, C) + \delta(fx_{2n}, z, C).$$

Therefore $\delta(Fx_{2n}, z, C) \to 0$. Similarly $\delta(Gx_{2n-1}, z, C) \to 0$.

By (2) for $n = 1, 2, 3, \dots$, we have

$$\begin{aligned}
&\delta^2(Fx_{2n}, Gv, C)\\
&\leq c_1 \max\left\{D^2(fx_{2n}, gv, C), \delta^2(fx_{2n}, Fx_{2n}, C), \delta^2(gv, Gv, C)\right\}\\
&\quad + c_2 \max\{\delta(fx_{2n}, Fx_{2n}, C).D(fx_{2n}, Gv, C), D(gv, Fx_{2n}, C).\delta(gv, Gv, C)\}\\
&\quad + c_3 D(fx_{2n}, Gv, C).D(gv, Fx_{2n}, C)\\
&\leq c_1 \max\left\{D^2(fx_{2n}, gv, C), \delta^2(fx_{2n}, Fx_{2n}, C), \delta^2(gv, Gv, C)\right\}\\
&\quad + c_2 \max\{\delta(fx_{2n}, Fx_{2n}, C).\delta(fx_{2n}, Gv, C), \delta(gv, Fx_{2n}, C).\delta(gv, Gv, C)\}\\
&\quad + c_3 \delta(fx_{2n}, Gv, C).\delta(gv, Fx_{2n}, C)
\end{aligned}$$

and since $\delta(fx_{2n}, Gv, C) \to \delta(z, Gv, C)$ when $fx_{2n} \to z$ we get as $n \to \infty$

$$\delta^2(z, Gv, C) \leq c_1 \delta^2(z, Gv, C).$$

Since $c_1 < 1$, we see that $Gv = \{z\} = \{gv\}$.

But as $G(X) \subseteq f(X)$, there exists $u \in X$ such that $\{fu\} = Gv = \{gv\} = \{z\}$. Now if $Fu \neq Gv, \delta(Fu, Gv, C) \neq 0$. So by (2), we have

$$\begin{aligned}
\delta^2(Fu, Gv, C) &\leq c_1 \max\left\{D^2(fu, gv, C), \delta^2(fu, Fu, C), \delta^2(gv, Gv, C)\right\}\\
&\quad + c_2 \max\{\delta(fu, Fu, C).D(fu, Gv, C), D(gv, Fu, C).\delta(gv, Gv, C)\}\\
&\quad + c_3 D(fu, Gv, C).D(gv, Fu, C)\\
&\leq c_1 \max\left\{D^2(fu, gv, C), \delta^2(fu, Fu, C), \delta^2(gv, Gv, C)\right\}\\
&\quad + c_2 \max\{\delta(fu, Fu, C).\delta(fu, Gv, C), \delta(gv, Fu, C).\delta(gv, Gv, C)\}\\
&\quad + c_3 \delta(fu, Gv, C).\delta(gv, Fu, C).
\end{aligned}$$

So we have $\delta^2(Fu, Gv, C) \leq c_1 \delta^2(Fu, Gv, C)$ and since $c_1 < 1$ we can see that

$$Fu = \{fu\} = \{gv\} = Gv = \{z\}.$$

Since $Fu = \{fu\}$ and the pair $\{F, f\}$ is weakly commuting of type (KB) at coincidence points in X, we obtain $\delta(ffu, Ffu, C) \leq R\delta(fu, Fu, C)$, which gives $\{fz\} = Fz$.

Again since $Gv = \{gv\}$ and the pair $\{G, g\}$ is weakly commuting of type (KB) at coincidence points in X, we obtain $\delta(ggv, Ggv, C) \leq R\delta(gv, Gv, C)$, which gives $\{gz\} = Gz$. By (2), we have

$$\begin{aligned}
\delta^2(Fz, z, C) &\leq \delta^2(Fz, Gv, C)\\
&\leq c_1 \max\left\{D^2(fz, gv, C), \delta^2(fz, Fz, C), \delta^2(gv, Gv, C)\right\}\\
&\quad + c_2 \max\{\delta(fz, Fz, C).D(fz, Gv, C), D(gv, Fz, C).\delta(gv, Gv, C)\}\\
&\quad + c_3 D(fz, Gv, C).D(gv, Fz, C)
\end{aligned}$$

$$
\begin{aligned}
&\le c_1 \max\left\{D^2(fz,gv,C), \delta^2(fz,Fz,C), \delta^2(gv,Gv,C)\right\}\\
&\quad + c_2 \max\{\delta(fz,Fz,C).\delta(fz,Gv,C), \delta(gv,Fz,C).\delta(gv,Gv,C)\}\\
&\quad + c_3\delta(fz,Gv,C).\delta(gv,Fz,C)\\
&\le (c_1+c_3)\delta^2(Fz,z,C).
\end{aligned}
$$

Since $c_1 + c_3 < 1$, it follows that $Fz = \{z\}$.

Consequently, we have $\{z\} = Fz = \{fz\}$. Similarly $\{z\} = Gz = \{gz\}$. Therefore we have

$$\{z\} = \{fz\} = \{gz\} = Fz = Gz.$$

Finally, we prove that z is unique. If not, let w be another common fixed point such that $z \neq w$ and $\{w\} = \{fw\} = \{gw\} = Fw = Gw$.

By (2), we have

$$
\begin{aligned}
D^2(z,w,C) &\le \delta^2(Fz,Gw,C)\\
&\le c_1 \max\left\{D^2(fz,gw,C), \delta^2(fz,Fz,C), \delta^2(gw,Gw,C)\right\}\\
&\quad + c_2 \max\{\delta(fz,Fz,C).D(fz,Gw,C), D(gw,Fz,C).\delta(gw,Gw,C)\}\\
&\quad + c_3D(fz,Gw,C).D(gw,Fz,C)\\
&\le (c_1+c_3)D^2(z,w,C).
\end{aligned}
$$

Since $c_1 + c_3 < 1$ it follows that $w = z$. This completes the proof. ■

If F and G are single valued mappings in the Theorem 1, then we get the following:

Corollary 1. *Let (X,d) be a 2-metric space and $f,g,F,G : X \to X$ be self mappings satisfying the conditions* (1), (3), (4) *and*

$$
\begin{aligned}
(5)\qquad d^2(Fx,Gy,c) &\le c_1 \max\left\{d^2(fx,gy,c), d^2(fx,Fx,c), d^2(gy,Gy,c)\right\}\\
&\quad + c_2 \max\{d(fx,Fx,c).d(fx,Gy,c), d(gy,Fx,c).d(gy,Gy,c)\}\\
&\quad + c_3d(fx,Gy,c).d(gy,Fx,c)
\end{aligned}
$$

for all $x,y,c \in X$ where $c_1 + 2c_2 < 1, c_1 + c_3 < 1, c_1, c_2, c_3 \ge 0$. Then there exists a unique fixed point z in X such that $z = fz = gz = Fz = Gz$.

If we put $c_2 = c_3 = 0$ in Theorem 1, we obtain:

Corollary 2. *Let (X,d) be a 2-metric space and $f,g : X \to X$ be self mappings, and $F,G : X \to B(X)$ be set valued mappings satisfying the conditions* (1), (3), (4), *and*

$$
(6)\qquad \delta^2(Fx,Gy,C) \le c_1 \max\left\{D^2(fx,gy,C), \delta^2(fx,Fx,C), \delta^2(gy,Gy,C)\right\}
$$

for all $x,y \in X$ where $c_1 \ge 0$.

Then there exists a unique fixed point z in X such that

$$\{z\} = \{fz\} = \{gz\} = Fz = Gz.$$

If we put $F = G$ and $f = g$ in Theorem 1, then we have the following:

Corollary 3. *Let (X, d) be a 2-metric space and $f : X \to X$ be self mapping, and $F : X \to B(X)$ be set valued mapping such that*

(7) $F(X) \subseteq f(X)$,

(8) $$\delta^2(Fx, Fy, C) \le c_1 \max\left\{D^2(fx, fy, C), \delta^2(fx, Fx, C), \delta^2(fy, Fy, C)\right\} + c_2 \max\{\delta(fx, Fx, C).D(fx, Fy, C), D(fy, Fx, C).\delta(fy, Fy, C)\} + c_3 D(fx, Fy, C).D(fy, Fx, C)$$

for all $x, y \in X$ and $C \in B(X)$ where $c_1 + 2c_2 < 1, c_1 + c_3 < 1, c_1, c_2, c_3 \ge 0$,

(9) *$f(X)$ is complete,*

(10) *the pair $\{F, f\}$ is weakly commuting of type (KB) at coincidence points in X.*

Then there exists a unique fixed point z in X such that $\{z\} = \{fz\} = Fz$.

For a set valued map $F : X \to B(X)$ (respectively a single valued map $f : X \to X$), F^* (respectively f^*) will denote the set of fixed points of F (respectively f).

Theorem 2. *Let (X, d) be a 2-metric space and $f, g : X \to X$ be self mappings, and $F, G : X \to B(X)$ be set valued mappings. If the condition* (2) *holds for all $x, y \in X$, then*

$$(f^* \cap g^*) \cap F^* = (f^* \cap g^*) \cap G^*.$$

Proof. Let $u \in (f^* \cap g^*) \cap F^*$, then

$$\begin{aligned} \delta^2(u, Gu, C) &= \delta^2(Fu, Gu, C) \\ &\le c_1 \max\left\{D^2(fu, gu, C), \delta^2(fu, Fu, C), \delta^2(gu, Gu, C)\right\} \\ &\quad + c_2 \max\{\delta(fu, Fu, C).D(fu, Gu, C), D(gu, Fu, C).\delta(gu, Gu, C)\} \\ &\quad + c_3 D(fu, Gu, C).D(gu, Fu, C) \\ &= c_1 \delta^2(u, Gu, C). \end{aligned}$$

Since $c_1 < 1$, it follows that $\{u\} = Gu$. Thus

$$(f^* \cap g^*) \cap F^* \subseteq (f^* \cap g^*) \cap G^*.$$

Similarly, we can show that

$$(f^* \cap g^*) \cap G^* \subseteq (f^* \cap g^*) \cap F^*.$$

Hence

$$(f^* \cap g^*) \cap F^* = (f^* \cap g^*) \cap G^*. \ \blacksquare$$

Theorem 1 and Theorem 2 imply the following:

Theorem 3. *Let $f, g : X \to X$ and $F_n : X \to B(X), n \in N$ be mappings satisfying the condition* (3) *and the following:*

(11) $F_1 x \subseteq g(X)$ *and* $F_2 x \subseteq f(X)$,

(12) $\delta^2(F_n x, F_{n+1} y, C)$

$$\le c_1 \max\left\{D^2(fx, gy, C), \delta^2(fx, F_n x, C), \delta^2(gy, F_{n+1} y, C)\right\}$$
$$+ c_2 \max\{\delta(fx, F_n x, C).D(fx, F_{n+1} y, C), D(gy, F_n x, C).\delta(gy, F_{n+1} y, C)\}$$
$$+ c_3 D(fx, F_{n+1} y, C).D(gy, F_n x, C)$$

for all $x, y \in X$ where $c_1 + 2c_2 < 1, c_2 + c_3 < 1, c_1, c_2, c_3 \ge 0, n \in N$,

(13) *the pairs $\{F_1, f\}$ and $\{F_2, g\}$ are weakly commuting of type (KB) at coincidence points in X.*

Then f, g and $\{F_n\}_{n \in N}$ have a unique common fixed point in X.

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