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GLOBAL SOLUTION OF REACTION DIFFUSION SYSTEM WITH NON DIAGONAL MATRIX

Abstract. The purpose of this paper is to prove the global existence in time of solutions for the coupled reaction-diffusion system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\Delta u - b\Delta v = f\ (u,v) & \text{in }]0,+\infty[\times\Omega \\ \dfrac{\partial v}{\partial t} - c\Delta v = g\,(u,v) & \text{in }]0,+\infty[\times\Omega \end{cases}$$

with triangular matrix of diffusion coefficients.

By combining the Lyapunov functional method with the regularizing effect, we show that global solutions exist. Our investigation applied for a wide class of the nonlinear terms f and g.

1. Introduction

In this paper we study the following semilinear parabolic system

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\Delta u - b\Delta v = f\ (u,v)\,, & \text{in }]0,+\infty[\times\Omega, \\ \dfrac{\partial v}{\partial t} - c\Delta v = g\,(u,v) & \text{in }]0,+\infty[\times\Omega, \end{cases} \tag{1.1}$$

where Ω is a regular and bounded domain of $\mathbb{R}^n$, $(n \geq 1)$, $u = u\,(t,x)$, $v = v\,(t,x)$, $x \in \Omega$, $t > 0$ are real valued functions, Δ denotes the Laplacian operator, and the constants of diffusion a, b, and c are assumed to be nonnegative.

System (1.1) is subjected to the following boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, \qquad \text{in }]0,+\infty[\times\partial\Omega, \tag{1.2}$$

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and the initial data

$$u(0,.) = u_0, \qquad v(0,.) = v_0 \qquad \text{in } \Omega \tag{1.3}$$

which are assumed to be continuous and nonnegative.

The above system (1.1)–(1.3) arises in physics, chemistry and various biological processes including population dynamics. (See [5], [17] and references therein).

Concerning the functions f and g, we assume the following hypothesis:

(H1) $f(r,s)$ and $g(r,s)$ are continuously differentiable on $\mathbb{R}^+ \times \mathbb{R}^+$, such that

$$f(0,s) \geq 0, \text{ and } g(r,0) \geq 0 \qquad \forall r,s \geq 0. \tag{1.4}$$

(H2) Assume further that

$$\sup\left(|f(r,s)|, |g(r,s)|\right) \leq C(r+s+1)^m, \quad \forall r,s \geq 0 \tag{1.5}$$

where C is a positive constant and $m \geq 1$.

Also, we suppose that, one of the following conditions is satisfied:

(C1) There exist $p \geq 2$, $c(p) > 0$ and positive numbers $(B_i(p))_{0 \leq i \leq p}$ such that

$$B_i(p) f(r,s) + B_{i-1}(p) g(r,s) \leq c(p)(r+s+1) \tag{1.6}$$

where

$$\left[bB_{i+1}(p) + (a+c)B_i(p)\right]^2 \leq 4aB_{i+1}(p)\left[cB_{i-1}(p) + bB_i(p)\right]. \tag{1.7}$$

(C2) There exist $c(1) > 0$ and $B_i(1)$, $0 \leq i \leq 1$ such that

$$\begin{cases} B_1(1) f(r,s) + B_0(1) g(r,s) \leq c(1)(r+s+1), \\ B_0(1), \ B_1(1) > 0. \end{cases} \tag{1.8}$$

The main question we want to address is the existence of global solutions for system (1.1)–(1.3). In fact the subject of the global existence of reaction diffusion systems has received a lot of attention in the last decades and several outstanding results have been proved by some of the major experts in the field. See [3, 4, 13].

For $b = 0$, this question has been investigated by many authors by considering special forms of the nonlinear terms f and g. Note that, Alikakos [1], treated the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\Delta u = f(u,v), \text{ in }]0,+\infty[\times \Omega, \\ \dfrac{\partial v}{\partial t} - c\Delta v = g(u,v), \text{ in }]0,+\infty[\times \Omega, \end{cases} \tag{1.9}$$

with the same boundary conditions (1.2) and initial condition (1.3), where $f(u,v) = -g(u,v) = -uv^{\sigma}$, and gave a positive answer to the problem of the global existence of system (1.9), (1.2), (1.3) under the assumption

$$1 < \sigma < \sigma_0 \tag{1.10}$$

where

$$\sigma_0 = 1 + \frac{2}{n}. \tag{1.11}$$

The method used in [1] is based on some Sobolev embedding theorems.

Note that the exponent σ_0 given in (1.11) is exactly the critical exponents given by Fujita [6] for the parabolic problem

$$\begin{cases} u_t = \Delta u + u^{\sigma}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.12}$$

where u_0 in (1.12) is a nonnegative. Fujita proved that if $1 < \sigma < \sigma_0$, then (1.12) possesses no global nonnegative solutions while if $\sigma > \sigma_0$, both global and nonglobal nonnegative solutions exist, depending on the nature of the initial energy.

In [15] Masuda obtained a global existence result for a large class of the parameter σ. In fact, by using some L^p estimates, he showed that the solution of problem (1.1)–(1.3) exists globally in time if $\sigma > 1$.

The same result in [15] was obtained by Hollis *et al* [10] by exploiting the duality arguments on L^p techniques, allowing to derive the uniform boundeness of the solution.

Following Masuda's approach, Haraux and Youkana [8] established a global existence result of system (1.1)–(1.3) for a large class of the function f and g. More precisely they showed that for

$$f(u,v) = -g(u,v) = -u\varphi(v) \tag{1.13}$$

the problem (1.1)–(1.3) admits a global solution provided that the following condition holds:

$$\lim_{v \to +\infty} \frac{[\operatorname{Log}(1+\varphi(v))]}{v} = 0.$$

In the general case, that is to say for

$$f(u,v) = -g(u,v) \tag{1.14}$$

the positivity of the function $g(u,v)$ together with the maximum principle of the heat operator give the following uniform estimate of the solution in $L^{\infty}(\Omega)$

$$\|u(t)\|_{\infty} \leq \|u_0(t)\|_{\infty}, \quad \forall t \in [0, T_{\max}[$$

where $T_{\max}$ is the maximal time of existence. See Pazy [18] for more details.

Based on the Lyapunov functional method and for f and g satisfying (1.14), Kouachi [12] proved that the solution of problem (1.1)–(1.3) exists globally in time if

$$\lim_{v\to+\infty}\frac{[\mathrm{Log}\,(1+\ f\ (u,v))]}{v}<\frac{8\alpha\beta}{n\,(\alpha-\beta)^2\,\|u_0\|_\infty}.$$

Recently, Moumeni and Salah Derradji [16] have established the existence of global solution using an approach that involves the Lyapunov's functional for the system (1.1)–(1.3) where the functions f and g are assumed to satisfy the condition (1.5) and $b=0$.

If $a\neq c$, an important particular case is that when $f\leq 0$, which means that the first substance is absorbed by the reaction, in this case, the problem of the global existence of system (1.9) reduces to obtaining a uniform estimate for v, since by the maximal principle we have

$$u(x,t)\leq\|u_0\|_\infty.$$

Here the global existence when $a>c$ has been treated by Kanel and Kirane [11] for a bounded domain Ω and by Martin and Pierre [14] for whole space $\mathbb{R}^n$.

Still in the case $a\neq c$, but without assuming $a>c$, the answer is again positive to the problem of the global existence of system (1.9) under condition (1.15) and a polynomial growth assumption on g :

$$g\,(u,v)\leq C\,(u+v+1)^\gamma\,,\quad \text{for all } u,v\ \geq 0 \text{ and some } \gamma\geq 1,$$

see [10] for more details.

If the diffusion coefficients are the same, that is, if $a=c$, then system (1.9) has a global solution under the condition

$$f(u,v)+g(u,v)\leq 0, \tag{1.15}$$

which is known as the mass dissipative structure condition. Indeed if $a=c$, then the solution (u,v) of (1.9) satisfies (by summing up the two equations in (1.9))

$$(u+v)_t-a\Delta(u+v)=f+g\leq 0.$$

Then the maximal principle implies

$$0\leq u+v\leq\|u_0\|_\infty+\|v_0\|_\infty.$$

Therefore, the global existence follows.

In the present work we consider problem (1.1)–(1.3) with $b>0$, and by adopting the Lyapunov method combined with some L^p estimates we establish a global existence result of the solution when $b>0$.

The plan of the paper is as follows. In section 2, we fix notations and for the convenience of the reader, we recall without proof the local existence result. In section 3, we state and prove our main result.

2. Notation and some preliminary observations

Throughout the text we shall denote by $\|.\|_p$ the $L^p(\Omega)$ norm for $1 \leq p \leq \infty$, i.e. $\| u\|_p^p = \frac{1}{|\Omega|}\int_\Omega |u(x)|^p\,dx$ and $\|u\|_\infty = \operatorname{ess\,sup}_{x\in\Omega} |u(x)|$, also we denote by $\|u\|_{C(\overline{\Omega})} = \max_{x\in\overline{\Omega}} |u(x)|$, the usual norms in $C(\overline{\Omega})$.

First, since the functions f and g are continuously differentiable on $\mathbb{R}^+ \times \mathbb{R}^+$ then, for any initial data in $C\left(\overline{\Omega}\right)$ it is easy to check the Lipschitz continuity on bounded subsets of the domain associated to the operator

$$A := \begin{pmatrix} a\Delta & b\Delta \\ 0 & c\Delta \end{pmatrix}.$$

Then, from the basic existence theory (see Pazy [18]) the problem admits unique classical solution (u, v) defined on $[0, T_{\max}[\times\Omega$. More precisely, under the above assumptions, we have the following theorem.

Theorem 2.1. *System* (1.1)–(1.3) *admits a unique classical solution* (u, v) *defined on* $(0, T_{\max}] \times \Omega$. *Moreover, if $T_{max} < \infty$, then*

$$\lim_{t\to T_{\max}} \{\|u(t, .)\|_\infty + \|v(t, .)\|_\infty\} = \infty.$$

In this case, $T_{\max}(\|u_0\|_\infty, \|v_0\|_\infty)$ is called the blowing up time.

Remark 2.1. Under condition (H1), it follows from the invariant region method that system (1.1)–(1.3) preserves positivity. In other words, if the initial data u_0 and v_0 in (1.3) are nonnegative, then the functions u and v of the corresponding solution of (1.1)–(1.3) are also nonnegative on $]0, T_{\max}[\times\Omega$. See [9].

3. Main results

In this section, we state and prove our global existence result of system (1.1)–(1.3). Our main theorem reads as follows.

Theorem 3.1. *Let $p > \frac{mn}{2}$. Assume that condition* (1.5) *holds and one of the conditions* (1.6) *or* (1.8) *are satisfied. Then the solution* $(u\,(t, .)\,, v\,(t, .))$ *of* (1.1)–(1.3) *exists globally in time.*

We note that to prove Theorem 3.1 it is sufficient to derive a uniform estimate of

$$\sup(\|f\,(u, v)\|_q\,, \|g\,(u, v)\|_q)$$

for some $q > n/2$. (See [9] for more details).

The following lemma is a useful tool in the proof of the Theorem 3.1.

Lemma 3.1. *Let* $(u(t,.), v(t,.))$ *be the solution of* (1.1)–(1.3). *If one of the conditions* (1.6) *or* (1.8) *are satisfied, then there exists an integer* $p \geq 1$ *and a continuous function* $C_p : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *such that*

$$\sup(\|u(t,.)\|_p, \|v(t,.)\|_p) \leq C_p(t), \qquad t < T_{\max}.$$

Proof. Let us consider the function L_p defined by

$$L_p(t) = \int_\Omega \Big(\sum_{i=0}^{p} C_p^i B_i(p) u^i v^{p-i}\Big) dx = \int_\Omega \Big(\sum_{i=0}^{p} \alpha_i(p) u^i v^{p-i}\Big) dx \tag{3.1}$$

where

$$\alpha_i(p) = C_p^i B_i(p), \qquad i = 0, \dots, p \tag{3.2}$$

and

$$C_p^i = \frac{p!}{i!(p-i)!}.$$

Differentiating L_p with respect to t yields

$$\begin{aligned} L_p'(t) &= \int_\Omega \Big[\Big(\sum_{i=1}^{p} i\alpha_i(p) u^{i-1} v^{p-i}\Big)\frac{\partial u}{\partial t} + \Big(\sum_{i=0}^{p-1} (p-i)\alpha_i(p) u^i v^{p-i-1}\Big)\frac{\partial v}{\partial t}\Big] dx \\ &= \int_\Omega \Big[\Big(\sum_{i=1}^{p} i\alpha_i(p) u^{i-1} v^{p-i}\Big)\frac{\partial u}{\partial t} + \Big(\sum_{i=1}^{p} (p-i+1)\alpha_{i-1}(p) u^{i-1} v^{p-i}\Big)\frac{\partial v}{\partial t}\Big] dx. \end{aligned}$$

A simple computation leads

$$\begin{aligned} L_p'(t) = &\int_\Omega \Big(\sum_{i=1}^{p} i\alpha_i(p) u^{i-1} v^{p-i}\Big)(f(u,v) + a\Delta u + b\Delta v)\, dx \\ &+ \int_\Omega \Big(\sum_{i=1}^{p} (p-i+1)\alpha_{i-1}(p) u^{i-1} v^{p-i}\Big)(g(u,v) + c\Delta v) dx. \end{aligned}$$

From the above equality, it follows that

$$\begin{aligned} &L_p'(t) \\ &= \int_\Omega \Big(\sum_{i=1}^{p} \{i\alpha_i(p) f(u,v) + (p-i+1)\alpha_{i-1}(p) g(u,v)\} u^{i-1} v^{p-i}\Big) dx \\ &\quad + \int_\Omega \Big(\sum_{i=1}^{p} \{a i\alpha_i(p)\Delta u + [b i\alpha_i(p) + c(p-i+1)\alpha_{i-1}(p)]\Delta v\} u^{i-1} v^{p-i}\Big) dx. \end{aligned} \tag{3.3}$$

We distinguish two cases:

Case 1: when $p=1$, we obtain from (3.3)

$$L_1'(t)=\int_\Omega [a\alpha_1(1)\Delta u+(b\alpha_1(1)+c\alpha_0(1))\Delta v]\,dx$$
$$+\int_\Omega (\alpha_1(1)f(u,v)+\alpha_0(1)g(u,v))\,dx.$$

By a simple use of Green's formula, we obtain

$$L_1'(t)=\int_\Omega (\alpha_1(1)f(u,v)+\alpha_0(1)g(u,v))\,dx$$
$$=\int_\Omega (B_1(1)f(u,v)+B_0(1)g(u,v))\,dx.$$

Using condition (1.8) we deduce,

$$L_1'(t)\le c(1)\int_\Omega (u+v+1)\,dx=c(1)\int_\Omega (u+v)\,dx+c(1)\,mes(\Omega).$$

Then the functional L_1 satisfies the following differential inequality

$$L_1'(t)\le c_1(1)L_1(t)+c_2(1),\quad \forall t<T_{\max}$$

where

$$c_1(1)=\frac{c(1)}{\min(\alpha_1(1),\alpha_0(1))},\qquad c_2(1)=c(1)\,mes(\Omega).$$

A simple integration of the above inequality gives

$$L_1(t)\le\left[L_1(0)+\frac{c_2(1)}{c_1(1)}\right]\exp(c_1(1)t)-\frac{c_2(1)}{c_1(1)},\quad \forall t<T_{\max}.$$

It's not hard to see that from (3.1) we obtain

$$L_1(t)\ge\min(\alpha_1(1),\alpha_0(1))\int_\Omega (u+v)\,dx$$
$$\ge\min(\alpha_1(1),\alpha_0(1))\sup(\|u(t,.)\|_1,\|v(t,.)\|_1).$$

Then we get

$$\sup\|u(t,.)\|_1,\|v(t,.)\|_1\le c_1(t),\quad \forall t<T_{\max}\tag{3.4}$$

where

$$c_1(t)=\frac{1}{\min(\alpha_1(1),\alpha_0(1))}\left\{\left[L_1(0)+\frac{c_2(1)}{c_1(1)}\right]\exp(c_1(1)t)-\frac{c_2(1)}{c_1(1)}\right\}.$$

Case 2: when $p\ge 2$, we set

$$T=\int_\Omega\Big(\sum_{i=1}^{p}\{ai\alpha_i(p)\Delta u+[bi\alpha_i(p)+c(p-i+1)\alpha_{i-1}(p)]\Delta v\}\,u^{i-1}v^{p-i}\Big)dx$$
$$=\sum_{i=1}^{p}\int_\Omega \Delta\{ai\alpha_i(p)u+[bi\alpha_i(p)+c(p-i+1)\alpha_{i-1}(p)]v\}\,u^{i-1}v^{p-i}dx.$$

Then, Green's formula gives

$$T=-\sum_{i=1}^{p}\int_{\Omega}\left[\nabla\left\{ai\alpha_i\left(p\right)u+b\left(p-i+1\right)\alpha_{i-1}\left(p\right)v\right\}\nabla\left(u^{i-1}v^{p-i}\right)\right]dx$$

which implies

$$\begin{aligned}T=-\int_{\Omega}\Big[&\sum_{i=2}^{p}a\left(i-1\right)i\alpha_i\left(p\right)u^{i-2}v^{p-i}\left(\nabla u\right)^2\\&+\sum_{i=1}^{p-1}ai\left(p-i\right)\alpha_i\left(p\right)u^{i-1}v^{p-i-1}\nabla u\nabla v\\&+\sum_{i=2}^{p}\left[bi\left(i-1\right)\alpha_i\left(p\right)+c\left(p-i+1\right)\left(i-1\right)\alpha_{i-1}\left(p\right)\right]u^{i-2}v^{p-i}\nabla u\nabla v+\\&\sum_{i=1}^{p-1}\left[bi\left(i-1\right)\alpha_i(p)+c(p-i+1)\left(p-i\right)\alpha_{i-1}(p)\right]u^{i-1}v^{p-i-1}\left(\nabla v\right)^2\Big]dx\end{aligned}$$

and therefore,

$$\begin{aligned}T=-\int_{\Omega}\Big\{\sum_{i=1}^{p-1}\Big[&ai\left(i+1\right)\alpha_{i+1}\left(p\right)\left(\nabla u\right)^2+\left[\left(a+c\right)i\left(p-i\right)\alpha_i\left(p\right)\right.\\&\left.+bi\left(i+1\right)\alpha_{i+1}\left(p\right)\right]\nabla u\nabla v+\left[c\left(p-i\right)\left(p-i+1\right)\alpha_{i-1}\left(p\right)\right.\\&\left.+bi\left(p-i\right)\alpha_i\left(p\right)\right]\left(\nabla v\right)^2\Big]u^{i-1}v^{p-i-1}\Big\}dx.\end{aligned}\tag{3.5}$$

Since

$$\alpha_i\left(p\right)=C_p^iB_i\left(p\right),\quad i=0,\ldots,p$$

then, (3.3) becomes

$$\begin{aligned}L_p'\left(t\right)=&\int_{\Omega}\Big(\sum_{i=1}^{p}\left\{iC_p^iB_i\left(p\right)f+\left(p-i+1\right)C_p^{i-1}B_{i-1}\left(p\right)g\right\}u^{i-1}v^{p-i}\Big)dx\\&-\int_{\Omega}\Big\{\sum_{i=1}^{p-1}\{ai\left(i+1\right)C_p^{i+1}B_{i+1}\left(p\right)\left(\nabla u\right)^2\\&+\left[i\left(i+1\right)bC_p^{i+1}B_{i+1}\left(p\right)+\left(a+c\right)i\left(p-i\right)C_p^iB_i\left(p\right)\right]\nabla u\nabla v\\&+\left[c(p-i)(p-i+1)C_p^{i-1}B_{i-1}(p)+bi(p-i)C_p^iB_i(p)\right](\nabla v)^2]u^{i-1}v^{p-i-1}\Big\}dx.\end{aligned}$$

Using the fact that

$$iC_p^i=\left(p-i+1\right)C_p^{i-1}=pC_{p-1}^{i-1},$$

and also since

$$i\,(i+1)\,C_p^{i+1} = i\,(p-i)\,C_p^{i} = (p-i)\,(p-i+1)\,C_p^{i-1} = p\,(p-1)\,C_{p-2}^{i-1},$$

we conclude

$$\begin{aligned}
L_p'(t) &= \int_\Omega \Big(\sum_{i=1}^{p} pC_{p-1}^{i-1}\left[B_i(p)\,f + B_{i-1}(p)\,g\right]u^{i-1}v^{p-i}\Big)dx \\
&- p(p-1)\int_\Omega \Big\{\sum_{i=1}^{p-1} C_{p-2}^{i-1}\Big[aB_{i+1}(p)(\nabla u)^2 + [bB_{i+1}(p) + (a+c)B_i(p)]\nabla u \nabla v \\
&+ \left[cB_{i-1}(p) + bB_i(p)\right](\nabla v)^2\Big]u^{i-1}v^{p-i-1}\Big\}dx.
\end{aligned}$$

The quadratic forms

$$aB_{i+1}(p)\,(\nabla u)^2 + \left[bB_{i+1}(p) + (a+c)\,B_i(p)\right]\nabla u \nabla v + \left[cB_{i-1}(p) + bB_i(p)\right](\nabla v)^2$$

are positive since from (1.7) we have

$$\left[bB_{i+1}(p) + (a+c)\,B_i(p)\right]^2 - 4aB_{i+1}(p)\left[cB_{i-1}(p) + bB_i(p)\right] \le 0.$$

Consequently,

$$L_p'(t) \le p\int_\Omega \Big(\sum_{i=1}^{p} C_{p-1}^{i-1}\left[B_i(p)\,f(u,v) + B_{i-1}(p)\,g(u,v)\right]u^{i-1}v^{p-i}\Big)dx.$$

Using condition (1.6), we deduce that

$$\begin{aligned}
L_p'(t) &\le c'(p)\int_\Omega \Big(\sum_{i=1}^{p} C_{p-1}^{i-1}(u+v+1)\,u^{i-1}v^{p-i}\Big)dx \\
&\le c'(p)\int_\Omega \Big(\sum_{i=1}^{p} C_{p-1}^{i-1}u^{i}v^{p-i} + \sum_{i=1}^{p} C_{p-1}^{i-1}u^{i-1}v^{p-i+1} + \sum_{i=1}^{p} C_{p-1}^{i-1}u^{i-1}v^{p-i}\Big)dx \\
&\le c'(p)\int_\Omega \Big(\sum_{i=1}^{p} C_{p-1}^{i-1}u^{i}v^{p-i} + \sum_{i=0}^{p-1} C_{p-1}^{i}u^{i}v^{p-i} + \sum_{i=0}^{p-1} C_{p-1}^{i}u^{i}v^{p-i-1}\Big)dx \\
&\le c'(p)\int_\Omega \Big(\sum_{i=0}^{p} C_{p}^{i}u^{i}v^{p-i}\Big)dx + c'(p)\int_\Omega \Big(\sum_{i=0}^{p-1} C_{p-1}^{i}u^{i}v^{p-i-1}\Big)dx.
\end{aligned}$$

Using the fact that

$$\sum_{i=0}^{p-1} C_{p-1}^{i}u^{i}v^{p-i-1} = (u+v)^{p-1}.$$

Therefore, the last inequality can be written as

$$L'_p(t) \leq c_1(p)\, L_p(t) + c'(p) \int_\Omega (u+v)^{p-1}\, dx.$$

Applying Hôlder's inequality to the second term in the right hand side of the above inequality, we obtain

$$L'_p(t) \leq c_1(p)\, L_p(t) + c'(p)\,(mes(\Omega))^{1/p} \left(\int_\Omega (u+v)^p\, dx\right)^{(p-1)/p}.$$

Since the following inequality holds,

$$(u+v)^p = \sum_{i=0}^{p} C_p^i u^i v^{p-i} \leq \frac{\sup_{0\leq i\leq p} C_p^i}{\min_{0\leq i\leq p} \alpha_i(p)} \sum_{i=0}^{p} \alpha_i(p)\, u^i v^{p-i}.$$

Then, we have

$$L'_p(t) \leq c_1(p) L_p(t) + c'(p)(mes\Omega)^{1/p} \left(\frac{\sup_{0\leq i\leq p} C_p^i}{\min_{0\leq i\leq p} \alpha_i(p)}\right)^{(p-1)/p} (L_p(t))^{(p-1)/p}.$$

Hence, the functional L_p satisfies the following differential inequality

$$L'_p(t) \leq c_1(p)\, L_p(t) + c_2(p)\,(L_p(t))^{(p-1)/p}\ ,\ \forall t < T_{\max} \tag{3.6}$$

$$c_2(p) = c'(p)\,(mes\Omega)^{1/p} \left(\frac{\sup_{\leq i\leq p} C_p^i}{\min_{0\leq i\leq p} \alpha_i(p)}\right)^{(p-1)/p}$$

which gives us, by a simple integration

$$(L_p(t))^{1/p} \leq \left[(L_p(0))^{1/p} + \frac{c'_2(p)}{c'_1(p)}\right] \exp\left(c'_1(p)\, t\right) - \frac{c'_2(p)}{c'_1(p)}$$

where

$$c'_1(p) = \frac{c_1(p)}{p} \text{ and } c'_2(p) = \frac{c_2(p)}{p}.$$

By using the inequality

$$L_p(t) = \int_\Omega \left(\sum_{i=0}^{p} \alpha_i(p)\, u^i v^{p-i}\right) dx \geq \int_\Omega \left[\alpha_p(p)\, u^p + \alpha_0(p)\, v^p\right] dx \tag{3.7}$$

it follows that

$$L_p(t) \geq \min\left(\alpha_0(p)\,,\ \alpha_p(p)\right) \sup\left(\int_\Omega u^p dx,\ \int_\Omega v^p dx\right).$$

Hence,

$$(L_p(t))^{1/p} \geq \left[\min\left(\alpha_0(p)\,, \alpha_p(p)\right)\right]^{1/p} \sup\left(\left(\int_\Omega u^p dx\right)^{1/p}, \left(\int_\Omega v^p dx\right)^{1/p}\right).$$

And therefore,

$$\text{(3.8)} \quad \sup\left(\|u(t,.)\|_p, \|v(t,.)\|_p\right) \le \frac{(L_p(t))^{1/p}}{[\min(\alpha_0(p),\ \alpha_p(p))]^{1/p}}, \quad \forall\, t < T_{\max}.$$

With (3.7) and (3.8) we obtain

$$\text{(3.9)} \qquad \sup\left(\|u(t,.)\|_p, \|v(t,.)\|_p\right) \le c_p(t), \quad \forall t < T_{\max}$$

where

(3.10)

$$c_p(t) = \frac{1}{[\min(\alpha_0(p), \alpha_p(p))]^{1/p}} \left\{ \left((L_p(0))^{1/p} + \frac{c_2'(p)}{c_1'(p)} \right) e^{\left(c_1'(p)t\right)} - \frac{c_2'(p)}{c_1'(p)} \right\}.$$

The proof of Lemma 3.1 is complete. ■

Proof of Theorem 3.1. From (1.5) we have

$$\sup(|f(u,v)|, |g(u,v)|) \le C(u+v+1)^m.$$

Then, it follows that

$$\sup\left(\int_\Omega |f(u,v)|^{p/m} dx, \int_\Omega |g(u,v)|^{p/m} dx\right) \le C^{p/m} \int_\Omega (u+v+1)^p dx$$

which implies

$$\text{(3.11)} \qquad \sup\left(\|f(u,v)\|_{p/m}^{p/m}, \|g(u,v)\|_{p/m}^{p/m}\right) \le C^{p/m} \int_\Omega (u+v+1)^p dx.$$

On the other hand, we have

$$\begin{aligned}\int_\Omega (u+v+1)^p dx &= \int_\Omega \Big[\sum_{k=0}^{p} C_p^k (u+v)^k \Big] dx \\ &= \int_\Omega [1+(u+v)^p]\, dx + \sum_{k=1}^{p-1} C_p^k \int_\Omega (u+v)^k dx.\end{aligned}$$

An application of Hôlder's inequality leads

$$\sum_{k=1}^{p-1} \int_\Omega (u+v)^k dx \le \sum_{k=1}^{p-1} C_p^k \left[\left(\int_\Omega 1^{p/(p-k)} dx \right)^{(p-k)/p} \left(\int_\Omega (u+v)^p dx \right)^{k/p} \right].$$

Hence

$$\begin{aligned}\text{(3.12)} \quad \int_\Omega (u+v+1)^p dx &\le mes(\Omega) + \int_\Omega (u+v)^p dx \\ &+ \sum_{k=1}^{p-1} C_p^k (mes(\Omega))^{(p-k)/p} \left(\int_\Omega (u+v)^p dx \right)^{k/p}\end{aligned}$$

using (3.9) we get

$$\left(\int_\Omega (u+v)^p\,dx\right)^{1/p} = \|u(t,.)+v(t,.)\|_p \le \|u(t,.)\|_p + \|v(t,.)\|_p \le 2c_p(t)$$

and the inequality (3.12) can be written as follows

$$\begin{aligned}\int_\Omega (u+v+1)^p\,dx &\le mes(\Omega) + 2^p(c_p(t))^p \\ &\quad + \sum_{k=1}^{p-1} 2^k C_p^k (mes(\Omega))^{(p-k)/p}(c_p(t))^k \\ &\le \sum_{k=0}^{p} 2^k C_p^k (mes(\Omega))^{(p-k)/p}(c_p(t))^k .\end{aligned}$$

Therefore

$$\begin{aligned}&\sup\left(\|f(u,v)\|_{p/m}^{p/m}, \|g(u,v)\|_{p/m}^{p/m}\right) \\ &\qquad \le C^{p/m}\Big[\sum_{k=0}^{p} 2^k C_p^k (mes(\Omega))^{(p-k)/p}(c_p(t))^k\Big]\end{aligned}$$

which gives that

$$\sup \|f(u,v)\|_{p/m}, \|g(u,v)\|_{p/m} \le c_{p,m}(t), \quad \forall t < T_{\max} \tag{3.13}$$

where

$$c_{p,m}(t) = c\Big[\sum_{k=0}^{p} 2^k C_p^k (mes(\Omega))^{(p-k)/p}(c_p(t))^k\Big]^{m/p} \quad \text{and} \quad \frac{p}{m} > \frac{n}{2}. \tag{3.14}$$

Remark 3.1. From both Lemma 3.1 and Theorem 3.1, we have obtained an uniform estimate of $\sup(\|f(u,v)\|_q, \|g(u,v)\|_q)$ with $q = p/m > n/2$. By the preliminary remarks, we conclude that the solution of the given problem exists globally in time.

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