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A ZERO-SUM COMPETITIVE MULTI-PLAYER GAME

Dedicated to Professor Agnieszka Plucińska
on the occasion of her 80th birthday

Abstract. A single period, zero-sum, multi-player game is constructed. Each player can either exit the game for a fixed payoff or stay and split the remaining payoff with the other non-exiting players. The emphasis is put on the rivalrous nature of the payoffs, meaning that the sum of all payoffs is fixed, but the exact allocation is based on the players' decisions. The value at which Nash and optimal equilibria are attained is shown to be unique and it is constructed explicitly.

1. Introduction

The classic *Dynkin game,* introduced by Dynkin [3] and extended by Neveu [12], is a zero-sum, optimal stopping game between two players where each player can stop the game for a payoff observable at that time. Much research has been done on this as well as its related problems (see, for instance, [1, 4, 9, 11, 13, 14, 15, 17] and the references therein). One application of Dynkin games is in the study of a game contingent claim, or a *game option*, as defined by Kifer [10] (see also Kallsen and Kühn [7]), who proved the existence and uniqueness of its arbitrage price.

Various formulations of multi-player Dynkin games can be found in the literature. Solan and Vieille [15] introduced a quitting game, which terminates when any player chooses to quit; then each player receives a payoff depending on the set of players quitting the game. Under certain payoff conditions, a subgame-perfect uniform ϵ-equilibrium using cyclic strategies can be found. In Solan and Vieille [16], another version is presented, in which the players are given the opportunity to stop the game in a turn-based fashion. A subgame-perfect ϵ-equilibrium was again shown to exist and consisted of

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pure strategies when the game is not degenerate. Hamadène and Hassani [6] presented a nonzero-sum Dynkin game where each player has his own separate payoff process. These processes are independent of the other players' decisions. Consequently, in the event where a player does not stop first, his payoff does not depend on who stopped the game exactly.

The purpose of this work is to formulate a simple variant of the Dynkin game with more than two players in a manner that allows for the construction of a multi-player extension of the financial game option. A zero-sum, simultaneous multi-player game is introduced, with a focus on designing the dependence between the payoffs of all players and their stopping decisions. In effect, we are modelling a multilateral 'contract' where all the players are sharing a fixed total sum of wealth. But each player also has the option to exit or terminate the contract for a predetermined benefit. The non-exiting players will receive an adjusted benefit to reflect the discrepancies caused by these exiting decisions. These adjustments are distributed among the remaining players according to their 'powers', a predetermined property of each player, to ensure that the total wealth is fixed.

Specifically, we present a new variant of a deterministic stopping game with a single opportunity for simultaneous exercise. The main result of the paper is Theorem 3.2 in which we show that the game has the unique payoff value, where Nash equilibria, as well as optimal equilibria, are achieved. The uniqueness of the value (but not its existence and explicit construction) can be seen as a special case of results due to Kats and Thisse [8] and De Wolf [2], who showed that, in any weakly unilaterally competitive game, all Nash equilibria and optimal equilibria achieve the same value. For completeness, an independent proof of this property for the particular game considered in this work is provided. The existence of the optimal equilibrium is shown by explicit construction and the value of the game is expressed as the projection onto a simplex in a fixed-sum coordinate system.

It is worth noting that a study of the game introduced in this paper is merely a preliminary step towards an analysis of nonzero-sum multi-player stochastic stopping games. Indeed, there are many opportunities for further generalisations and extensions. In particular, all results can be applied to the stochastic case where both terminal and exercise payoffs are allowed to be random and adapted to some filtration. Analogous multi-player financial game options can be constructed as well, where the properties of the optimal equilibrium will be reflected in the super-hedging arguments used to price these options within the framework of a complete model of the financial market. Some of these extensions are presented in the follow-up paper by Guo [5].

The paper is organised as follows. In Section 2, we construct the game and discuss its basic features. Various concepts of equilibrium are introduced and some preliminary results are established. In Section 3, we present the existence and uniqueness of the value of the game by analysing projections in a fixed sum coordinate system. The crucial feature of the game is a constructive way of obtaining its value. Throughout this paper, the game option terminology of 'exercise' will be utilised when referring to the 'stopping' or 'quitting' of the game by the players. The corresponding payoff from doing so will be called 'exercise payoffs'.

2. Preliminary results

Let us first recall briefly the mechanism of a two player game option, as defined in Kifer [10]. The *game option* is a contract where the buyer can exercise the option at any time t for a payoff X_t, while the seller can cancel (or also 'exercise') the option at any time t for a cancellation fee of Y_t paid to the buyer. In the case of a two player game option, it is common to postulate that the inequality $X_t \leq Y_t$ holds for every t. In other words, the cancellation fee should always at least as great as the exercise payoff. This ensures that the outcome of the contract will always be well defined. When the buyer exercises, it will cost the seller at least as much if he also cancels. Similarly, when the seller cancels, the buyer can only lose by exercising. If the players are exercising optimally, simultaneous exercise only occurs when the equality $X_t = Y_t$ is true, in which case the payoff is still well defined.

There are various way to generalise the exercising mechanism of the two player version. In this work, we restrict ourselves to the special case where the players are only allowed to exercise at one predetermined time. Another issue is the 'burden' of one player exercising. In the two player version, when one player exercises, the entirety of that payoff (or 'cost'), whether positive of negative, is paid for by the other player. In the multi-player version, this burden will be paid for by all of the non-exercising players. It will be split up according to the 'power' of each player, which is agreed before the game is played. The restriction $X_t \leq Y_t$ in the two player contract translates to $\sum_k X_k \leq 0$ with k running over the set of all players. This is required to prevent the possibility of everyone from exercising simultaneously without being able to fund the cost of the outcome. In fact, the value of the sum can be changed from 0 to any constant C, but we still refer to it as a zero-sum game.

We will now set up the notation for a zero-sum, single-period, deterministic game with m players. There are m *players*, that is, parties involved in the contract, enumerated by the indices $1, 2, \dots, m$; the set of all players is denoted by $\mathcal{M}$. The *terminal payoff* P_k is the amount received by player k if no player exercises. The vector of terminal payoffs $[P_1, \dots, P_m]$ is denoted by $\boldsymbol{P}$. The *exercise payoff* X_k is the amount received by player k if he exercises at time 0. The vector of exercise payoffs $[X_1, \dots, X_m]$ is denoted by $\boldsymbol{X}$.

We define the *total value* of the game as the sum of the terminal payoffs and we denote it by C, that is, $C := \sum_{k\in\mathcal{M}} P_k$. Intuitively, the total value of the game can be understood as the total contribution by the players to enter the game in the first place. The following standing assumption is crucial, since it ensures that, no matter how the players exercise, the total payoffs can be financed by the total initial investment C (see Lemma 2.1).

Assumption 2.1. We postulate that

$$\sum_{k\in\mathcal{M}} X_k \leq \sum_{k\in\mathcal{M}} P_k = C.$$

The *strategy* $s_k \in \mathcal{S}_k$ of player k specifies whether player k will exercise, where $\mathcal{S}_k = \{0,1\}$ is the space of strategies for player k. In particular, $s_k = 0$ means that player k will *exercise* at time 0, whereas $s_k = 1$ means that player k will not exercise. We define

$$\mathcal{U} = \prod_{k\in\mathcal{M}} \mathcal{S}_k$$

to be the space of all m-tuples of strategies and we call any $u \in \mathcal{U}$ a *strategy profile*. Given a strategy profile $u \in \mathcal{U}$, the *exercise set*, denoted by $\mathcal{E}(u)$, is the set of players who exercised at time 0.

The main difference between the game introduced here and its counterparts studied in [6, 15, 16] is the *rivalrous* nature of the payoffs, meaning that the total value of the game is fixed and shared between the players. Hence each exercise action causes a suitable redistribution of the payoffs. The *difference due to exercise*, denoted by $D(u)$, equals

$$D(u) = \sum_{k\in\mathcal{E}(u)} (X_k - P_k).$$

For any strategy profile $u \in \mathcal{U}$, we define the *weights* $w_k(u)$ for all $k \in \mathcal{M} \setminus \mathcal{E}(u)$ as real numbers such that $w_k(u) \geq 0$ and $\sum_{k\in\mathcal{M}\setminus\mathcal{E}(u)} w_k(u) = 1$.

Given a strategy profile $u \in \mathcal{U}$, the *modified payoff* of player k, denoted by $V_k(u)$, is the actual payoff received by player k if a strategy profile u is carried out. By definition, it equals

$$V_k(u) = \begin{cases} X_k, & k \in \mathcal{E}(u), \\ P_k(u), & k \in \mathcal{M} \setminus \mathcal{E}(u), \end{cases}$$

where $P_k(u) := P_k - w_k(u)D(u)$ specifies the payoff for a non-exercising player k. The vector of modified payoffs $[V_1(u), \dots, V_m(u)]$ is denoted as $\boldsymbol{V}(u)$. We are in a position to define the class of games examined in this work.

Definition 2.1. The *zero-sum m-player game*, denoted by $\mathcal{G}$, is defined by:

(i) the set of m players $\mathcal{M} = \{1, 2, \dots, m\}$,
(ii) the real valued exercise and terminal payoffs, X_k and P_k for every $k \in \mathcal{M}$,
(iii) the class $\mathcal{U} = \prod_{k \in \mathcal{M}} \mathcal{S}_k$ of strategy profiles,
(iv) the weights $w_k(u) \geq 0, k \in \mathcal{M} \setminus \mathcal{E}(u)$ for any strategy profile $u \in \mathcal{U}$, where $\mathcal{E}(u)$ is the exercise set of a strategy profile u,
(v) the modified payoffs $V_k(u)$ for all $k \in \mathcal{M}$ and each strategy profile u.

The following result justifies the term *zero-sum* used in Definition 2.1.

Lemma 2.1. (i) *If a strategy profile u is such that not all players exercise at time 0, that is, $\mathcal{E}(u) \neq \mathcal{M}$, then $\sum_{k \in \mathcal{M}} V_k(u) = C$.*
(ii) *If all players exercise at time 0, that is, $\mathcal{E}(u) = \mathcal{M}$, then $\sum_{k \in \mathcal{M}} V_k(u) \leq C$.*

Proof. The first statement is an immediate consequence of the definition of the vector of modified payoffs. The second statement follows from Assumption 2.1, since $\sum_{k \in \mathcal{M}} V_k(u) = \sum_{k \in \mathcal{M}} X_k \leq C$. ∎

2.1. Nash equilibrium. A strategy profile is referred to as a *Nash equilibrium* if no player can improve his modified payoff by altering only his own strategy.

Definition 2.2. A strategy profile $u^* \in \mathcal{U}$ is a *Nash equilibrium* if, for all $k \in \mathcal{M}$,

$$V_k\left(\left[s_k^*, s_{-k}^*\right]\right) \geq V_k\left(\left[s_k, s_{-k}^*\right]\right), \quad \forall\, s_k \in \mathcal{S}_k. \tag{1}$$

We will now examine some basic features of a Nash equilibrium for our game. We show, in particular, that any player, whose terminal payoff is less than his exercise payoff, will exercise in any Nash equilibrium.

Proposition 2.1. *Let a strategy profile u^* be a Nash equilibrium. Then:*

(i) $\sum_{k \in \mathcal{M}} V_k(u^*) = C$,
(ii) $D(u^*) \geq 0$,
(iii) *for each player $k \in \mathcal{M}$ we have that $X_k \leq V_k(u^*)$,*
(iv) *for each player $k \in \mathcal{M} \setminus \mathcal{E}(u^*)$ we have that $V_k(u^*) \leq P_k$,*
(v) *if $X_k > P_k$ then $k \in \mathcal{E}(u^*)$.*

Proof. To prove part (i), we argue by contradiction. Assume that u^* is a Nash equilibrium and $\sum_{k \in \mathcal{M}} V_k(u^*) < C$. Then, by Lemma 2.1, we must have $\mathcal{E}(u^*) = \mathcal{M}$, so every player exercised. If player i decides to not exercise instead, his new payoff becomes $C - \sum_{k \neq i} V_k(u^*) > V_i(u^*)$, which shows that u^* is not a Nash equilibrium.

We will now show that part (ii) is valid. Suppose the contrary, that is, $D(u^*) < 0$. Then there must exists a player $k \in \mathcal{E}(u^*)$ with $X_k - P_k < 0$. Let us write $u^* = [0, s_{-k}^*]$ and let us consider the modified strategy profile

$\widehat{u} = [1, s^*_{-k}]$, meaning that player k chooses not to exercise. Then $\mathcal{E}(\widehat{u}) = \mathcal{E}(u^*) \setminus \{k\}$ and his modified payoff equals

$$P_k(\widehat{u}) = P_k - w_k(\widehat{u}) \sum_{i \in \mathcal{E}(\widehat{u})} (X_i - P_i).$$

Since u^* is a Nash equilibrium, we have $V_k(\widehat{u}) = P_k(\widehat{u}) \leq X_k = V_k(u^*)$. Therefore,

$$X_k - P_k(\widehat{u}) = X_k - P_k + w_k(\widehat{u}) \sum_{i \in \mathcal{E}(\widehat{u})} (X_i - P_i) \geq 0,$$

which in turn implies that $\sum_{i \in \mathcal{E}(\widehat{u})}(X_i - P_i) \geq 0$ (since $X_k - P_k < 0$). Finally, $0 \leq w_k(\widehat{u}) \leq 1$ and thus

$$D(u^*) = \sum_{i \in \mathcal{E}(u^*)} (X_i - P_i) \geq X_k - P_k + w_k(\widehat{u}) \sum_{i \in \mathcal{E}(\widehat{u})} (X_i - P_i) \geq 0,$$

which contradicts the assumption that $D(u^*) < 0$.

To establish (iii), assume that $X_k > V_k(u^*)$ for some $k \in \mathcal{M}$. Then we have

$$V_k\left(\left[s^*_k, s^*_{-k}\right]\right) < X_k = V_k\left(\left[0, s^*_{-k}\right]\right)$$

so that u^* is not a Nash equilibrium and thus (iii) is valid. For part (iv), we note that for every $k \in \mathcal{M} \setminus \mathcal{E}(u^*)$

$$V_k(u^*) = P_k(u^*) = P_k - w_k(u^*) D(u^*) \leq P_k,$$

since $D(u^*) \geq 0$, by part (ii). To prove (v), assume that $k \in \mathcal{M} \setminus \mathcal{E}(u^*)$, meaning that player k did not exercise. Since u^* is a Nash equilibrium, the modified payoff of player k should be at least as great as his exercise payoff, that is,

$$P_k(u^*) = P_k - w_k(u^*) D(u^*) \geq X_k > P_k.$$

This implies that $D(u^*) < 0$, which in turn contradicts part (ii). ∎

2.2. Optimal equilibrium. The next definition strengthens the concept of a Nash equilibrium.

Definition 2.3. A strategy profile $u^* \in \mathcal{U}$ is an *optimal equilibrium* if, for all $k \in \mathcal{M}$,

(2)

$$V_k\left(\left[s^*_k, s_{-k}\right]\right) \geq V_k\left(\left[s^*_k, s^*_{-k}\right]\right) \geq V_k\left(\left[s_k, s^*_{-k}\right]\right), \quad \forall\, s_k \in \mathcal{S}_k, \forall\, s_{-k} \in \mathcal{S}_{-k}.$$

A vector $\boldsymbol{V}^* \in \mathbb{R}^m$ is called a *value* of the game $\mathcal{G}$ if there exists an optimal equilibrium u^* with $\boldsymbol{V}^* = \boldsymbol{V}(u^*) = [V_1(u^*), \ldots, V_m(u^*)]$.

The right-hand side inequality in (2) makes it clear that an optimal equilibrium is a Nash equilibrium. Let us examine the basic features of an optimal equilibrium in the present context.

Lemma 2.2. *Let u^* be any optimal equilibrium. For each player k, it is not possible to guarantee a payoff greater than $V_k(u^*)$.*

Proof. Assume, on the contrary, that it is possible for player k to guarantee a payoff greater than $V_k(u^*)$ with some strategy $s'_k \in \mathcal{S}_k$, that is,

$$\min_{s_{-k}\in\mathcal{S}_{-k}} V_k\left([s'_k, s_{-k}]\right) > V_k\left([s^*_k, s^*_{-k}]\right) = V_k(u^*). \tag{3}$$

Consider the strategy profile $u' = [s^*_1, \ldots, s^*_{k-1}, s'_k, s^*_{k+1}, \ldots, s^*_m]$, where every other player uses their optimal equilibrium strategy. In view of (3), we should thus have $V_k(u') > V_k(u^*)$. By the left-hand side inequality in (2), $V_i(u') \geq V_i(u^*)$ for all $i \neq k$. By Lemma 2.1, the inequality $\sum_{i\in\mathcal{M}} V_i(u') \leq C$ holds, whereas part (i) in Proposition 2.1 yields $\sum_{i\in\mathcal{M}} V_i(u^*) = C$. Consequently,

$$V_k(u') \leq C - \sum_{i\neq k} V_i(u') \leq C - \sum_{i\neq k} V_i(u^*) = V_k(u^*),$$

which contradicts the inequality $V_k(u') > V_k(u^*)$. Hence the assertion of the lemma follows. ∎

Proposition 2.2. *The value $\boldsymbol{V}^* = [V^*_1, \ldots, V^*_m]$ of the game $\mathcal{G}$ is unique. Moreover, the vector $\boldsymbol{V}^*$ satisfies $X_k \leq V^*_k$ and $\sum_{k\in\mathcal{M}} V^*_k = C$.*

Proof. Assume there exists a value $\boldsymbol{V}^* = \boldsymbol{V}(u^*)$ with a corresponding optimal equilibrium u^*. If there is a second value $\boldsymbol{V}' = \boldsymbol{V}(u') \neq \boldsymbol{V}^*$, corresponding to an optimal equilibrium u', then there must exist a player k for which $V_k(u^*) \neq V_k(u')$. Without loss of generality, we may assume that $V_k(u') > V_k(u^*)$. By Lemma 2.2, it is not possible for player k to guarantee a payoff greater than $V_k(u^*)$. However, since u' is an optimal equilibrium, it is possible to guarantee a payoff of $V_k(u') > V_k(u^*)$, yielding an immediate contradiction. In view of Proposition 2.1, the value $\boldsymbol{V}^*$ satisfies $X_k \leq V^*_k$ and $\sum_{k\in\mathcal{M}} V^*_k = C$. ∎

There are a couple of other properties worth noting. We will merely state these properties, since their proofs are immediate consequences of definitions.

Proposition 2.3. (i) *A strategy profile $u^* \in \mathcal{U}$ is an optimal equilibrium if and only if for any proper subset $\mathcal{E} \subset \mathcal{M}$ we have that*

$$\sum_{k\in\mathcal{E}} V_k\left([s^*_{\mathcal{E}}, s^*_{-\mathcal{E}}]\right) \geq \sum_{k\in\mathcal{E}} V_k\left([s_{\mathcal{E}}, s^*_{-\mathcal{E}}]\right), \quad \forall\, s_{\mathcal{E}} \in \mathcal{S}_{\mathcal{E}} := \prod_{k\in\mathcal{E}} \mathcal{S}_k.$$

(ii) *If there exists an optimal equilibrium u^* then the corresponding unique value $\boldsymbol{V}^* = \boldsymbol{V}(u^*)$ satisfies*

$$V_k(u^*) = V^*_k = \min_{s_{-k}\in\mathcal{S}_{-k}} \max_{s_k\in\mathcal{S}_k} V_k\left([s_k, s_{-k}]\right) = \max_{s_k\in\mathcal{S}_k} \min_{s_{-k}\in\mathcal{S}_{-k}} V_k\left([s_k, s_{-k}]\right)$$

where $\mathcal{S}_{-k} = \prod_{i\neq k} \mathcal{S}_i$.

At the intuitive level, the 'fair value' of the game for player k is the highest amount one would offer to take up the position of player k. We claim that this value is equal to the unique value V_k^*, provided that it exists. Indeed, by the definition of an optimal equilibrium, this amount not only can be guaranteed by player k by playing optimally no matter what the decisions of all other players are, but it is also the highest possible payoff for player k, if everyone else also plays perfectly.

2.3. Weakly unilaterally competitive games. In order to proceed further, we need to be more explicit about the way in which the weights $w_k(u)$ are specified. We find it convenient to express them in terms of players' *powers*. The power of each player is used to compute the weight and thus to determine how the payoffs are redistributed among non-exercising players. For the rest of the paper, we work under the following standing assumption.

Assumption 2.2. For any strategy profile $u \in \mathcal{U}$ and any $k \in \mathcal{M} \setminus \mathcal{E}(u)$, the weight $w_k(u)$ is given by the equality

$$w_k(u) = \frac{a_k}{\sum_{i \in \mathcal{M} \setminus \mathcal{E}(u)} a_i}, \tag{4}$$

where, for each $i \in \mathcal{M}$, a strictly positive number a_i represents the *power* of player i.

In the distribution of the difference due to exercise, $D(u)$, the weights given by (4) induce an important property. Assume some players change their strategies from non-exercising to exercising, thus changing the strategy profile from u to u'. Instead of recalculating the distribution of $D(u')$, we can simply split up $D(u') - D(u)$ according to the powers and adjust the modified payoffs of the remaining non-exercising players. These weights can also be described to be invariant under *projection*, allowing for the construction of *subgames* (see Section 3.3).

In papers by De Wolf [2] and Kats and Thisse [8], a game is said to be *weakly unilaterally competitive* if for arbitrary $k, l \in \mathcal{M}$ and all $s_k, s'_k \in \mathcal{S}_k$ and $s_{-k} \in \mathcal{S}_{-k}$

$$V_k\big([s_k, s_{-k}]\big) > V_k\big([s'_k, s_{-k}]\big) \;\Rightarrow\; V_l\big([s_k, s_{-k}]\big) \le V_l\big([s'_k, s_{-k}]\big),$$
$$V_k\big([s_k, s_{-k}]\big) = V_k\big([s'_k, s_{-k}]\big) \;\Rightarrow\; V_l\big([s_k, s_{-k}]\big) = V_l\big([s'_k, s_{-k}]\big).$$

As shown in [2, 8], in a weakly unilaterally competitive game, all Nash equilibria must have the same value, where optimal equilibria are also achieved. When the weights are defined in terms of powers through formula (4), it is easy to check that the game introduced in Definition 2.1 is weakly unilaterally competitive, irrespective of a choice of the exercise and terminal payoffs. Consequently, the Nash equilibria must coincide with the optimal equilibria in terms of the value. For the sake of completeness, we present in the next section an independent proof of this result for our game. More importantly, we provide also explicit constructions of the game's value and an optimal equilibrium.

3. Construction of an optimal equilibrium

The goal of this section is to establish the existence and uniqueness of the game's value. We first examine, in Theorem 3.1, the case of the *degenerate game*. Next, in Proposition 3.3, any Nash equilibrium is shown to also be an optimal equilibrium. Finally, we show in Theorem 3.2 that an optimal equilibrium always exists and any optimal equilibrium attains the unique value. We also provide an explicit construction of the value by projection and we propose an algorithm for finding an optimal equilibrium. Recall that we work under the standing Assumptions 2.1 and 2.2.

3.1. Value of the degenerate game. Recall that the strategy of each player consists of a binary choice of whether to exercise at time 0 or not. Hence the map $\mathcal{E} : \mathcal{U} \to 2^{\mathcal{M}}$ is a bijection between the class of all strategy profiles and the class of all exercise sets. The *degenerate game* is characterised by the equality $\sum_{k\in\mathcal{M}} X_k = C$ where $C := \sum_{k\in\mathcal{M}} P_k$. It is worth noting that Theorem 3.1 can be seen as a special case of Theorem 3.2 when the simplex $\mathbb{S}$ and the hyperplane $\mathbb{H}_{\mathcal{M}}$, introduced in Section 3.2 below, are degenerate, specifically, $\mathbb{S} = \mathbb{H}_{\mathcal{M}} = \boldsymbol{X}$. This feature motivates the terminology *degenerate game*.

Theorem 3.1. *If $\sum_{k\in\mathcal{M}} X_k = C$ then the unique value $\boldsymbol{V}^*$ to the game $\mathcal{G}$ satisfies $\boldsymbol{V}^* = \boldsymbol{X}$. Moreover, the strategy profile $u^* = [0, \dots, 0]$ is an optimal equilibrium.*

Proof. The uniqueness is a consequence of Proposition 2.2, but we will demonstrate it anyway for this case. Assume there exists an optimal equilibrium u^* with

$$\boldsymbol{V}(u^*) = [V_1(u^*), \dots, V_m(u^*)] \neq [X_1, \dots, X_m] = \boldsymbol{X}.$$

By part (i) in Proposition 2.1, $\sum_{k\in\mathcal{M}} V_k(u^*) = C = \sum_{k\in\mathcal{M}} X_k$, so that $V_i(u^*) < X_i$ for some $i \in \mathcal{M}$ (indeed, otherwise $V_k(u^*) = X_k$ for all $k \in \mathcal{M}$, which contradicts the assumption). By part (iii) in Proposition 2.1, a strategy profile u^* cannot be an optimal equilibrium.

To show the existence of an optimal equilibrium, let us consider the strategy profile $u^* = [0, \dots, 0]$ that corresponds to all players exercising at time 0. Since each player exercises, his payoff is guaranteed to be X_k, regardless of the other players' decisions. In fact, one easily check that (2) holds, that is, u^* is an optimal equilibrium. It is also obvious that u^* attains the required values, since manifestly $\boldsymbol{V}(u^*) = \boldsymbol{X} = \boldsymbol{V}^*$. ∎

Remark 3.1. Let us stress that when $\sum_{k\in\mathcal{M}} X_k = C$ not every strategy profile u such that $\boldsymbol{V}(u) = \boldsymbol{V}^*$ is an optimal equilibrium. Consider, for instance, the two player game where $\boldsymbol{P} = [0,0]$ and $\boldsymbol{X} = [1,-1]$. An optimal equilibrium occurs when both players exercise, so that $u^* = [0,0]$, and the value is $\boldsymbol{V}^* = [1,-1]$. But if only player 2 exercises, so that $u = [1,0]$, the same value is reached since $\boldsymbol{V}(u) = [1,-1]$. The strategy profile u is not an optimal equilibrium, however, as player 2 could now do better by not exercising. Indeed, for the strategy profile $\widehat{u} = [1,1]$ we obtain $\boldsymbol{V}(\widehat{u}) = [0,0]$, and thus $V_2(\widehat{u}) > V_2(u)$.

3.2. Value space and projections. Our next goal is to examine the *non-degenerate game*, meaning that $\sum_{k\in\mathcal{M}} X_k < C$. Recall that $\boldsymbol{P}$ and $\boldsymbol{V}$ can be seen as vectors in $\mathbb{R}^m$. We define the hyperplane

$$\mathbb{H} = \left\{ \boldsymbol{x} \in \mathbb{R}^m : \sum_{k=1}^{m} x_k = C \right\}, \tag{5}$$

so that $\boldsymbol{P}$ lies on $\mathbb{H}$. In addition, by part (i) in Proposition 2.1 and Definition 2.3, any value $\boldsymbol{V}$ of the game must also lie on $\mathbb{H}$. The *value space* of the game $\mathcal{G}$ is the hyperplane $\mathbb{H}$ in $\mathbb{R}^m$.

We endow the space $\mathbb{R}^m$ with the the norm $\|\cdot\|$ generated by the following inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{k=1}^{m} \left(\frac{x_k y_k}{a_k} \right).$$

The following auxiliary result summarises some standard properties of projections.

Lemma 3.1. *For any vector $\boldsymbol{P}$ and any closed convex set $\mathbb{K}$ in $\mathbb{R}^m$, there exists a unique projection of $\boldsymbol{P}$ onto $\mathbb{K}$, denoted by $\pi_{\mathbb{K}}(\boldsymbol{P})$, such that $\pi_{\mathbb{K}}(\boldsymbol{P}) \in \mathbb{K}$ and*

$$\|\pi_{\mathbb{K}}(\boldsymbol{P}) - \boldsymbol{P}\| \leq \|\boldsymbol{Q} - \boldsymbol{P}\|, \quad \forall\, \boldsymbol{Q} \in \mathbb{K}.$$

If $\mathbb{K}$ is a hyperplane then the projection is orthogonal, that is, $\pi_{\mathbb{K}}(\boldsymbol{P})$ is the unique vector in $\mathbb{K}$ such that

$$\langle \pi_{\mathbb{K}}(\boldsymbol{P}) - \boldsymbol{P}, \boldsymbol{Q} - \pi_{\mathbb{K}}(\boldsymbol{P}) \rangle = 0, \quad \forall\, \boldsymbol{Q} \in \mathbb{K}.$$

Let $\mathbb{J}$ be a closed convex subset of the hyperplane $\mathbb{K}$. Then for any vector $\boldsymbol{P}$ we have $\pi_{\mathbb{J}}(\boldsymbol{P}) = \pi_{\mathbb{J}}(\pi_{\mathbb{K}}(\boldsymbol{P}))$.

For any proper subset $\mathcal{E} \subset \mathcal{M}$, we define the hyperplane

$$\mathbb{H}_{\mathcal{E}} = \left\{ \boldsymbol{x} \in \mathbb{R}^m : x_i = X_i \text{ for all } i \in \mathcal{E} \text{ and } \sum_{k=1}^{m} x_k = C \right\}. \tag{6}$$

It is clear from (5) that $\mathbb{H}_{\emptyset} = \mathbb{H}$ and $\mathbb{H}_{\mathcal{E}} \subseteq \mathbb{H}$. For completeness, we also denote

$$\mathbb{H}_{\mathcal{M}} = \left\{ \boldsymbol{x} \in \mathbb{R}^m : x_i = X_i \text{ for all } i \in \mathcal{M} \right\} = \boldsymbol{X}.$$

Observe that $\mathbb{H}_{\mathcal{M}} = \boldsymbol{X} \notin \mathbb{H}$, unless the equality $\sum_{k \in \mathcal{M}} X_k = C$ holds.

Lemma 3.2. *Let $\mathcal{E}$ be a proper subset of $\mathcal{M}$ and let c_i, $i \in \mathcal{E}$ and α be any real numbers. Then the vector $\boldsymbol{v} = [v_1, \ldots, v_m]$ is orthogonal to the hyperplane $\widehat{\mathbb{H}}$ given by*

$$\widehat{\mathbb{H}} = \left\{ \boldsymbol{x} \in \mathbb{R}^m : x_i = c_i \text{ for all } i \in \mathcal{E} \text{ and } \sum_{k \in \mathcal{M} \setminus \mathcal{E}} \left(\frac{v_k x_k}{a_k} \right) = \alpha \right\}.$$

Proof. It suffices to check that for every $\boldsymbol{x_1}, \boldsymbol{x_2} \in \widehat{\mathbb{H}}$ we have that $\langle \boldsymbol{x_1} - \boldsymbol{x_2}, \boldsymbol{v} \rangle = 0$. ∎

Proposition 3.1 gives us an elegant and alternate way of representing and computing the modified payoffs when the exercise set is known. It also justifies ex post our choice of the norm.

Proposition 3.1. *Let $u \in \mathcal{U}$ be any strategy profile such that $\mathcal{E}(u)$ is a proper subset of $\mathcal{M}$. Then the vector $\boldsymbol{V}(u)$ of modified payoffs equals*

$$\boldsymbol{V}(u) = \pi_{\mathbb{H}_{\mathcal{E}(u)}}(\boldsymbol{P}).$$

Proof. For the sake of brevity, we denote $\mathcal{E} := \mathcal{E}(u)$. If $\mathcal{E} = \mathcal{M}$ then, clearly, the equalities $\boldsymbol{V}(u) = \boldsymbol{X} = \mathbb{H}_{\mathcal{M}}$ hold. If $\mathcal{E}$ is a proper subset of $\mathcal{M}$ then the vector $\boldsymbol{V}(u)$ equals

$$\boldsymbol{V}(u) = \big[V_i(u) = X_i,\, i \in \mathcal{E},\, V_i(u) = P_i(u) = P_i - w_i(u) D(u),\, i \in \mathcal{M} \setminus \mathcal{E} \big].$$

Note that $\boldsymbol{V}(u)$ lies on $\mathbb{H}_{\mathcal{E}}$ since, by Lemma 2.1(i), $\sum_{i \in \mathcal{M}} V_i(u) = C$. Let

$$\boldsymbol{v} := \boldsymbol{P} - \boldsymbol{V}(u) = \big[v_i = P_i - X_i,\, i \in \mathcal{E},\, v_i = w_i(u) D(u),\, i \in \mathcal{M} \setminus \mathcal{E} \big].$$

Upon setting $c_i = X_i$ and

$$\alpha = \frac{D(u)\big(C - \sum_{i \in \mathcal{E}} X_i\big)}{\sum_{i \in \mathcal{M} \setminus \mathcal{E}} a_i},$$

we deduce from Lemma 3.2 that $\boldsymbol{v} = \boldsymbol{P} - \boldsymbol{V}(u)$ is orthogonal to $\mathbb{H}_{\mathcal{E}}$. We conclude that $\boldsymbol{V}(u)$ is the orthogonal projection of $\boldsymbol{P}$ onto $\mathbb{H}_{\mathcal{E}}$, as required. ∎

Consider the simplex $\mathbb{S}$ given by the formula

$$\mathbb{S} = \Big\{ \boldsymbol{x} \in \mathbb{R}^m : x_k \geq X_k,\ 1 \leq k \leq m \text{ and } \sum_{k=1}^{m} x_k = C \Big\}. \tag{7}$$

Remark 3.2. In the case where $\sum_{k \in \mathcal{M}} X_k < C$, the simplex $\mathbb{S}$ is non-degenerate. If, on the contrary, $\sum_{k \in \mathcal{M}} X_k = C$ then $\mathbb{S} = \mathbb{H}_{\mathcal{M}} = \boldsymbol{X}$.

By Proposition 2.1, if a strategy profile u^* is a Nash equilibrium then necessarily $\boldsymbol{V}(u^*) \in \mathbb{S}$. Moreover, by Proposition 2.2, the unique value $\boldsymbol{V}^*$ of the game $\mathcal{G}$ belongs to $\mathbb{S}$.

Let us observe that for any proper subset $\mathcal{E} \subset \mathcal{M}$ we have that $\mathbb{H}_{\mathcal{E}} \cap \mathbb{S} \neq \emptyset$. Note also that the inclusions $\mathbb{H}_{\mathcal{E}} \subset \mathbb{S}$ and $\mathbb{S} \subset \mathbb{H}_{\mathcal{E}}$ do not hold, in general. The following auxiliary results will be used in the proof of the main result of this work, Theorem 3.2. Lemma 3.3 shows that the projection of a point inside the simplex still lies in the simplex. From Lemma 3.4, it follows that if a point lies on the side of a face opposite the simplex, then the projection must lie on that face. Finally, Lemma 3.5 demonstrates that projecting onto the simplex is the same as projecting onto a particular hyperplane of the simplex.

Lemma 3.3. *Assume that $\boldsymbol{P} \in \mathbb{S}$. Then $\pi_{\mathbb{H}_{\mathcal{E}}}(\boldsymbol{P}) \in \mathbb{S}$ for any proper subset $\mathcal{E} \subset \mathcal{M}$.*

Proof. By Proposition 3.1, the projection $\pi_{\mathbb{H}_{\mathcal{E}}}(\boldsymbol{P})$ corresponds to the modified payoff when $\mathcal{E}$ is the set of exercising players. Let u be the corresponding strategy profile. In particular, for any $i \in \mathcal{E}$

$$\left[\pi_{\mathbb{H}_{\mathcal{E}}}(\boldsymbol{P})\right]_i = X_i \leq P_i$$

and thus $D(u) = \sum_{i \in \mathcal{E}} (X_i - P_i) \leq 0$. Consequently, for any $i \in \mathcal{M} \setminus \mathcal{E}$

$$\left[\pi_{\mathbb{H}_{\mathcal{E}}}(\boldsymbol{P})\right]_i = V_i(u) = P_i - w_i(u) D(u) \geq P_i \geq X_i$$

and thus $\pi_{\mathbb{H}_{\mathcal{E}}}(\boldsymbol{P}) \in \mathbb{S}$. ∎

Lemma 3.4. *Let $k \in \mathcal{M}$. If $\pi_{\mathbb{S}}(\boldsymbol{P}) \notin \mathbb{H}_{\{k\}}$ then $P_k > X_k$. Equivalently, if $P_k \leq X_k$ then $\pi_{\mathbb{S}}(\boldsymbol{P}) \in \mathbb{H}_{\{k\}}$.*

Proof. Suppose that $P_k \leq X_k$ and assume that $\pi_{\mathbb{S}}(\boldsymbol{P}) \notin \mathbb{H}_{\{k\}}$. Then the projection $\boldsymbol{Q} = \pi_{\mathbb{H}_{\{k\}}}(\pi_{\mathbb{S}}(\boldsymbol{P}))$ is still in $\mathbb{S}$ (by Lemma 3.3) and it is distinct from $\pi_{\mathbb{S}}(\boldsymbol{P})$ (since $\pi_{\mathbb{S}}(\boldsymbol{P}) \notin \mathbb{H}_{\{k\}}$). We will show that

$$\|\boldsymbol{P} - \boldsymbol{Q}\| < \|\boldsymbol{P} - \pi_{\mathbb{S}}(\boldsymbol{P})\|, \tag{8}$$

which contradicts the definition of $\pi_{\mathbb{S}}(\boldsymbol{P})$. In the case of $P_k = X_k$, we have $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{H}_{\{k\}}$ and $\pi_{\mathbb{S}}(\boldsymbol{P}) - \boldsymbol{Q}$ being orthogonal to $\boldsymbol{P} - \boldsymbol{Q}$. Hence

$$\|\boldsymbol{P} - \boldsymbol{Q}\|^2 < \|\boldsymbol{P} - \boldsymbol{Q}\|^2 + \|\pi_{\mathbb{S}}(\boldsymbol{P}) - \boldsymbol{Q}\|^2 = \|\boldsymbol{P} - \pi_{\mathbb{S}}(\boldsymbol{P})\|^2.$$

To establish (8) in the case $P_k < X_k$, we introduce a hyperplane $\widehat{\mathbb{H}}_{\{k\}}$ parallel to $\mathbb{H}_{\{k\}}$ by setting

$$\widehat{\mathbb{H}}_{\{k\}} = \left\{ \boldsymbol{x} \in \mathbb{R}^m : x_k = P_k \text{ and } \sum_{i=1}^{m} x_i = C \right\},$$

so that, in particular, $\boldsymbol{P} \in \widehat{\mathbb{H}}_{\{k\}}$. Let $\boldsymbol{R} = \pi_{\widehat{\mathbb{H}}_{\{k\}}}(\pi_{\mathbb{S}}(\boldsymbol{P}))$, so that also

$$\boldsymbol{R} = \pi_{\widehat{\mathbb{H}}_{\{k\}}}\big(\pi_{\mathbb{H}_{\{k\}}}(\pi_{\mathbb{S}}(\boldsymbol{P}))\big) = \pi_{\widehat{\mathbb{H}}_{\{k\}}}(\boldsymbol{Q})\,.$$

Since $P_k < X_k$, $\boldsymbol{R} \in \widehat{\mathbb{H}}_{\{k\}}$ and $\pi_{\mathbb{S}}(\boldsymbol{P}) \in \mathbb{S} \setminus \mathbb{H}_{\{k\}}$ lie on opposite sides of the hyperplane $\mathbb{H}_{\{k\}}$. It is thus clear that

$$\|\boldsymbol{R} - \boldsymbol{Q}\| < \|\boldsymbol{R} - \boldsymbol{Q}\| + \|\boldsymbol{Q} - \pi_{\mathbb{S}}(\boldsymbol{P})\| = \|\boldsymbol{R} - \pi_{\mathbb{S}}(\boldsymbol{P})\|\,. \tag{9}$$

Finally, since $\boldsymbol{P} - \boldsymbol{R}$ is orthogonal to both $\boldsymbol{R} - \boldsymbol{Q}$ and $\boldsymbol{R} - \pi_{\mathbb{S}}(\boldsymbol{P})$, we have

$$\|\boldsymbol{P} - \boldsymbol{Q}\|^2 = \|\boldsymbol{P} - \boldsymbol{R}\|^2 + \|\boldsymbol{R} - \boldsymbol{Q}\|^2$$

and

$$\|\boldsymbol{P} - \pi_{\mathbb{S}}(\boldsymbol{P})\|^2 = \|\boldsymbol{P} - \boldsymbol{R}\|^2 + \|\boldsymbol{R} - \pi_{\mathbb{S}}(\boldsymbol{P})\|^2\,.$$

Therefore, (9) implies (8), as required. ∎

Lemma 3.5. *For any $\boldsymbol{P}$, there exists a proper subset $\mathcal{E} \subset \mathcal{M}$ such that $\pi_{\mathbb{S}}(\boldsymbol{P}) = \pi_{\mathbb{H}_{\mathcal{E}}}(\boldsymbol{P})$. In particular, if $\boldsymbol{P} \in \mathbb{S}$ then $\mathcal{E} = \emptyset$, so that $\mathbb{H}_{\mathcal{E}} = \mathbb{H}_{\emptyset} = \mathbb{H}$.*

Proof. For $\boldsymbol{P} \in \mathbb{S}$ the statement is trivial. For $\boldsymbol{P} \notin \mathbb{S}$, we will proceed by induction with respect to m. The base case when the number of players $m = 2$ is easily verified.

Let us assume that $\boldsymbol{P} \notin \mathbb{S}$. Then, by definition of $\mathbb{S}$, there exists $k \in \mathcal{M}$ such that $P_k < X_k$, and thus, by Lemma 3.4, the projection $\pi_{\mathbb{S}}(\boldsymbol{P}) \in \mathbb{H}_{\{k\}}$. Hence by Lemma 3.1

$$\pi_{\mathbb{S}}(\boldsymbol{P}) = \pi_{\mathbb{S} \cap \mathbb{H}_{\{k\}}}(\boldsymbol{P}) = \pi_{\mathbb{S} \cap \mathbb{H}_{\{k\}}}\big(\pi_{\mathbb{H}_{\{k\}}}(\boldsymbol{P})\big).$$

By applying the induction hypothesis to $\boldsymbol{P'} = \pi_{\mathbb{H}_{\{k\}}}(\boldsymbol{P})$ and $\boldsymbol{S'} = \mathbb{S} \cap \mathbb{H}_{\{k\}}$, while working under the domain of $\mathbb{H}_{\{k\}}$ (instead of $\mathbb{R}^m$), we can find $\mathcal{E}' \subset \mathcal{M} \setminus \{k\}$ such that

$$\begin{aligned} \pi_{\boldsymbol{S'}}\big(\boldsymbol{P'}\big) &= \pi_{\mathbb{H}_{\mathcal{E}'} \cap \mathbb{H}_{\{k\}}}\big(\boldsymbol{P'}\big) = \pi_{\mathbb{H}_{\mathcal{E}'} \cap \mathbb{H}_{\{k\}}}\big(\pi_{\mathbb{H}_{\{k\}}}(\boldsymbol{P})\big) \\ &= \pi_{\mathbb{H}_{\mathcal{E}'} \cap \mathbb{H}_{\{k\}}}(\boldsymbol{P}) = \pi_{\mathbb{H}_{\mathcal{E}' \cup \{k\}}}(\boldsymbol{P})\,. \end{aligned}$$

To complete the induction step, it suffices to set $\mathcal{E} = \mathcal{E}' \cup \{k\}$. ∎

3.3. Subgames. Consider a proper subset $\mathcal{E} \subset \mathcal{M}$ and assume that every player in $\mathcal{E}$ exercises at time 0. Then the game $\mathcal{G}$ reduces to the subgame $\mathcal{G}_{\mathcal{M} \setminus \mathcal{E}}$ for players from $\mathcal{M} \setminus \mathcal{E}$. In particular, $\mathcal{G} = \mathcal{G}_{\mathcal{M}}$. Let us denote by $\mathcal{U}_{\mathcal{M} \setminus \mathcal{E}}$ the class of all strategy profiles for $\mathcal{G}$ such that $s_i = 0$ for $i \in \mathcal{E}$. Formally, by the subgame $\mathcal{G}_{\mathcal{M}\setminus\mathcal{E}}$ we mean the game $\mathcal{G}$ with the reduced class $\mathcal{U}_{\mathcal{M}\setminus\mathcal{E}}$ of strategy profiles. Since $x_i = X_i$ for $i \in \mathcal{E}$, the value space of the subgame $\mathcal{G}_{\mathcal{M}\setminus\mathcal{E}}$ is equal to $\mathbb{H}_{\mathcal{E}}$.

Lemma 3.6. *The terminal payoffs of the subgame $\mathcal{G}_{\mathcal{M}\setminus\mathcal{E}}$ can be identified with $\boldsymbol{P}' = \pi_{\mathbb{H}_{\mathcal{E}}}(\boldsymbol{P})$.*

Proof. Let u be any strategy profile of $\mathcal{G}$ such that $\mathcal{E}(u)$ is a proper subset of $\mathcal{M}$ such that $\mathcal{E} \subseteq \mathcal{E}(u)$. This means, in particular, that u belongs to $\mathcal{U}_{\mathcal{M}\setminus\mathcal{E}}$. Set $\mathcal{E}'(u) = \mathcal{E}(u) \setminus \mathcal{E}$. By Proposition 3.1, the vector of modified payoffs $\boldsymbol{V}(u)$ is given by

$$\boldsymbol{V}(u) = \pi_{\mathbb{H}_{\mathcal{E}(u)}}(\boldsymbol{P}) = \pi_{\mathbb{H}_{\mathcal{E}\cup\mathcal{E}'(u)}}(\boldsymbol{P}).$$

Since $\mathbb{H}_{\mathcal{E}\cup\mathcal{E}'(u)} \subseteq \mathbb{H}_{\mathcal{E}}$ are both hyperplanes, from Lemma 3.1, we obtain

$$\boldsymbol{V}(u) = \pi_{\mathbb{H}_{\mathcal{E}\cup\mathcal{E}'(u)}}(\pi_{\mathbb{H}_{\mathcal{E}}}(\boldsymbol{P})) = \pi_{\mathbb{H}_{\mathcal{E}\cup\mathcal{E}'(u)}}(\boldsymbol{P}').$$

By applying again Proposition 3.1, we conclude that we may identify the vector $\boldsymbol{P}'$ with the vector of terminal payoffs of the subgame $\mathcal{G}_{\mathcal{M}\setminus\mathcal{E}}$. ■

According to Lemma 3.6, the subgame $\mathcal{G}_{\mathcal{M}\setminus\mathcal{E}}$ can be seen as the game with the class $\mathcal{U}_{\mathcal{M}\setminus\mathcal{E}}$ of strategy profiles (i.e., with active players from $\mathcal{M}\setminus\mathcal{E}$) and the terminal payoffs $\pi_{\mathbb{H}_{\mathcal{E}}}(\boldsymbol{P})$. In view of Lemma 3.6, the following result can be easily checked by the definition of an optimal (or Nash) equilibrium.

Proposition 3.2. *Let $u^* = [s_1^*, \dots, s_m^*] \in \mathcal{U}$ be an optimal (or Nash) equilibrium of $\mathcal{G}$. If for some proper subset $\mathcal{E} \subset \mathcal{M}$ the inclusion $\mathcal{E} \subseteq \mathcal{E}(u^*)$ holds then u^* is an optimal (or Nash) equilibrium of the subgame $\mathcal{G}_{\mathcal{M}\setminus\mathcal{E}}$, that is, the game with the class $\mathcal{U}_{\mathcal{M}\setminus\mathcal{E}}$ of strategy profiles and the terminal payoffs $\pi_{\mathbb{H}_{\mathcal{E}}}(\boldsymbol{P})$.*

3.4. Value of the non-degenerate game. The following result shows that any Nash equilibrium u^* is also an optimal equilibrium. In other words, if player k chooses his Nash equilibrium strategy s_k^*, he is guaranteed a payoff at least as much as the value payoff, regardless of other players' strategies.

Proposition 3.3. *Let $u^* = [s_1^*, \dots, s_m^*]$ be a Nash equilibrium of the game $\mathcal{G}$. Then it is also an optimal equilibrium, meaning that for any $k \in \mathcal{M}$, we also have that*

$$V_k[s_1, \dots, s_{k-1}, s_k^*, s_{k+1}, \dots, s_m] \geq V_k[s_1^*, \dots, s_{k-1}^*, s_k^*, s_{k+1}^*, \dots, s_m^*] \tag{10}$$

for all $s_1, \dots, s_{k-1}, s_{k+1}, \dots, s_m \in \mathcal{S}$.

Proof. Consider a particular player k. If $s_k^* = 0$ then he is exercising at time 0 and his payoff X_k is then independent of the decisions of other players, so that (10) is valid. If $s_k^* = 1$, we will argue by contradiction. Suppose there exists $u = [s_1, \dots, s_{k-1}, 1, s_{k+1}, \dots, s_m]$ such that $V_k(u) < V_k(u^*)$. We proceed by induction on the number of players. For the base case of $m = 2$ players, it can be easily verified that such a u does not exist.

Let us then consider any $m \geq 3$. Assume first that $\mathcal{E}(u) \cap \mathcal{E}(u^*) \neq \emptyset$. Then there exists $j \in \mathcal{E}(u) \cap \mathcal{E}(u^*)$, meaning that player j is exercising under both u and u^* and, obviously, $j \neq k$. Hence we can consider the reduced subgame $\mathcal{G}_{\{-j\}}$ in which we deal with decisions of $m - 1$ remaining players. By Lemma 3.6, the vector of terminal payoffs for the subgame $\mathcal{G}_{\{-j\}}$ can be identified with $\pi_{\mathbb{H}_{\{j\}}}(\boldsymbol{P})$. Next, by Proposition 3.2, any optimal equilibrium in $\mathcal{G}$ is also an optimal equilibrium in $\mathcal{G}_{\{-j\}}$, after considering exercise decision of player j. By the induction hypothesis on $\mathcal{G}_{\{-j\}}$, which only has $m - 1$ players, the inequality $V_k(u) \geq V_k(u^*)$ holds, and thus we arrive at a contradiction.

Assume now that $\mathcal{E}(u) \cap \mathcal{E}(u^*) = \emptyset$, so that for every $i \in \mathcal{E}(u)$ we have that $i \notin \mathcal{E}(u^*)$. By the contra-positive of part (v) in Proposition 2.1, $P_i \geq X_i$ for every $i \in \mathcal{E}(u)$. Then the difference due to exercising

$$D(u) = \sum_{i \in \mathcal{E}(u)} (X_i - P_i) \leq 0.$$

Consequently, the payoff for player k equals

$$V_k(u) = P_k - w_k(u) D(u) \geq P_k \geq V_k(u^*),$$

where the last inequality follows from part (iv) in Proposition 2.1. This again yields a contradiction. ∎

We are in a position to establish the main result of this paper, which shows that the unique value of the game can be computed by projecting $\boldsymbol{P}$ on the simplex $\mathbb{S}$. Recall that we now assume that $\sum_{k \in \mathcal{M}} X_k < C$.

Theorem 3.2. (i) *A strategy profile $u^* \in \mathcal{U}$ is an optimal equilibrium for the game $\mathcal{G}$ if and only if the set of exercising players $\mathcal{E}(u^*)$ is such that*

$$\pi_{\mathbb{H}_{\mathcal{E}(u^*)}}(\boldsymbol{P}) = \pi_{\mathbb{S}}(\boldsymbol{P}) \,. \tag{11}$$

(ii) *A strategy profile u^* satisfying* (11) *always exists and the unique value of the game $\mathcal{G}$ equals*

$$\boldsymbol{V}^* = \boldsymbol{V}(u^*) = [V_1(u^*), \dots, V_m(u^*)] = \pi_{\mathbb{S}}(\boldsymbol{P}) \,.$$

Proof. We first demonstrate part (i) of the theorem.
($\Leftarrow$) Let $u^* \in \mathcal{U}$ be any strategy profile, such that $\mathcal{E}(u^*)$ satisfies

$$\pi_{\mathbb{S}}(\boldsymbol{P}) = \pi_{\mathbb{H}_{\mathcal{E}(u^*)}}(\boldsymbol{P}) = \boldsymbol{V}(u^*),$$

where the second equality follows from Proposition 3.1. We will prove that u^* is a Nash equilibrium and thus, by Proposition 3.3, it is also an optimal equilibrium. Let us fix $k \in \mathcal{M}$. We need to show that condition (1) is satisfied.

We first assume that k is not in $\mathcal{E}(u^*)$, so that $s_k^* = 1$. Observe that, by the definition of $\mathbb{S}$, the condition $\boldsymbol{V}(u^*) = \pi_{\mathbb{S}}(\boldsymbol{P}) \in \mathbb{S}$ implies that $V_k(u^*) \geq X_k$. Consequently,

$$V_k\left(\left[1, s_{-k}^*\right]\right) = V_k(u^*) \geq X_k = V_k\left(\left[0, s_{-k}^*\right]\right)$$

and thus (1) holds. Suppose now that k belongs to $\mathcal{E}(u^*)$, so that $s_k^* = 0$. We need to show that

$$V_k\left(\left[0, s_{-k}^*\right]\right) = X_k \geq V_k\left(\left[1, s_{-k}^*\right]\right) = V_k(u'), \tag{12}$$

where u' is the strategy profile where k no longer exercises, whereas all other players follow the strategy profile u^*. We thus have that $\mathcal{E}(u') = \mathcal{E}(u^*) \setminus \{k\}$ and $\boldsymbol{V}(u') = \pi_{\mathbb{H}_{\mathcal{E}(u')}}(\boldsymbol{P})$ is the new modified payoff. It is clear that (12) fails to hold whenever $V_k(u') > X_k$. To complete the proof of the first implication, it thus suffices to show that the inequality $V_k(u') > X_k$ leads to a contradiction. To this end, it suffices to show that there exists $\boldsymbol{R} \in \mathbb{S} \cap \mathbb{H}_{\mathcal{E}(u')}$ such that

$$\left\|\boldsymbol{R} - \boldsymbol{V}(u')\right\| < \left\|\boldsymbol{V}(u^*) - \boldsymbol{V}(u')\right\|. \tag{13}$$

Indeed, let us suppose that such an $\boldsymbol{R}$ exists. Then, by Proposition 3.1, $\boldsymbol{V}(u') = \pi_{\mathbb{H}_{\mathcal{E}(u')}}(\boldsymbol{P})$ and thus $\boldsymbol{P} - \boldsymbol{V}(u')$ is orthogonal to $\mathbb{H}_{\mathcal{E}(u')}$. Recall that $\boldsymbol{R}, \boldsymbol{V}(u') \in \mathbb{H}_{\mathcal{E}(u')}$ and also $\boldsymbol{V}(u^*) \in \mathbb{H}_{\mathcal{E}(u^*)} \subset \mathbb{H}_{\mathcal{E}(u')}$. Consequently,

$$\|\boldsymbol{P} - \boldsymbol{R}\| < \|\boldsymbol{P} - \boldsymbol{V}(u^*)\|,$$

and this clearly contradicts the assumption that $\pi_{\mathbb{S}}(\boldsymbol{P}) = \boldsymbol{V}(u^*)$. We conclude that a strategy profile u^* is an equilibrium, as desired.

It thus remains to show that if $V_k(u') > X_k$ then there exists $\boldsymbol{R} \in \mathbb{S} \cap \mathbb{H}_{\mathcal{E}(u')}$ such that (13) holds. Let $\boldsymbol{Q}$ be the point representing the zero-dimensional hyperplane $\mathbb{H}_{\mathcal{M}\setminus\{k\}}$, that is,

$$\boldsymbol{Q} = \left[X_1, \ldots, X_{k-1}, \widehat{X}_k, X_{k+1}, \ldots, X_m\right],$$

where $\widehat{X}_k = C - \sum_{i \in \mathcal{M}\setminus\{k\}} X_i > 0$. Since the simplex $\mathbb{S}$ is non-degenerate, $\boldsymbol{Q}$ and $\boldsymbol{V}(u')$ both lie on the same side of $\mathbb{H}_{\{k\}}$.

If $\boldsymbol{Q}, \boldsymbol{V}(u^*)$ and $\boldsymbol{V}(u')$ are collinear, as shown in Figure 1, $\boldsymbol{Q}$ and $\boldsymbol{V}(u')$ must lie on the same side of $\boldsymbol{V}(u^*)$ and distinct from $\boldsymbol{V}(u^*)$. Choose $\boldsymbol{R}$ to be any point on the interval joining $\boldsymbol{V}(u')$ and $\boldsymbol{V}(u^*)$. Then (13) is manifestly satisfied.

If $\boldsymbol{Q}, \boldsymbol{V}(u^*)$ and $\boldsymbol{V}(u')$ are not collinear, as shown in Figure 2, let the two-dimensional plane containing them be $\mathbb{A}$. Note that $\mathbb{A}$ is a subset of $\mathbb{H}_{\mathcal{E}(u')}$. Let the intersection of $\mathbb{A}$ and $\mathbb{H}_{\{k\}}$ be the line ℓ. Now, by an application of

Lemma 3.1, we obtain

$$\begin{aligned}\boldsymbol{V}(u^*) &= \pi_{\mathbb{H}_{\mathcal{E}(u^*)}}(\boldsymbol{P}) = \pi_{\mathbb{H}_{\{k\}}\cap\mathbb{H}_{\mathcal{E}(u')}}\Big(\pi_{\mathbb{H}_{\mathcal{E}(u')}}(\boldsymbol{P})\Big)\\ &= \pi_{\mathbb{H}_{\{k\}}\cap\mathbb{H}_{\mathcal{E}(u')}}\big(\boldsymbol{V}(u')\big) = \pi_\ell\big(\boldsymbol{V}(u')\big)\,.\end{aligned}$$

The last equality holds because $\boldsymbol{V}(u^*) \in \ell$ and

$$\ell = \big(\mathbb{H}_{\{k\}} \cap \mathbb{A}\big) \subset \big(\mathbb{H}_{\{k\}} \cap \mathbb{H}_{\mathcal{E}(u')}\big)\,.$$

Therefore, $\boldsymbol{V}(u^*)$ is the orthogonal projection of $\boldsymbol{V}(u')$ onto ℓ.

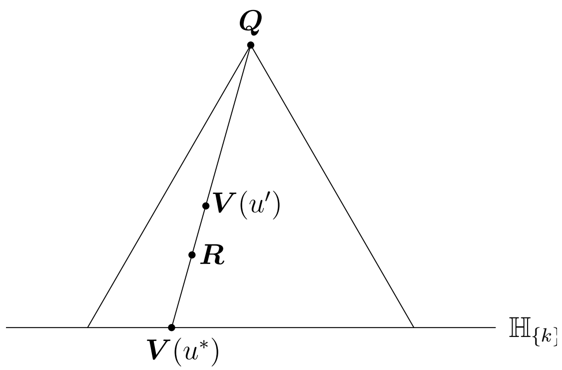

Fig. 1. A plane containing $\boldsymbol{Q}, \boldsymbol{V}(u^*)$ and $\boldsymbol{V}(u')$ when they are collinear

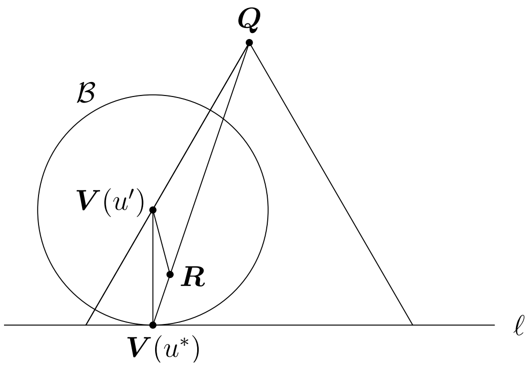

Fig. 2. The plane $\mathbb{A}$ when $\boldsymbol{Q}, \boldsymbol{V}(u^*)$ and $\boldsymbol{V}(u')$ are not collinear

Consider the closed ball on $\mathbb{A}$, centered at $\boldsymbol{V}(u')$ with radius

$$\big\|\boldsymbol{V}(u^*) - \boldsymbol{V}(u')\big\|\,,$$

name it $\mathcal{B}$. Under our norm, $\mathcal{B}$ is an ellipse and ℓ is the tangent at $\boldsymbol{V}(u^*)$. Since $\boldsymbol{Q}$ lies on the same side of ℓ as $\boldsymbol{V}(u')$, the interval joining $\boldsymbol{Q}$ and $\boldsymbol{V}(u^*)$

must contain a point $\boldsymbol{R}$ lying on the interior of $\mathcal{B}$. Hence

$$\|\boldsymbol{R} - \boldsymbol{V}(u')\| < \|\boldsymbol{V}(u^*) - \boldsymbol{V}(u')\|$$

which completes this part of the proof.
($\Rightarrow$) We start by noting that Lemma 3.5 yields the existence of a strategy profile u^* such that $\pi_{\mathbb{H}_{\mathcal{E}(u^*)}}(\boldsymbol{P}) = \pi_{\mathbb{S}}(\boldsymbol{P})$. The first part of this proof implies that u^* is also an optimal equilibrium. Suppose that there exists another optimal equilibrium u' corresponding to the value $\boldsymbol{V}(u')$. From Proposition 2.2, we deduce that $\boldsymbol{V}(u') = \boldsymbol{V}(u^*)$. Consequently,

$$\pi_{\mathbb{S}}(\boldsymbol{P}) = \boldsymbol{V}(u') = \pi_{\mathbb{H}_{\mathcal{E}(u')}}(\boldsymbol{P}),$$

where the last equality follows from Proposition 3.1. Part (ii) is an immediate consequence of Lemma 3.5, Proposition 2.2 and Proposition 3.1. ∎

Recall that in the degenerate case, we have $\mathbb{S} = \boldsymbol{X} = \mathbb{H}_{\mathcal{M}}$ (see Remark 3.2). Hence, for any $\boldsymbol{P}$ we obtain $\pi_{\mathbb{S}}(\boldsymbol{P}) = \boldsymbol{X} = \pi_{\mathbb{H}_{\mathcal{M}}}(\boldsymbol{P})$. This shows that Theorem 3.1 can be reformulated as a special case of Theorem 3.2. The result for the non-degenerate case $\sum_{k\in\mathcal{M}} X_k < C$ is similar to the result obtained for the degenerate case $\sum_{k\in\mathcal{M}} X_k = C$, except that in the former case any strategy profile u corresponding to the value $\boldsymbol{V}^*$ is an equilibrium, which is not necessarily true in the latter case (see Remark 3.1 for a counterexample). Recall also that, by Lemma 2.2, for any optimal equilibrium u^*, the value payoff of $V_k(u^*)$ is the greatest payoff player k can guarantee. Player k can ensure this payoff by simply carrying out his optimal equilibrium strategy s_k^* implicit in the strategy profile u^*. To summarise, the game has the unique value $\boldsymbol{V}^*$ at which the equilibria are achieved. The value $\boldsymbol{V}^*$ also represents the largest possible payoff each player can guarantee, irrespective of strategies of other players. Hence the value $\boldsymbol{V}^*$ is also the unique value of the game at time 0. Furthermore, $\boldsymbol{V}^*$ is financed precisely by the total initial investment, meaning that $\sum_{k\in\mathcal{M}} V_k(u^*) = C$.

Remark 3.3. An algorithm of finding an optimal equilibrium u^* with the property $\pi_{\mathbb{H}_{\mathcal{E}(u^*)}}(\boldsymbol{P}) = \pi_{\mathbb{S}}(\boldsymbol{P})$ can be inferred from Lemma 3.5 and its proof. Basically, if $\boldsymbol{P}$ does not belong to $\mathbb{S}$ then there exists $k \in \mathcal{M}$ with $P_k < X_k$. This means that $s_k^* = 0$ and thus player k should exercise at time 0. The game is then reduced to the subgame $\mathcal{G}_{\{-k\}}$ with one less player, with the new terminal payoff $\pi_{\mathbb{H}_{\{k\}}}(\boldsymbol{P})$, the value space $\mathbb{H}_{\{k\}}$ and the sub-simplex $\mathbb{S} \cap \mathbb{H}_{\{k\}}$. This is repeated recursively until the point lies inside the simplex, which will occur within $m - 1$ iterations.

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