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## PRINCIPLE OF CONDITIONING REVISITED

*Dedicated to Professor Agnieszka Plucińska*

**Abstract.** Principle of Conditioning is a well-known heuristic rule which allows constructing limit theorems for sums of *dependent* random variables from existing limit theorems for *independent* summands. In the paper we state a general limit theorem on convergence to stable laws, which is valid for stationary sequences and provides a link between the Principle of Conditioning and ergodic theorems.

### 1. Introduction

Principle of Conditioning is a heuristic rule which allows constructing limit theorems for sums of *dependent* random variables from existing limit theorems for *independent* summands. For example, applying the Principle of Conditioning to the Lindeberg–Feller Central Limit Theorem, one obtains the Martingale Central Limit Theorem due to Brown and Eagleson [4]. Indeed, in the Lindeberg–Feller theorem we deal with a triangular array  $\{X_{n,j}; j = 1, 2, \dots, j_n, n \in \mathbb{N}\}$  of random variables, which are independent in rows and satisfy  $EX_{n,j} = 0$ ,  $EX_{n,j}^2 < +\infty$ ,  $j = 1, 2, \dots, j_n$ ,  $n \in \mathbb{N}$ . Then the conditions

$$\sum_{j=1}^{j_n} EX_{n,j}^2 \rightarrow 1,$$

$$\sum_{j=1}^{j_n} EX_{n,j}^2 I\{|X_{n,j}| > \varepsilon\} \rightarrow 0, \quad \text{for every } \varepsilon > 0,$$

held as  $n \rightarrow \infty$ , imply the convergence in distribution of sums in rows:

$$(1) \quad \sum_{j=1}^{j_n} X_{n,j} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1), \quad \text{as } n \rightarrow \infty.$$

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The Principle of Conditioning says that if in the above conditions we replace:

- the expectations by conditional expectations with respect to the past,
- the summation to constants by summation to stopping times,
- the convergence of numbers by convergence in probability of random variables appearing in conditions,

then still the conclusion (1) will hold. In particular, we obtain a slight refinement of the Martingale Central Limit Theorem given in [4]:

**THEOREM 1.1.** *Let  $\{X_{n,j}; j \in \mathbb{N}, n \in \mathbb{N}\}$  be an array of random variables, which are row-wise adapted to a sequence of filtrations  $\{\{\mathcal{F}_{n,j}; j = 0, 1, \dots\}; n \in \mathbb{N}\}$ . For each  $n \in \mathbb{N}$ , let  $\tau_n$  be a stopping time with respect to  $\{\mathcal{F}_{n,j}; j = 0, 1, \dots\}$ .*

*If  $E(X_{n,j} | \mathcal{F}_{n,j-1}) = 0$ ,  $j, n \in \mathbb{N}$ , (i.e.  $\{X_{n,j}\}$  is a martingale difference array) and, as  $n \rightarrow \infty$ ,*

$$\sum_{j=1}^{\tau_n} E(X_{n,j}^2 | \mathcal{F}_{n,j-1}) \xrightarrow{\mathcal{P}} 1,$$

$$\sum_{j=1}^{\tau_n} E(X_{n,j}^2 I\{|X_{n,j}| > \varepsilon\} | \mathcal{F}_{n,j-1}) \xrightarrow{\mathcal{P}} 0, \text{ for every } \varepsilon > 0,$$

then

$$\sum_{j=1}^{\tau_n} X_{n,j} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1), \text{ as } n \rightarrow \infty.$$

The Principle of Conditioning is valid for more general theorems, including convergence to infinitely divisible laws with finite variance [4], general infinitely divisible laws [16], [2], limit theorems in Hilbert spaces [11], [13] and in certain Banach spaces [21] and an extension to functional limit theorems for sums [6] and for semimartingales [10]. We refer the reader to the expository paper [12], where detailed formulations, discussion and examples are contained. Later developments related to the Principle of Conditioning can be found in the well-known books by Kwapień and Woyczyński [17] and de la Peña and Giné [5].

From the contemporary perspective the Principle of Conditioning looks like an old-fashioned chapter of research, intensively developed in the period 1970–1985 and essentially closed nowadays.

This is not so, at least by two reasons. One of them is the fact that there seems to be again a growing interest in limit theorems related to martingale methods. Young people approach various problems in the limit theory by advanced tools like the Malliavin calculus, sometimes with referencing to the history [20] and sometimes not (see [19], where the results for quadratic

forms from [14] were rediscovered). On the other hand the emergence of new econometric models (see e.g. [22]) or physical motivations (see e.g. [15]) show that there are still many problems where limit theorems “with conditioning” lie in the heart of the reasoning.

The other reason is the tool that stands behind the verbal expression for the Principle of Conditioning.

**MAIN LEMMA.** [11] *Let  $\{X_{n,j}; j = 1, 2, \dots, k_n, n \in \mathbb{N}\}$  be an array of random variables which are row-wise adapted to a sequence of filtrations  $\{\{\mathcal{F}_{n,j}\}_{j=0,1,\dots,k_n}\}_{n \in \mathbb{N}}$ . Define*

$$\phi_{n,j}(\theta) = E(e^{i\theta X_{n,j}} | \mathcal{F}_{n,j-1}), \quad \phi_n(\theta) = \phi_{n,1}(\theta) \cdot \phi_{n,2}(\theta) \cdot \dots \cdot \phi_{n,j_n}(\theta).$$

If

$$(2) \quad \phi_n(\theta) \xrightarrow{\mathcal{P}} C(\theta) \neq 0,$$

where  $C(\theta)$  is a constant, then also

$$E e^{i\theta(X_{n,1} + X_{n,2} + \dots + X_{n,j_n})} \rightarrow C(\theta).$$

The Main Lemma admits a useful variant for normalized sequences (see [12, Theorem 3.1]).

**MAIN LEMMA FOR SEQUENCES.** *Let  $\{X_j; j \in \mathbb{N}\}$  be a sequence of random variables adapted to a filtration  $\{\mathcal{F}_j; j = 0, 1, 2, \dots\}$ . Let  $B_n \rightarrow \infty$  be a sequence of constants. Define*

$$\phi_{n,j}(\theta) = E(e^{i\theta X_j / B_n} | \mathcal{F}_{j-1}), \quad \phi_n(\theta) = \phi_{n,1}(\theta) \cdot \phi_{n,2}(\theta) \cdot \dots \cdot \phi_{n,j_n}(\theta).$$

If

$$(3) \quad \phi_n(\theta) \xrightarrow{\mathcal{P}} C(\theta),$$

where  $C(\theta)$  is a random variable such that  $P(C(\theta) \neq 0) = 1$ , then also

$$E e^{i\theta(X_1 + X_2 + \dots + X_{j_n}) / B_n} \rightarrow EC(\theta).$$

If (3) holds for every  $\theta \in \mathbb{R}^1$  and

$$C(\theta) = C(\theta, \omega) = \int_{\mathbb{R}^1} e^{i\theta x} \mu(dx, \omega),$$

for some random probability measure  $\mu(\cdot, \omega)$ , then  $(X_1 + X_2 + \dots + X_{j_n}) / B_n$  converges in distribution to the mixture  $\nu(\cdot) = E\mu(\cdot, \omega)$  and the convergence is stable in the Renyi sense.

We are going to demonstrate that the two above results can often be applied directly, i.e. without verifying the whole list of conditions of the type given in Theorem 1.1.

## 2. Two special models

**2.1. GARCH(1,1) processes.** A GARCH(1,1) process is defined by the system of recurrence equations

$$(4) \quad X_j = \sigma_j Z_j,$$

$$(5) \quad \sigma_j^2 = \beta + \lambda X_{j-1}^2 + \delta \sigma_{j-1}^2,$$

where the constants  $\beta, \lambda, \delta$  are nonnegative,  $\{Z_j\}$  is an i.i.d. noise,  $\sigma_j \geq 0$  and  $X_0$  and  $\sigma_0^2$  are given and independent of  $\{Z_j\}_{j \geq 1}$ . If  $\delta = 0$  in (5) then the corresponding process is called ARCH(1) process.

The terminology (ARCH stands for "Autoregressive Conditionally Heteroskedastic", while GARCH is the "Generalized ARCH") was introduced by Engle [8] and Bollerslev [3] in the context of modelling volatility phenomena in econometric time series. There exists a huge literature on both theoretical and practical aspects of GARCH processes. We refer the reader to the mathematical introduction to ARCH processes given in [7] and to the recent advanced source [9].

It is well known that if  $\lambda + \delta < 1$ , then there exists a strictly stationary sequence  $\{(X_j, \sigma_j^2)\}$  built on the i.i.d. noise  $\{Z_j\}$ , satisfying (4) and (5) and such that

$$(6) \quad E\sigma_j^2 = EX_j^2 = \frac{\beta}{1 - \lambda - \delta}.$$

Following Engle, let us assume that

$$(7) \quad \text{the noise variables are standard normal: } Z_j \sim \mathcal{N}(0, 1).$$

**THEOREM 2.1.** *Suppose that  $\lambda + \delta < 1$ ,  $\beta > 0$ , (7) holds and  $(X_0, \sigma_0^2)$  are chosen to make the whole sequence  $\{(X_n, \sigma_n^2)\}$  stationary.*

*Then*

$$\frac{X_1 + X_2 + \dots + X_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{\beta}{1 - \lambda - \delta}\right).$$

The above theorem seems to be well-known and can be proved in various ways. One possible direction is based on mixing properties of GARCH processes. Mikosch and Stărică [18] proved that GARCH(1,1) processes with Gaussian multiplicative renewals are strongly (or  $\alpha$ -)mixing with exponential rate. This means that  $\alpha(n) \leq K\rho^n$  for some constants  $K > 0$  and  $\rho \in (0, 1)$ , where for a stochastic process  $\{Y_j\}_{j \in \mathbb{N}}$  the well-known coefficient  $\alpha(n) = \alpha(n, \{Y_j\})$  is defined as

$$\alpha(n) = \sup \{|P(A \cap B) - P(A)P(B)|; A \in \mathcal{F}_1^m, B \in \mathcal{F}_{m+n}^\infty, m \in \mathbb{N}\},$$

with  $\mathcal{F}_1^m = \sigma\{Y_j; j \leq m\}$  and  $\mathcal{F}_{m+n}^\infty = \{Y_j; j \geq m+n\}$ . Since we have exponential  $\alpha$ -mixing and there exist moments higher than 2 (due to  $\lambda + \delta < 1$ ), it is easy to use the CLT for strongly mixing sequences. On the other

hand  $\{X_{n,j} = \frac{X_j}{\sqrt{n}}; j = 1, 2, \dots, n, n \in \mathbb{N}\}$  is a square integrable martingale difference array, so one might use also the Martingale CLT.

We prefer another proof, based on our Main Lemma from Section 1. Indeed, by (4), (5) and (7), the regular distribution of  $X_j$  given the “past” is of the form

$$\mathcal{N}(0, \beta + \lambda X_{j-1}^2 + \delta \sigma_{j-1}^2).$$

Hence

$$\phi_{n,j}(\theta) = E(e^{i\theta X_j/\sqrt{n}} | \mathcal{F}_{j-1}) = \exp\left(-\frac{1}{2n}\theta^2(\beta + \lambda X_{j-1}^2 + \delta \sigma_{j-1}^2)\right),$$

and by the Individual Ergodic Theorem

$$\begin{aligned} -\ln \phi_n(\theta) &= \frac{1}{n} \sum_{j=1}^n \frac{1}{2}\theta^2(\beta + \lambda X_{j-1}^2 + \delta \sigma_{j-1}^2) \\ &\rightarrow \frac{1}{2}\theta^2(\beta + \lambda EX_0^2 + \delta E\sigma_0^2) \text{ a.s.} \\ &= \frac{\beta}{1 - \lambda - \delta} \quad \text{by (6).} \end{aligned}$$

If  $\lambda + \delta = 1$  then the stationary solution to (4) and (5) has infinite variance and things become rather delicate. To bring out the flavour of possible complications we state here a particular result for this case.

**THEOREM 2.2.** *Assume  $\delta = 0, \lambda = 1$  and  $\beta > 0$ . Then*

$$\frac{X_1 + X_2 + \dots + X_n}{\sqrt{n \ln n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, C_{\beta,1}),$$

where

$$C_{\beta,1} = \frac{\beta}{E[(Z_1^2) \ln(Z_1^2)]} \approx 1.3705 \cdot \beta.$$

**Proof.** Writing down  $\ln \phi_n(\theta)$  as before we obtain

$$-\ln \phi_n(\theta) = \frac{1}{n \ln n} \sum_{j=1}^n \frac{1}{2}\theta^2(\beta + X_{j-1}^2) \sim \frac{1}{2}\theta^2 \frac{1}{n \ln n} \sum_{j=1}^n X_{j-1}^2$$

and the convergence in probability of  $\phi_n(\theta)$  is not obvious at all, unless we apply the proper tool - the weak law of large numbers due to Szewczak [23]. ■

The reader is referred to [1] for discussion of the radically different case  $\lambda + \delta > 1$ .

**2.2. Martingale transforms with stable multiplicative noise.** Surgailis [22] discussed a class of complex GARCH-like models (“Quadratic ARCH( $\infty$ )”) and showed that in some cases their limit behaviour can be

reduced to limit theorems for “martingale transforms” of special shape. Following Appendix B in [22] let us consider the following class of stationary processes.

- $X_j = V_j Z_j$ ,  $j \in \mathbb{Z}$ , is a stationary sequence adapted to a filtration  $\{\mathcal{F}_j\}$ .
- The sequence  $\{V_j\}$  is stationary and *predictable* with respect to  $\{\mathcal{F}_j\}$  (i.e.  $V_j$  is  $\mathcal{F}_{j-1}$ -measurable,  $j \in \mathbb{Z}$ ).
- The sequence  $\{Z_j\}$  is i.i.d., adapted to  $\{\mathcal{F}_j\}$  and for each  $j \in \mathbb{Z}$ ,  $Z_j$  is *independent* of  $\mathcal{F}_{j-1}$  (hence of  $V_j$ ).
- The law of  $Z_j$  belongs to the domain of *normal* attraction of a strictly  $\alpha$ -stable law  $\nu_\alpha$ ,  $\alpha \in (1, 2)$ . This means that as  $n \rightarrow \infty$

$$(8) \quad \frac{Z_1 + Z_2 + \dots + Z_n}{n^{1/\alpha}} \xrightarrow{\mathcal{D}} \nu_\alpha,$$

where

$$(9) \quad \hat{\nu}_\alpha(\theta) = \exp \left( -c|\theta|^\alpha (1 - i\beta \operatorname{sgn}(\theta) \tan(\pi\alpha/2)) \right),$$

for some  $c > 0$  and  $\beta \in [-1, 1]$  and, in particular,

$$(10) \quad EZ_j = 0, \quad j \in \mathbb{Z}.$$

Surgailis [22, Theorem B.1] proved the following result.

**THEOREM 2.3.** *For the model described above, if  $\{V_j\}$  is ergodic and for some  $r > \alpha$*

$$(11) \quad E|V_0|^r < +\infty,$$

then

$$\frac{X_1 + X_2 + \dots + X_n}{n^{1/\alpha}} \xrightarrow{\mathcal{D}} \mu_\alpha,$$

where

$$\hat{\mu}_\alpha(\theta) = \exp \left( -c'|\theta|^\alpha (1 - i\beta' \operatorname{sgn}(\theta) \tan(\pi\alpha/2)) \right),$$

and

$$c' = cE|V_0|^\alpha, \quad \beta' = \frac{\beta E(|V_0|^\alpha \operatorname{sgn}(V_0))}{E|V_0|^\alpha}.$$

We are going to show that assumption (11) can be relaxed to

$$E|V_0|^\alpha < +\infty,$$

and it is possible to provide a formula for the limiting mixture of stable distributions also in the non-ergodic case. This will be done in the next section. Here we shall examine the simplest possible case via our Main Lemma for Sequences.

**EXAMPLE 1.** Let  $Z_j$ 's be *symmetric*  $\alpha$ -stable,  $\alpha \in (0, 2]$ . Then

$$\phi_{n,j}(\theta) = E(e^{i\theta X_j/n^{1/\alpha}} | \mathcal{F}_{j-1}) = \exp \left( -\frac{c}{n} |V_j|^\alpha |\theta|^\alpha \right),$$

and by the Individual Ergodic Theorem

$$\begin{aligned}
 -\ln \phi_n(\theta) &= c\theta^\alpha \frac{1}{n} \sum_{j=1}^n |V_j|^\alpha \\
 &\rightarrow c|\theta|^\alpha E(|V_1|^\alpha | \mathcal{I}) \text{ a.s.},
 \end{aligned}$$

where  $\mathcal{I}$  is the invariant  $\sigma$ -field for the stationary process  $\{V_j\}$ . Hence

$$\frac{X_1 + X_2 + \dots + X_n}{n^{1/\alpha}} \xrightarrow{\mathcal{D}} X_\infty,$$

where

$$Ee^{i\theta X_\infty} = E \exp \left( -c|\theta|^\alpha E(|V_1|^\alpha | \mathcal{I}) \right).$$

### 3. Convergence to mixtures of stable distributions

Let us consider a stationary sequence  $\{X_j\}$  adapted to a filtration  $\{\mathcal{F}_j\}$ . For each  $j$ , let  $\mu_j$  be a version of the regular conditional distribution of  $X_j$  given  $\mathcal{F}_{j-1}$ . In what follows we will assume that  $\{\mu_j\}$  is a stationary sequence. In other words, for every bounded Borel function  $f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ , the sequence of real random variables

$$(12) \quad \left\{ E(f(X_j) | \mathcal{F}_{j-1}) \right\}_{j \in \mathbb{N}} \text{ is stationary.}$$

Such assumption is not extra demanding: it is enough that  $\{X_j\}$  is a sequence indexed by  $\mathbb{Z}$  and that  $\mathcal{F}_j = \sigma\{(X_l, A_l); l \leq j\}$  for some auxillary stationary sequence  $\{A_j\}_{j \in \mathbb{Z}}$ .

**THEOREM 3.1.** *Let  $\{X_j\}$  be a stationary sequence adapted to a filtration  $\{\mathcal{F}_j\}$ . Suppose that (12) holds and for some  $B_n \rightarrow \infty$  the following conditions are satisfied.*

$$(13) \quad L_1\text{-}\lim_{n \rightarrow \infty} n \left| 1 - E(e^{i\theta X_1/B_n} | \mathcal{F}_0) \right|^2 = 0, \theta \in \mathbb{R}^1,$$

$$(14) \quad L_1\text{-}\lim_{n \rightarrow \infty} n \left( 1 - E(e^{i\theta X_1/B_n} | \mathcal{F}_0) \right) = \Psi_1(\theta), \theta \in \mathbb{R}^1,$$

where  $L_1$ -lim means taking the limit in  $L_1$ .

Then for each  $\theta \in \mathbb{R}^1$

$$(15) \quad \prod_{j=1}^n E(e^{i\theta X_j/B_n} | \mathcal{F}_{j-1}) \xrightarrow{\mathcal{P}} \exp \left( -\bar{\Psi}(\theta) \right),$$

where

$$(16) \quad \bar{\Psi}(\theta) = L_1\text{-}\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \Psi_j(\theta),$$

with

$$(17) \quad \Psi_j(\theta) = L_1\text{-}\lim_{n \rightarrow \infty} n \left( 1 - E(e^{i\theta X_j/B_n} | \mathcal{F}_{j-1}) \right).$$

Moreover, if the function

$$\theta \mapsto E \exp \left( - \bar{\Psi}(\theta) \right) \in \mathbb{C}$$

is continuous at  $\theta = 0$ , then  $(X_1 + X_2 + \dots + X_n)/B_n$  converges in distribution to a random variable  $X_\infty$  with the characteristic function given by

$$E e^{i\theta X_\infty} = E \exp \left( - \bar{\Psi}(\theta) \right),$$

and the convergence is stable in the Renyi sense.

**Proof.** Fix  $\theta \in \mathbb{R}^1$  and for simplicity define  $\psi_{n,j} = E(e^{i\theta X_j/B_n} | \mathcal{F}_{j-1})$ . It is an easy computation to show that if  $z_1, z_2, \dots, z_n$  are complex numbers such that  $|z_j| \leq 1, j = 1, 2, \dots, n$ , then

$$(18) \quad \left| \prod_{j=1}^n z_j - \exp \left( \sum_{j=1}^n (z_k - 1) \right) \right| \leq 5 \sum_{j=1}^n |1 - z_j|^2.$$

It follows that

$$\left| \prod_{j=1}^n \psi_{n,j} - \exp \left( - \sum_{j=1}^n (1 - \psi_{n,j}) \right) \right| \leq 5 \sum_{j=1}^n |1 - \psi_{n,j}|^2.$$

By (12) and (13)

$$E \sum_{j=1}^n |1 - \psi_{n,j}|^2 = n E |1 - \psi_{n,1}|^2 \rightarrow 0,$$

hence it suffices to show that

$$(19) \quad L_1\text{-}\lim_{n \rightarrow \infty} \sum_{j=1}^n (1 - \psi_{n,j}) = \bar{\Psi}(\theta).$$

First let us observe that random variables  $\Psi_j(\theta)$  do exist, are in  $L_1$  by stationarity and (14), and form a stationary sequence. Hence by the Mean Ergodic Theorem there exists

$$\bar{\Psi}(\theta) = L_1\text{-}\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \Psi_j(\theta).$$

It is now enough to notice that

$$E \left| \sum_{j=1}^n (1 - \psi_{n,j}) - \frac{1}{n} \sum_{j=1}^n \Psi_j(\theta) \right| = \frac{1}{n} E \left| \sum_{j=1}^n (n(1 - \psi_{n,j}) - \Psi_j(\theta)) \right|$$

$$\begin{aligned} &\leq \frac{1}{n} E \sum_{j=1}^n |n(1 - \psi_{n,j}) - \Psi_j(\theta)| \\ &= E |n(1 - \psi_{n,1}) - \Psi_1(\theta)| \rightarrow 0. \end{aligned}$$

The other statements are contained in the Main Lemma for Sequences given in Section 1. ■

We are now ready to prove the announced result on convergence of martingale transforms.

**THEOREM 3.2.** *Let  $(X_j, V_j, Z_j)$  be the model described in Section 2.2. If*

$$(20) \quad E|V_0|^\alpha < +\infty,$$

*then stably in the Renyi sense*

$$\frac{X_1 + X_2 + \dots + X_n}{n^{1/\alpha}} \xrightarrow{\mathcal{D}} X_\infty,$$

where

$$(21) \quad Ee^{i\theta X_\infty} = \exp(-\bar{\Psi}(\theta)),$$

$$(22) \quad \bar{\Psi}(\theta) = c|\theta|^\alpha E\left(|V_1|^\alpha (1 - i\beta \operatorname{sgn}(V_1\theta) \tan(\pi\alpha/2)) \Big| \mathcal{I}\right),$$

and  $\mathcal{I}$  is the invariant  $\sigma$ -field for the stationary process  $\{V_j\}$ .

**Proof.** According to Theorem 3.1 we have to check (13) and (14) and identify the limit  $\bar{\Psi}(\theta)$  in (16).

Let us first apply the standard estimates to  $|1 - E(e^{i\theta X_1/B_n} | \mathcal{F}_0)|$ , taking into account relation (10).

$$\begin{aligned} &\left| 1 - E\left(e^{i\theta V_1 Z_1/n^{1/\alpha}} \Big| \mathcal{F}_0\right) \right| \\ &= \left| E\left(1 + \frac{i\theta V_1 Z_1}{n^{1/\alpha}} - e^{i\theta V_1 Z_1/n^{1/\alpha}} \Big| \mathcal{F}_0\right) \right| \\ &\leq \left| E\left(\left(1 + \frac{i\theta V_1 Z_1}{n^{1/\alpha}} - e^{i\theta V_1 Z_1/n^{1/\alpha}}\right) I\{|V_1 Z_1| \leq n^{1/\alpha}\} \Big| \mathcal{F}_0\right) \right| \\ &\quad + \left| E\left(\left(1 - e^{i\theta V_1 Z_1/n^{1/\alpha}}\right) I\{|V_1 Z_1| > n^{1/\alpha}\} \Big| \mathcal{F}_0\right) \right| \\ &\quad + \left| E\left(\left(\frac{i\theta V_1 Z_1}{n^{1/\alpha}} I\{|V_1 Z_1| > n^{1/\alpha}\}\right) \Big| \mathcal{F}_0\right) \right| \\ &\leq \frac{|\theta|^2}{2n^{2/\alpha}} V_1^2 E(Z_1^2 I\{|V_1 Z_1| \leq n^{1/\alpha}\} | \mathcal{F}_0) \\ &\quad + 2P(|V_1 Z_1| > n^{1/\alpha}) | \mathcal{F}_0 + \frac{|\theta|}{n^{1/\alpha}} |V_1| E(|Z_1| I\{|V_1 Z_1| > n^{1/\alpha}\} | \mathcal{F}_0) \\ &= W_1(n, \theta) + W_2(n, \theta) + W_3(n, \theta). \end{aligned}$$

It follows now from (8) that

$$(23) \quad nP(|Z_1| > n^{1/\alpha}) \rightarrow \gamma > 0,$$

hence there exists a constant  $K > 0$  such that

$$(24) \quad x^\alpha P(|Z_1| > x) \leq K, \quad x \in \mathbb{R}^+.$$

Using (24) we can estimate the quantities  $W_1(n, \theta)$ ,  $W_2(n, \theta)$  and  $W_3(n, \theta)$ .

$$\begin{aligned} W_1(n, \theta) &\leq \frac{|\theta|^2}{2n^{2/\alpha}} V_1^2 2 \int_0^{n^{1/\alpha}/|V_1|} tP(|Z_1| > t) dt \\ &\leq \frac{|\theta|^2 V_1^2}{n^{2/\alpha}} \frac{Kn^{(2-\alpha)/\alpha}}{(2-\alpha)|V_1|^{2-\alpha}} = \frac{1}{n} K|\theta|^2 |V_1|^\alpha. \\ W_2(n, \theta) &\leq \frac{1}{n} K|V_1|^\alpha. \end{aligned}$$

$$\begin{aligned} W_3(n, \theta) &= \frac{|\theta||V_1|}{n^{1/\alpha}} \left( \frac{n^{1/\alpha}}{|V_1|} P(|V_1 Z_1| > n^{1/\alpha} | \mathcal{F}_0) + \int_{\frac{n^{1/\alpha}}{|V_1|}}^\infty P(|Z_1| > t) dt \right) \\ &\leq \frac{1}{n} \frac{\alpha}{\alpha - 1} K|\theta||V_1|^\alpha. \end{aligned}$$

We conclude that

$$(25) \quad \left| 1 - E(e^{i\theta V_1 Z_1/n^{1/\alpha}} | \mathcal{F}_0) \right| \rightarrow 0, \text{ a.s.},$$

and

$$(26) \quad n \left| 1 - E(e^{i\theta V_1 Z_1/n^{1/\alpha}} | \mathcal{F}_0) \right| \leq K \left( |\theta|^2 + 1 + \frac{\alpha}{\alpha - 1} |\theta| \right) |V_1|^\alpha,$$

$$(27) \quad n \left| 1 - E(e^{i\theta V_1 Z_1/n^{1/\alpha}} | \mathcal{F}_0) \right|^2 \leq 2K \left( |\theta|^2 + 1 + \frac{\alpha}{\alpha - 1} |\theta| \right) |V_1|^\alpha.$$

By (25) and (26)

$$n \left| 1 - E(e^{i\theta V_1 Z_1/n^{1/\alpha}} | \mathcal{F}_0) \right|^2 \rightarrow 0, \text{ a.s.}$$

Since  $E|V|^\alpha < \infty$ , relation (27) gives the  $L_1$ -domination and (13) follows. Similarly, relation (26) gives the  $L_1$ -domination for (14) and so it is enough to show that for some  $\Psi_1(\theta)$ ,

$$n \left( 1 - E(e^{i\theta V_1 Z_1/n^{1/\alpha}} | \mathcal{F}_0) \right) \rightarrow \Psi_1(\theta), \text{ a.s.}$$

Here the crucial observation is that (8) - valid for independent centered summands - implies the convergence

$$n \left( 1 - Ee^{i\theta Z_1/n^{1/\alpha}} \right) \rightarrow c|\theta|^\alpha \left( 1 - i\beta \operatorname{sgn}(\theta) \tan(\pi\alpha/2) \right),$$

(see (9)). Substituting in the above  $V_1\theta$  in place of  $\theta$  we get that

$$\begin{aligned} & n\left(1 - E\left(e^{i\theta V_1 Z_1/n^{1/\alpha}} \mid \mathcal{F}_0\right)\right) \\ & \rightarrow c|V_1|^\alpha |\theta|^\alpha (1 - i\beta \operatorname{sgn}(V_1\theta) \tan(\pi\alpha/2)) =: \Psi_1(\theta) \text{ a.s.} \end{aligned}$$

Now formula (22) clearly follows from (16). ■

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