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# CONDITIONAL VERSION OF THE DONATI-MARTIN AND YOR FORMULA AND ITS APPLICATIONS

*Dedicated to Professor Agnieszka Plucińska*
*on the occasion of her 80th birthday*

**Abstract.** We present a new elementary and elegant probabilistic proof of conditional version of the Donati-Martin and Yor formula, more precisely, the conditional version of the Laplace transform of $\int_0^t B_u^2 du$ given $B_t = y$, where $B$ is a Brownian motion. Next, using our form of conditional formula, we obtain the new results concerning Laplace transforms of some processes and a kind of reflection principle.

## 1. Introduction

The functionals of Brownian motion play a central role in many applications of mathematics. Books of Borodin and Salminen [1], Revuz and Yor [6] or Mansuy and Yor [5] are, among others, the excellent sources of references for a lot of formulae associated with different functionals of Brownian motion. The technique of establishing conditional functionals of Markov processes using the knowledge of its semi-group is presented in [6] (for details see, e.g., Proposition 3.1 p. 350). In [1], the main tool for establishing such functionals is the Feynman–Kac theorem. In this note, we present a probabilistic proof of conditional version of the Donati-Martin and Yor formula [2] for the Laplace transform of $\int_0^t B_u^2 du$ given $B_t = y$ for a Brownian motion $B$ starting from $x \in \mathbb{R}$. More precisely, in Theorem 2.1 we find the closed-form formula for

$$H_{b,t,x}(y) := \mathbb{E}\left[\exp\left(-\frac{b^2}{2}\int_0^t B_u^2 du\right)\Big| B_t = y\right], \tag{1}$$

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where $b > 0,\ y \in \mathbb{R}$. This result can be also derived by a straightforward calculation from the following formula from Borodin and Salminen [1, formula 1.9.7 p. 168]

$$\text{(2)}\qquad \mathbb{E}\left\{\exp\left(-\frac{b^2}{2}\int_0^t B_u^2 du\right); B_t \in dy\right\} = \frac{\sqrt{b}}{\sqrt{2\pi\sinh(tb)}}\exp\left(-\frac{(x^2+y^2)b\cosh(tb)-2xyb}{2\sinh(tb)}\right)dy,$$

where for random variables $\xi$ and $\eta$ the following notation is used

$$\mathbb{E}(\xi;\eta \in dy) := \frac{\partial}{\partial y}\mathbb{E}(\xi 1_{\{\eta<y\}})\ dy.$$

Our proof of Theorem 2.1 uses probability methods and is different from that in [1] used for the proof of (2), and we believe that it is interesting by itself. The formula presented by us can be very useful, especially in computing Laplace transforms of vectors $(g(B_t), \int_0^t B_u^2 du)$ for a Brownian motion starting from an arbitrary $x \in \mathbb{R}$ and some function $g$. Indeed, by conditioning, we have the following formula

$$\mathbb{E}e^{-cg(B_t)-\frac{b^2}{2}\int_0^t B_u^2 du} = \mathbb{E}\big(e^{-cg(B_t)}H_{b,t,x}(B_t)\big),$$

where the explicit form of $H_{b,t,x}$ is given by Theorem 2.1. For the example of usefulness of the above argumentation see, e.g., Jakubowski and Wiśniewolski [3, Prop. 4.6].

Theorem 2.1 enables also to deduce some interesting property of Brownian motion path. In Corollary 2.2 we show that Theorem 2.1 implies formula (2) obtained by Borodin and Salminen. In Corollary 2.3 and Theorem 2.4 we investigate the conditional law of $\int_0^t B_s^2 ds$ given the set $\{B_t = y\}$. In Theorem 2.4 we prove that the conditional law of $\int_0^t B_s^2 ds$ given the set $\{B_t = y\}$ is equal to the conditional law of $\int_0^t W_s^2 ds$ given $\{W_t = y\}$ for a Brownian motion $W$ starting from $-y$. Theorem 2.5 uses the result of Theorem 2.1 in establishing the representation of

$$\mathbb{E}\left[\exp\left(-\lambda\frac{\int_0^t B_u^2 du}{1+\int_0^t B_u^2 du}\right)\middle| B_t = y\right]$$

in terms of function $H_{b,t,x}$ and a squared Bessel process of index $(-1)$.

## 2. Results

We start from giving the explicite form of conditional expectation (1).

**Theorem 2.1.** *Let $B$ be a Brownian motion starting from $x \in \mathbb{R}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, for any $b > 0$, $t > 0$ and $y \in \mathbb{R}$,*

$$H_{b,t,x}(y) = \sqrt{\frac{bt}{\sinh(bt)}} \exp\left(-\frac{1}{2t}\left[y^2(bt\coth(bt)-1) + 2btx(x+y)\frac{\cosh(bt)-1}{\sinh(bt)}\right]\right).$$

**Proof.** We use the well known formula for Laplace transform of vector $(B_t^2, \int_0^t B_u^2 du)$ (see Mansuy and Yor [5, p. 18]). For any $c > 0$, $b > 0$ we have

$$\mathbb{E}\left(\exp\left(-cB_t^2 - \frac{b^2}{2}\int_0^t B_u^2 du\right)\right) = \frac{1}{\sqrt{\cosh(bt) + \frac{2c}{b}\sinh(bt)}} \exp\left(x^2\left[\frac{b}{2} - \frac{(b/2+c)e^{bt}}{\cosh(bt) + \frac{2c}{b}\sinh(bt)}\right]\right).$$

After some algebra, we obtain the equivalent formula which is suitable for our purpose

$$\mathbb{E}\left(\exp\left(-cB_t^2 - \frac{b^2}{2}\int_0^t B_u^2 du\right)\right) = \frac{1}{\sqrt{\cosh(bt) + \frac{2c}{b}\sinh(bt)}} \exp\left(-cx^2 + x^2\frac{\frac{2}{b}(c^2 - b^2/4)}{\coth(bt) + 2c/b}\right). \tag{3}$$

It is clear that for $t = 0$ we have $H_{b,0,x}(y) = 1$ for any $y \in \mathbb{R}$, and $0 < H_{b,t,x} \leq 1$. The function $H_{b,t,x}(\cdot)$ must satisfy, for all $c > 0$,

$$\mathbb{E}\left(\exp\left(-cB_t^2 - \frac{b^2}{2}\int_0^t B_u^2 du\right)\right) = \mathbb{E}\left[e^{-cB_t^2} H_{b,t,x}(B_t)\right]. \tag{4}$$

It is easy to see that there can be only one such function $H_{b,t,x}(y), y \in \mathbb{R}$. Let us try to find it in the form

$$H_{b,t,x}(y) = F(b,t,x)\exp\left(-\frac{1}{2t}\left[(K(b,t,x)-1)(y-x)^2 + 2L(b,t,x)(y-x) + R(b,t,x)\right]\right) \tag{5}$$

for some functions $K, L, R, F$ of three variables $b, t, x$. To make calculations more clear we omit the arguments of the functions $K, L, R, F$. Since $B$ is a Brownian motion starting from $x$,

$$\mathbb{E}\big[e^{-cB_t^2} H_{b,t,x}(B_t)\big] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} F \exp\left(-c(x+y)^2 - \frac{1}{2t}[Ky^2 + 2Ly + R]\right) dy$$
$$= F \frac{1}{\sqrt{K+2tc}} \exp\left(-cx^2 + \frac{1}{2t}\left[\frac{(L+2ctx)^2}{K+2ct} - R\right]\right).$$

We now match the last expression with (3) and guess that

$$\begin{aligned} F &= F(b,t,x) = F(b,t) = \sqrt{\frac{bt}{\sinh(bt)}}, \\ K &= K(b,t,x) = K(b,t) = bt\coth(bt), \\ R &= 2Lx, \\ 0 &= L^2 - 2LKx + b^2x^2t^2. \end{aligned}$$

The last equation has two solutions $L_1 = xbt(\cosh(bt)+1)/\sinh(bt)$ and $L_2 = xbt(\cosh(bt)-1)/\sinh(bt)$ but the first one tends to infinity when $t$ tends to zero (causing $H$ to be 0 in the limit). But from the Lebesgue theorem we clearly see that

$$\lim_{t\to 0} \mathbb{E} H_{b,t,x}(B_t) = \lim_{t\to 0} \mathbb{E}\left( \exp\left(-\frac{b^2}{2}\int_0^t B_u^2 du\right)\right) = 1,$$

so what is left is the second solution that fits perfectly our computation. Hence, with

$$\begin{aligned} L &= L(b,t,x) = xbt\frac{\cosh(bt)-1}{\sinh(bt)}, \\ R &= R(b,t,x) = 2xB(b,t,x) = 2x^2bt\frac{\cosh(bt)-1}{\sinh(bt)}, \end{aligned}$$

and $F, K$ defined above, the function $H_{b,t,x}$ satisfies the condition (4) by construction. To conclude the proof we insert $F, K, L, R$ just computed in (5) and obtain the assertion of the theorem. ∎

**Corollary 2.2.** *Theorem* 2.1 *implies formula* (2).

**Proof.** Using the notation of Theorem 2.1, we have

$$\mathbb{E}\left[\exp\left(-\frac{b^2}{2}\int_0^t B_u^2 du\right) 1_{\{B_t \le y\}}\right] = \mathbb{E}\big[H_{b,t,x}(B_t) 1_{\{B_t \le y\}}\big]$$
$$= \int_{-\infty}^{y-x} \frac{1}{\sqrt{2\pi t}}\, e^{-z^2/2t} H_{b,t,x}(z) dz$$

$$= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi t}} \, e^{-(s-x)^2/2t} H_{b,t,x}(s-x)ds,$$

and after inserting in the last expression the form of $H_{b,t,x}$ given by Theorem 2.1 and some simple algebra the result follows. ∎

The next corollary gives closed-form formula, which is much simpler then formula 1.9.8 in [1] (page 169).

**Corollary 2.3.** *Let $t \geq 0$ be fixed and $g$ be the density function of the vector $(B_t, \int_0^t B_u^2 du)$, where $B$ is a Brownian motion starting from $x \in \mathbb{R}$. Then for $y \in \mathbb{R}$ and $c > 0$,*

$$\int_0^{\infty} e^{-cz} g(y,z)dz = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\sqrt{2c}}{\sinh(t\sqrt{2c})}} \exp\left(-\frac{y^2}{2}\sqrt{2c}\coth(t\sqrt{2c})\right)$$
$$\times \exp\left(-2(x+y)x\sqrt{2c}\frac{\cosh(t\sqrt{2c})-1}{\sinh(t\sqrt{2c})} - \frac{2yx-x^2}{2t}\right).$$

**Proof.** This is an immediate consequence of formula (1) with $b = \sqrt{2c}$, the fact that

$$\int_0^{\infty} e^{-cz} g(y,z)dz = H_{\sqrt{2c},t,x}(y) g_1(y),$$

where $g_1(y) = \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}$ denotes the density function of the random variable $B_t + x$, and the form of $H_{b,t,x}$ given by Theorem 2.1. ∎

Now, we deduce from Theorem 2.1 a very interesting property of Brownian motion path. In a sense, it is a kind of reflection principle (see, e.g. [4, & 2.2.6]).

**Theorem 2.4.** *Fix $t \geq 0$ and $y \in \mathbb{R}$. Let $B$ be a standard Brownian motion. Then the conditional law of random variable $\int_0^t B_s^2 ds$ given $\{B_t = y\}$ is equal to the conditional law of random variable $\int_0^t W_s^2 ds$ given $\{W_t = y\}$, where $W$ is a Brownian motion such that $W_0 = -y$.*

**Proof.** We use the notation of Theorem 2.1. Let $X = \int_0^t B_s^2 ds$ and $Y = \int_0^t W_s^2 ds$. The key observation which leads to prove the theorem is that for any $\lambda > 0$ we have $H_{\lambda,t,0}(y) = H_{\lambda,t,-y}(y)$. It means that the conditional Laplace transform of $X$ given $\{B_t = y\}$ is equal to the conditional Laplace transform of $Y$ given $\{W_t = y\}$, which means that these random variables have the same conditional law. ∎

The next theorem is an interesting application of Theorem 2.1.

**Theorem 2.5.** *Let $\lambda > 0$ and $B$ be a Brownian motion starting from $x \in \mathbb{R}$. Then*

$$\mathbb{E}\left[\exp\left(-\frac{\lambda\int_0^t B_u^2 du}{1+\int_0^t B_u^2 du}\right)\Big| B_t = y\right] = \mathbb{E}(f_{R^\lambda_{\frac{1}{2}}}(y)),$$

*where $f_\alpha(y) = H_{\sqrt{2\alpha},t,x}(y)$ for $\alpha > 0$, and $R^\lambda$ is a squared Bessel process of index $(-1)$ starting from $\lambda$.*

**Proof.** Let $R^\lambda$ be a squared Bessel process of index $(-1)$ starting from $\lambda$ and independent of $B$. Using the Laplace transform of squared Bessel process (see [6, Chapter XI, p. 441]), we infer that

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(-R^\lambda_{\frac{1}{2}}\int_0^t B_u^2 du\Big)\Big| B_t\Big] &= \mathbb{E}\Big[\mathbb{E}\Big[\exp\Big(-R^\lambda_{\frac{1}{2}}\int_0^t B_u^2 du\Big)\Big|\sigma(B_u, u\le t)\Big] B_t\Big] \\ &= \mathbb{E}\left[\exp\left(-\frac{\lambda\int_0^t B_u^2 du}{1+\int_0^t B_u^2 du}\right)\Big| B_t\right].\end{aligned}$$

Let us denote by $\rho_R = \rho_R(t,\lambda,z)dz$ the density function of $R^\lambda_t$. Since processes $R$ and $B$ are independent, we observe that for $A \in \sigma(B_t)$

$$\begin{aligned}&\mathbb{E}\Big[1_A \exp\Big(-R^\lambda_{\frac{1}{2}}\int_0^t B_u^2 du\Big)\Big] \\ &\qquad = \int_0^\infty \mathbb{E}\Big[1_A \exp\Big(-z\int_0^t B_u^2 du\Big)\Big]\rho_R(1/2,\lambda,z)dz \\ &\qquad = \int_0^\infty \mathbb{E}\Big[1_A \mathbb{E}\Big(\exp\Big(-z\int_0^t B_u^2 du\Big)\Big| B_t\Big)\Big]\rho_R(1/2,\lambda,z)dz \\ &\qquad = \int_0^\infty \mathbb{E}\big(1_A f_z(B_t)\big)\rho_R(1/2,\lambda,z)dz = \mathbb{E}\big(1_A f_{R^\lambda_{\frac{1}{2}}}(B_t)\big),\end{aligned}$$

where in the last equation we use Theorem 2.1. This concludes the proof. ∎

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