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Diana Dziewa-Dawidczyk, Zbigniew Pasternak-Winiarski

DIFFERENTIAL COMPLETIONS AND COMPACTIFICATIONS OF A DIFFERENTIAL SPACE

Abstract. Differential completions and compactifications of differential spaces are introduced and investigated. The existence of the maximal differential completion and the maximal differential compactification is proved. A sufficient condition for the existence of a complete uniform differential structure on a given differential space is given.

1. Introduction

This article is the forth of the series of papers concerning integration of differential forms and densities on differential spaces (the first three are [4], [5] and [6]). We describe differential completions and differential compactifications of differential spaces which are used in our theory of integration.

Section 2 of the paper contains basic definitions and the description of preliminary facts concerning theory of differential spaces. In Section 3 we give basic definitions and describe the standard facts concerning theory of uniform spaces. Results contained in Propositions 3.1–3.4 and in Corollary 3.1 are well known but we give descriptions and proofs of this results in the form which is convenient to our purpose. We introduce the notion of a differential completion of a differential space. We construct differential completions of a differential space using families of generators of its differential structure (Proposition 3.5, Definition 3.8). Section 4 is devoted to the investigation of properties of differential completions. We define some natural order in the set of all differential completions of a given differential space. We prove that for any differential space $(M, \mathcal{C})$ there exists the maximal differential completion with respect to this order (Theorem 4.1). If for the uniform structure defined by some family of generators the space M is complete then the appropriate differential completion of $(M, \mathcal{C})$ is maximal and coincides with $(M, \mathcal{C})$ (Theorem 4.2). At the end we prove that

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if a differential structure $\mathcal{C}$ posses a countable family of generators, then it coincides with its maximal differential completion (Theorem 4.3). As a corollary we obtain general topological result about the existence on a given topological space a complete uniform structure defining the initial topology (Corollary 4.1). In Section 5 we introduce and investigate the notion of a differential compactification of a differential space. Similarly as in Section 4 we prove the existence of the maximal differential compactification of a given differential space with respect to the suitable order.

Without any other explanation we use the following symbols: $\mathbb{N}$-the set of natural numbers; $\mathbb{R}$-the set of reals.

2. Differential spaces

Let M be a nonempty set and let $\mathcal{C}$ be a family of real valued functions on M. Denote by $\tau_{\mathcal{C}}$ the weakest topology on M with respect to which all functions of $\mathcal{C}$ are continuous. A function $f : M \to \mathbb{R}$ is called *a local $\mathcal{C}$-function on M* if for every $m \in M$ there is a neighborhood V of m and $\alpha \in \mathcal{C}$ such that $f_{|V} = \alpha_{|V}$. The set of all local $\mathcal{C}$-functions on M is denoted by $\mathcal{C}_M$. It is easy to see that $\tau_{\mathcal{C}_M} = \tau_{\mathcal{C}}$ (see [4], [5]).

A function $f : M \to \mathbb{R}$ is called *$\mathcal{C}$-smooth function on M* if there exist $n \in \mathbb{N}$, $\omega \in C^\infty(\mathbb{R}^n)$ and $\alpha_1, \ldots, \alpha_n \in \mathcal{C}$ such that $f = \omega \circ (\alpha_1, \ldots, \alpha_n)$. The set of all $\mathcal{C}$-smooth functions on M is denoted by sc$\mathcal{C}$. Since $\mathcal{C} \subset sc\mathcal{C}$ and any superposition $\omega \circ (\alpha_1, \ldots, \alpha_n)$ is continuous with respect to $\tau_{\mathcal{C}}$ we obtain $\tau_{sc\mathcal{C}} = \tau_{\mathcal{C}}$ (see [4], [5]).

A set $\mathcal{C}$ of real functions on M is said to be a *(Sikorski's) differential structure* if: (i) $\mathcal{C}$ is *closed with respect to localization* i.e. $\mathcal{C} = \mathcal{C}_M$; (ii) $\mathcal{C}$ is *closed with respect to superposition with smooth functions* i.e. $\mathcal{C} =$ sc$\mathcal{C}$. In this case a pair $(M, \mathcal{C})$ is said to be a *(Sikorski's) differential space* (see [9]). Any element of $\mathcal{C}$ is called a *smooth function on M* (with respect to $\mathcal{C}$).

One can easily prove that the intersection of any family of differential structures defined on a set $M \neq \emptyset$ is a differential structure on M (see [4], [5], Proposition 2.1). Then the intersection $\mathcal{C}$ of all differential structures on M containing a given set $\mathcal{F}$ of real functions on M is a differential structure on M. It is the smallest differential structure on M containing $\mathcal{F}$. It can be proved that $\mathcal{C} = (sc\mathcal{F})_M$ (see [10]). This structure is called *the differential structure generated by $\mathcal{F}$* and is denoted by *gen*$(\mathcal{F})$. Functions of $\mathcal{F}$ are called *generators* of the differential structure $\mathcal{C}$. We have also $\tau_{(sc\mathcal{F})_M} = \tau_{sc\mathcal{F}} = \tau_{\mathcal{F}}$.

Let $(M, \mathcal{C})$ and $(N, \mathcal{D})$ be differential spaces. A map $F : M \to N$ is said to be *smooth* if for any $\beta \in \mathcal{D}$ the superposition $\beta \circ F \in \mathcal{C}$. We will denote the fact that F is smooth writing $F : (M, \mathcal{C}) \to (N, \mathcal{D})$. If $F : (M, \mathcal{C}) \to (N, \mathcal{D})$ is a bijection and $F^{-1} : (N, \mathcal{D}) \to (M, \mathcal{C})$ then F is called *a diffeomorphism.*

If A is a nonempty subset of M and $\mathcal{C}$ is a differential structure on M then $\mathcal{C}_A$ denotes the differential structure on A generated by the family of restrictions $\{\alpha_{|A} : \alpha \in \mathcal{C}\}$. The differential space $(A, \mathcal{C}_A)$ is called *a differential subspace* of $(M, \mathcal{C})$. One can easy prove that $F : (M, \mathcal{C}) \to (N, \mathcal{D})$ iff $F : (M, \mathcal{C}) \to (F(M), F(M)_{\mathcal{D}})$.

If the map $F : (M, \mathcal{C}) \to (F(M), F(M)_{\mathcal{D}})$ is a diffeomorphism then we say that $F : M \to N$ is *a diffeomorphism onto its range* (in $(N, \mathcal{D})$). In particular the natural embedding $A \ni m \mapsto i(m) := m \in M$ is a diffeomorphism of $(A, \mathcal{C}_A)$ onto its range in $(M, \mathcal{C})$.

If $\{(M_i, \mathcal{C}_i)\}_{i\in I}$ is an arbitrary family of differential spaces then we consider the Cartesian product $\prod\limits_{i\in I} M_i$ as a differential space with the differential structure $\hat{\bigotimes\limits_{i\in I}}\mathcal{C}_i$ generated by the family of functions $\mathcal{F} := \{\alpha_i \circ pr_i : i \in I, \alpha_i \in \mathcal{C}_i\}$, where $\prod\limits_{i\in I} M_i \ni (m_i) \mapsto pr_j((m_i)) =: m_j \in M_j$ for any $j \in I$. The topology $\tau_{\hat{\bigotimes}_{i\in I}\mathcal{C}_i}$ coincides with the standard product topology on $\prod\limits_{i\in I} M_i$. We will denote the differential structure $\hat{\bigotimes\limits_{i\in I}} C^\infty(\mathbb{R})$ on $\mathbb{R}^I$ by $C^\infty(\mathbb{R}^I)$. In the case when I is an n-element finite set, the differential structure $C^\infty(\mathbb{R}^I)$ coincides with the ordinary differential structure $C^\infty(\mathbb{R}^n)$ of all real-valued functions on $\mathbb{R}^n$ which posses partial derivatives of any order (see [9]). In any case a function $\alpha : \mathbb{R}^I \to \mathbb{R}$ is an element of $C^\infty(\mathbb{R}^I)$ iff for any $a = (a_i) \in \mathbb{R}^I$ there are $n \in \mathbb{N}$, elements $i_1, i_2, \ldots, i_n \in I$, a set U open in $\mathbb{R}^n$ and a function $\omega \in C^\infty(\mathbb{R}^n)$ such that $a \in U[i_1, i_2, \ldots, i_n] := \{(x_i) \in \mathbb{R}^I : (x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \in U\}$ and for any $x = (x_i) \in U[i_1, i_2, \ldots, i_n]$ we have

$$\alpha(x) = \omega(x_{i_1}, x_{i_2}, \ldots, x_{i_n}).$$

Let $\mathcal{F}$ be a family of generators of a differential structure $\mathcal{C}$ on a set M. *The generator embedding* of the differential space $(M, \mathcal{C})$ into the Cartesian space defined by $\mathcal{F}$ is a mapping $\phi_{\mathcal{F}} : (M, \mathcal{C}) \to (\mathbb{R}^{\mathcal{F}}, C^\infty(\mathbb{R}^{\mathcal{F}}))$ given by the formula

$$\phi_{\mathcal{F}}(m) = (\alpha(m))_{\alpha\in\mathcal{F}}, \qquad m \in M$$

(for example if $\mathcal{F} = \{\alpha_1, \alpha_2, \alpha_3\}$ then $\phi_{\mathcal{F}}(m) = (\alpha_1(m), \alpha_2(m), \alpha_3(m)) \in \mathbb{R}^3 \cong \mathbb{R}^{\mathcal{F}}$). If $\mathcal{F}$ separates points of M the generator embedding is a diffeomorphism onto its image. On that image we consider a differential structure of a subspace of $(\mathbb{R}^{\mathcal{F}}, C^\infty(\mathbb{R}^{\mathcal{F}}))$ (see [5], Proposition 2.3).

Let $\mathbb{K}$ be a field. A differential structure $\mathcal{C}_{\mathbb{K}}$ is called *a field differential structure* if the field operations are smooth with respect to $\mathcal{C}_{\mathbb{K}}$ and $\mathcal{C}_{\mathbb{K}}\hat{\otimes}\mathcal{C}_{\mathbb{K}}$. Then the pair $(\mathbb{K}, \mathcal{C}_{\mathbb{K}})$ is called *a differential field.* If V is a vector space over the field $\mathbb{K}$ then a differential structure $\mathcal{C}$ on V is said to be *a vector space differential structure* if the suitable vector space operations are smooth with respect to $\mathcal{C}$, $\mathcal{C}\hat{\otimes}\mathcal{C}$ and $\mathcal{C}_{\mathbb{K}}$, where $\mathcal{C}_{\mathbb{K}}$ is a field differential structure on $\mathbb{K}$. In this case the pair $(V, \mathcal{C})$ is called *a differential vector space.* If $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ we take $\mathcal{C}_{\mathbb{K}} = C^{\infty}(\mathbb{K})$ as a standard field differential structure. If $\mathcal{F}$ is a family of constant functions and (some) linear functionals defined on V then the differential structure $\mathcal{C}$ generated by $\mathcal{F}$ on V is a vector space differential structure (see [4], Proposition 2.3).

By *a tangent vector* to a differential space $(M, \mathcal{C})$ at a point $m \in M$ we call an $\mathbb{R}$-linear mapping $v : \mathcal{C} \to \mathbb{R}$ satisfying the Leibniz condition: $v(\alpha \cdot \beta) = \alpha(m)v(\beta) + \beta(m)v(\alpha)$ for any $\alpha, \beta \in \mathcal{C}$. We denote by T_mM the set of all vectors tangent to $(M, \mathcal{C})$ at the point $m \in M$ and call it *the tangent space to* $(M, \mathcal{C})$ *at the point* m. The union $TM := \bigcup\limits_{m\in M} T_mM$ is called *the tangent space to* $(M, \mathcal{C})$.

The set TM can be endowed with a differential structure in the following standard way. We define *the projection* $\pi : TM \to M$ such that for any $m \in M$ and any $v \in T_mM$

$$\pi(v) = m.$$

For any $\alpha \in \mathcal{C}$ we define *the differential* (or *the exterior derivative*) of α as a map $d\alpha : TM \to \mathbb{R}$ given by the following formula

$$d\alpha(v) := v(\alpha), \qquad v \in TM.$$

Then we define $\mathcal{T}C$ as the differential structure on TM generated by the family of functions $\mathcal{T}C_0 := \{\alpha \circ \pi : \alpha \in \mathcal{C}\} \cup \{d\alpha : \alpha \in \mathcal{C}\}$. From now on we will consider TM as a differential space with the differential structure $\mathcal{T}C$. We have $\pi : (TM, \mathcal{T}C) \to (M, \mathcal{C})$.

For any $m \in M$ we will denote by $d\alpha_m$ the restriction $d\alpha_{|T_mM}$. It is clear that $d\alpha_m$ is a linear functional on T_mM. Hence the pair $(T_mM, \mathcal{T}C_{T_mM})$ is a differential vector space. It is easy to show that T_mM is a Hausdorff space (with respect to the topology induced by $\mathcal{T}C_{T_mM}$ – see [4], Theorem 3.1).

Let $(M, \mathcal{C})$ and $(N, \mathcal{D})$ be differential spaces and let $F : (M, \mathcal{C}) \to (N, \mathcal{D})$. Then for any $v \in TM$ the linear functional $TF(v) : \mathcal{D} \to \mathbb{R}$ given by the formula

$$[TF(v)](\beta) := v(\beta \circ F), \qquad \beta \in \mathcal{D}, \tag{1}$$

is an element of $T_{F(\pi_M(v))}$, where $\pi_M : TM \to M$ is the natural projection (see [6], Proposition 2.4). The map $TF : TM \to TN$ given by the formula (1) is called *the map tangent to* F. It is well known that $TF : (TM, \mathcal{T}C) \to (TN, \mathcal{T}D)$ and $\pi_N \circ TF = F \circ \pi_M$, where $\pi_M : TM \to M$ and $\pi_N : TN \to N$ are natural projections (see [9] or [6], Proposition 2.5).

Let us consider the differential space $(\mathbb{R}^I, C^\infty(\mathbb{R}^I))$. The differential structure $C^\infty(\mathbb{R}^I)$ is generated by the family of projections $\mathcal{F} := \{pr_i\}_{i\in I}$, where

$$pr_j((x_i)) := x_j, \qquad (x_i) \in \mathbb{R}^I, \;\; j \in I.$$

For any $x = (x_i),\; v = (v_i) \in \mathbb{R}^I$ the functional $\vec{v} : C^\infty(\mathbb{R}^I) \to \mathbb{R}$ given by the formula

$$\vec{v}(\alpha) := \sum_{i\in I} v_i \frac{\partial \alpha}{\partial x_i}(x)$$

is well defined (in some neighbourhood of x the function α depends on finite number of variables x_i) and is a vector tangent to $\mathbb{R}^I$ at x. On the other hand, if $u \in T_x\mathbb{R}^I$ and for any $i \in I$ we denote $v_i := u(pr_i)$ then for any $\alpha \in C^\infty(\mathbb{R}^I)$ we have $\vec{v}(\alpha) = u(\alpha)$. Then we identify the set $T_x\mathbb{R}^I$ with $\{x\} \times \mathbb{R}^I$. Consequently we identify the set $T\mathbb{R}^I$ with $\mathbb{R}^I \times \mathbb{R}^I$. In this case the differential structure $\mathcal{T}C^\infty(\mathbb{R}^I)$ is generated by the family of functions $\mathcal{T}F := \{pr_i \circ \pi\}_{i\in I} \cup \{dpr_i\}_{i\in I}$, where

$$\pi(x, v) = x, \qquad (x, v) \in \mathbb{R}^I \times \mathbb{R}^I.$$

Hence for any $j \in I$

$$pr_j \circ \pi((x_i), (v_i)) = x_j \;\; \text{and} \;\; dpr_j((x_i), (v_i)) = v_j.$$

It means that $\mathcal{T}C^\infty(\mathbb{R}^I) = C^\infty(\mathbb{R}^I \times \mathbb{R}^I)$ and consequently for any $x \in \mathbb{R}^I$ the differential structure $\mathcal{T}C^\infty(\mathbb{R}^I)_{T_x\mathbb{R}^I}$ is generated by the family of projections $\{pr'_i : \{x\} \times \mathbb{R}^I \to \mathbb{R}\}_I$, where

$$pr'_j(x, (v_i)) = v_j.$$

Then we can identify $\mathcal{T}C^\infty(\mathbb{R}^I)_{T_x\mathbb{R}^I}$ with $C^\infty(\mathbb{R}^I)$.

Let $\phi_{\mathcal{F}} : (M, \mathcal{C}) \to (\mathbb{R}^{\mathcal{F}}, C^\infty(\mathbb{R}^{\mathcal{F}}))$ be the generator embedding of the differential Hausdorff space $(M, \mathcal{C})$ defined by some family of generators $\mathcal{F}$. Then we can identify differential spaces $(M, \mathcal{C})$ and $(\phi_{\mathcal{F}}(M), C^\infty(\mathbb{R}^{\mathcal{F}})_{\phi_{\mathcal{F}}(M)})$ ($\phi_{\mathcal{F}}$ is a diffeomorphism). We also identify tangent spaces T_mM and $T_{\phi_{\mathcal{F}}(m)}\phi_{\mathcal{F}}(M)$ using *the tangent map* $T\phi_{\mathcal{F}}$.

Theorem 2.1. *Let I be an arbitrary nonempty set and let X be a nonempty subset of the Cartesian space $\mathbb{R}^I$. Then for any $x = (x_i) \in X$ the space T_xX tangent to the differential space $(X, C^\infty(\mathbb{R}^I)_X)$ at the point x is a closed subspace of the space $T_x\mathbb{R}^I$ tangent to the differential space $(\mathbb{R}^I, C^\infty(\mathbb{R}^I))$ at x.*

For the proof see [4], Theorem 3.2. ∎

A map $X : M \to TM$ such that for any $m \in M$ the value $X(m) \in T_mM$ is called *a vector field* on M. A vector field X on M is *smooth* if $X : (M, \mathcal{C}) \to (TM, \mathcal{T}C)$.

3. Uniform structures and completions of a differential space defined by families of generators

For the general theory of uniform structures and completions see [7], Chapter 8 or [1]. It is also described in [8], [5] and [6]. Here we introduce notions and collect results which are necessary to develop our theory of differential completions of differential spaces. We start with the definition of the uniform structure given on a differential space by a family $\mathcal{F}$ of generators of its differential structure.

Let $\mathcal{F}$ be a family of real-valued functions on a set M and let $(M, \mathcal{C})$ be a differential space such that $\mathcal{C} = (\mathrm{sc}\mathcal{F})_M$ and $(M, \tau_{\mathcal{C}})$ is a Hausdorff space (the last is true iff the family $\mathcal{C}$ separates points in X iff the family $\mathcal{F}$ separates points in X). On the set M the family $\mathcal{F}$ defines the uniform structure $\mathcal{U}_{\mathcal{F}}$ such that the base $\mathcal{B}$ of $\mathcal{U}_{\mathcal{F}}$ is given as follows:

$$\mathcal{B} = \{V(f_1, \ldots, f_k, \varepsilon) \subset M \times M; k \in \mathbb{N}; f_1, \ldots, f_k \in \mathcal{F}, \varepsilon > 0\}, \tag{2}$$

where

$$V(f_1, \ldots, f_k, \varepsilon) = \{(x, y) \in M \times M : \forall 1 \leq i \leq k \quad |f_i(x) - f_i(y)| < \varepsilon\}$$

(see [5], Proposition 3.1).

Definition 3.1. The uniform structure $\mathcal{U}$ on a set M is said to be *a differential uniform structure* on the differential space $(M, \mathcal{C})$ if there exists a family $\mathcal{F}$ of generators of $\mathcal{C}$ such that $\mathcal{U} = \mathcal{U}_{\mathcal{F}}$, where $\mathcal{U}_{\mathcal{F}}$ is defined by the base (2). The uniform space $(M, \mathcal{U}_{\mathcal{F}})$ is said to be *the uniform space given by the family of generators* $\mathcal{F}$.

If we have two different families $\mathcal{F}_1$ and $\mathcal{F}_2$ of generators of a differential space $(M, \mathcal{C})$, then the uniform structures $\mathcal{U}_{\mathcal{F}_1}$ and $\mathcal{U}_{\mathcal{F}_1}$ can be different too.

Example 3.1. Let $M = \mathbb{R}$, $\mathcal{C} = C^\infty(\mathbb{R})$, $\mathcal{F}_1 = \{id_{\mathbb{R}}\}$ and $\mathcal{F}_2 = \{id_{\mathbb{R}}, f\}$, where

$$id_{\mathbb{R}}(x) = x, \quad and \quad f(x) = x^2, \qquad x \in \mathbb{R}.$$

Then does not exist $\varepsilon > 0$ such that $V(id_{\mathbb{R}}, \varepsilon) \subset V(f, 1)$. Hence $V(f, 1) \notin \mathcal{U}_{\mathcal{F}_1}$ and $\mathcal{U}_{\mathcal{F}_1} \neq \mathcal{U}_{\mathcal{F}_2}$.

If $\mathcal{F}$ is a family of generators of a differential structure $\mathcal{C}$ on a set M, then we define a uniform structure $\mathcal{U}_{\mathcal{T}\mathcal{F}}$ on the space TM tangent to the differential space $(M, \mathcal{C})$ using the family of real-valued functions

$$\mathcal{T}F = \{f \circ \pi : f \in \mathcal{F}\} \cup \{df : f \in \mathcal{F}\},$$

where $\pi : TM \to M$ is the natural projection and $df : TM \to \mathbb{R}$, $df(v) = v(f)$. As we know from the previous section, the family $\mathcal{T}\mathcal{F}$ generates the natural differential structure $\mathcal{TC}$ on TM. The base $\mathcal{D}$ of $U_{\mathcal{TF}}$ is given by:

$$\mathcal{D} = \{V(\pi \circ f_1, \ldots, \pi \circ f_k, df_{k+1}, \ldots, df_m, \varepsilon) \subset TM \times TM : k, m \in \mathbb{N}, \\ f_1, \ldots, f_m \in \mathcal{F}, \varepsilon > 0\}.$$

Let $(X,\mathcal{U})$, and $(Y,\mathcal{V})$ be uniform spaces.

Definition 3.2. A mapping $f : X \to Y$ is said to be *uniform* with respect to uniform structures $\mathcal{U}$ and $\mathcal{V}$ if

$$\forall V \in \mathcal{V}\ \exists U \in \mathcal{U}\ \forall x, x' \in X\ [(x, x') \in U \Rightarrow (f(x), f(x')) \in V].$$

In other words, for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $U \subset (f \times f)^{-1}(V)$. We denote it by

$$f : (X,\mathcal{U}) \to (Y,\mathcal{V}).$$

It is clear that if $\mathcal{F}$ is a family of generators of the differential structure $\mathcal{C}$ on the set M then any element of $\mathcal{F}$ is a uniform mapping with respect to the uniform structure $\mathcal{U}_{\mathcal{F}}$ on M and the standard uniform structure on $\mathbb{R}$.

It is easy to prove that: (i) any uniform mapping $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is continuous with respect to topologies $\tau_{\mathcal{U}}$ and $\tau_{\mathcal{V}}$; (ii) a superposition of uniform mappings is a uniform mapping; (iii) $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ iff for any bases $\mathcal{B}$ and $\mathcal{D}$ of $\mathcal{U}$ and $\mathcal{V}$ respectively and for each $V \in \mathcal{D}$ there exists $U \in \mathcal{B}$ such that $U \subset (f \times f)^{-1}(V)$ (see [7]).

A mapping f, which is uniform with respect to uniform structures $\mathcal{U}$ and $\mathcal{V}$, could not be uniform with respect to another uniform structures $\overline{\mathcal{U}}$ and $\overline{\mathcal{V}}$ defined on X and Y respectively, even if the topologies $\tau_{\mathcal{U}}, \tau_{\overline{\mathcal{U}}}, \tau_{\mathcal{V}}$ and $\tau_{\overline{\mathcal{V}}}$ fulfil equalities: $\tau_{\mathcal{U}} = \tau_{\overline{\mathcal{U}}}$ and $\tau_{\mathcal{V}} = \tau_{\overline{\mathcal{V}}}$ (see [6], Example 3.2).

A bijective mapping $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is *a uniform homeomorphism* if f^{-1} is a uniform mapping. Then we say that $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are *uniformly homeomorphic*. By (i) it is obvious that if $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a uniform homeomorphism then f is a homeomorphism of the topological spaces $(X, \tau_{\mathcal{U}})$ and $(Y, \tau_{\mathcal{V}})$.

Let $(X,\mathcal{U})$ be a uniform space and let $A \subset X$. Then the family $\mathcal{U}_A := \{(A \times A) \cap U : U \in \mathcal{U}\}$ is a uniform structure on A. The uniform space $(A,\mathcal{U}_A)$ is called *the uniform subspace* of the uniform space $(X,\mathcal{U})$. Note that if $\mathcal{F}$ is a family of generators of a differential structure $\mathcal{C}$ on a set M, $A \subset M$ and $\mathcal{F}_{|A} = \{f_{|A} : f \in \mathcal{F}\}$, then the uniform space $(A,\mathcal{U}_{\mathcal{F}_{|A}})$ is a uniform subspace of the uniform space $(M,\mathcal{U}_{\mathcal{F}})$.

Let $(X, \mathcal{D})$ be a uniform space and $V \in \mathcal{D}$. A set $U \subset X$ is said to be *small of rank* V if $\exists x \in U\ \forall y \in U\ [(x, y) \in V]$ (see [5], Definition 2.2). If we define *the ball* $K(x, V)$ as a set:

$$K(x, V) = \{y \in X : (x, y) \in V\},$$

then a set $U \subset X$ is small of rank V iff $\exists x \in U\ [U \in K(x, V)]$.

If $F \subset X$ and $V \in \mathcal{D}$ we define *the V-neighbourhood* of F as a set

$$K(F,V) := \bigcup_{x \in F} K(x,V) = \{y \in X : \exists x \in F\ [(x,y) \in V]\}.$$

Now we recall basic notions and facts concerning the theory of filters, Cauchy filters and completions of uniform spaces.

A nonempty family $\mathcal{F}$ of subsets of a set X is said to be *a filter on* X if: (F1) $(F \in \mathcal{F}\ \wedge\ F \subset U \subset X)\ \Rightarrow\ (U \in \mathcal{F})$; (F2) $(F_1, F_2 \in \mathcal{F}) \Rightarrow (F_1 \cap F_2 \in \mathcal{F})$; (F3) $\emptyset \notin \mathcal{F}$. A filtering base on X is a nonempty family $\mathcal{B}$ of subsets of X such that: (FB1) $\forall A_1, A_2 \in \mathcal{B}\ \exists A_3 \in \mathcal{B}\ [A_3 \subset A_1 \cap A_2]$; (FB2) $\emptyset \notin \mathcal{B}$. If $\mathcal{B}$ is a filtering base on X then

$$\mathcal{F} = \{F \subset X : \exists A \in \mathcal{B}\ [A \subset F]\}$$

is a filter on X. It is called *the filter defined by* $\mathcal{B}$ and $\mathcal{B}$ is called *the base of* $\mathcal{F}$.

It is easy to show that if $\{\mathcal{F}_i\}_{i \in I}$ is the family of filters on the set X then the intersection $\bigcap_{i \in I} \mathcal{F}_i$ is a filter on X (see [6], Proposition 3.1).

Let X be a topological space. We say that a filter $\mathcal{F}$ on X *is convergent to* $x \in X$ ($\mathcal{F} \to x$) if for any neighbourhood U of x there exists $F \in \mathcal{F}$ such that $F \subset U$ (i.e. $U \in \mathcal{F}$). If for any $i \in I$ the filter $\mathcal{F}_i \to x$ then $\bigcap_{i \in I} \mathcal{F}_i \to x$ (see [6], Proposition 3.2).

Definition 3.3. Let $(X,\mathcal{U})$ be a uniform space. A filter $\mathcal{F}$ on X is *a Cauchy filter* if

$$\forall V \in \mathcal{U}\ \ \exists F \in \mathcal{F}\ \ [F \times F \subset V].$$

We say that two Cauchy filters $\mathcal{F}_1$ and $\mathcal{F}_2$ *are in the relation* R if

$$\forall V \in \mathcal{U}\ \ \exists F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\ \ [F_1 \times F_2 \subset V].$$

Proposition 3.1. *Two filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on the uniform space $(X,\mathcal{U})$ are in the relation R iff $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_1 \cap \mathcal{F}_2$ are Cauchy filters on X.*

Proof. ($\Rightarrow$) Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ to be in the relation R and fix $V \in \mathcal{U}$. Let $W \in \mathcal{U}$ be such that $4W \subset V$. There exist $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ such that $F_1 \times F_2 \subset W$. Then $F_2 \subset K(F_1, W)$ which implies that $K(F_1,W) \in \mathcal{F}_2$. Since $F_1 \subset K(F_1,W)$ we have $K(F_1,W) \in \mathcal{F}_1$. Hence $K(F_1,W) \in \mathcal{F}_1 \cap \mathcal{F}_2$. On the other hand, for any $y_1, y_2 \in K(F_1,W)$ there are $x_1, x_2 \in F_1$ such that $(y_1,x_1), (y_2,x_2) \in W$. For an arbitrarily chosen $z \in F_2$ we have $(z,x_1), (z,x_2) \in W$. Hence $(y_1,y_2) \in 4W \subset V$. It means that $K(F_1,W) \times K(F_1,W) \subset V$. Since $K(F_1,W)$ is an element of $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_1 \cap \mathcal{F}_2$, all these filters are Cauchy filters.

($\Leftarrow$) Suppose $\mathcal{F}_1 \cap \mathcal{F}_2$ to be Cauchy filter on X. Fix $V \in \mathcal{U}$, choose $F \in \mathcal{F}_1 \cap \mathcal{F}_2$ such that $F \times F \subset V$ and put $F_1 := F_2 := F$. Then $F_1 \in \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$ and $F_1 \times F_2 \subset V$. Hence $\mathcal{F}_1 R \mathcal{F}_2$. ∎

Proposition 3.2. *The relation R described in Definition 3.8 is an equivalence relation on the set $CF(X)$ of all Cauchy filters on the uniform space $(X,\mathcal{U})$.*

Proof. It is obvious that for any Cauchy filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on X we have $\mathcal{F}_1 R \mathcal{F}_1$, and if $\mathcal{F}_1 R \mathcal{F}_2$ then $\mathcal{F}_2 R \mathcal{F}_1$. Suppose now $\mathcal{F}_3$ to be such a Cauchy filter on X that $\mathcal{F}_1 R \mathcal{F}_2$ and $\mathcal{F}_2 R \mathcal{F}_3$. Fix $V \in \mathcal{U}$ and choose $W \in \mathcal{U}$ such that $2W \subset V$. There exist $F_1 \in \mathcal{F}_1$, $F_2', F_2'' \in \mathcal{F}_2$ and $F_3 \in \mathcal{F}_3$ such that $F_1 \times F_2' \subset W$ and $F_2'' \times F_3 \subset W$. Let $F_2 := F_2' \cap F_2''$. Then $F_2 \in \mathcal{F}_2$, $F_1 \times F_2 \subset W$ and $F_2 \times F_3 \subset W$. Since $2W \subset V$ we obtain $F_1 \times F_3 \subset V$. Hence $\mathcal{F}_1 R \mathcal{F}_3$. ∎

For any Cauchy filter $\mathcal{F}$ on X we denote by $[\mathcal{F}]$ the equivalence class of $\mathcal{F}$ with respect to the equivalence relation R given in Definition 3.8.

Proposition 3.3. *If $\{\mathcal{F}_i\}_{i\in I}$ is a family of Cauchy filters on an uniform space X contained in an equivalence class $[\mathcal{F}]$ then $\bigcap\limits_{i\in I} \mathcal{F}_i \in [\mathcal{F}]$ i.e. $\bigcap\limits_{i\in I} \mathcal{F}_i$ is a Cauchy filter and it is equivalent to $\mathcal{F}$.*

Proof. Let V be an arbitrary element of the uniform structure $\mathcal{U}$ on X. Let $W \in \mathcal{U}$ be such that $4W \subset V$. Choose $F \in \mathcal{F}$ such that $F \times F \subset W$. Similarly as in the proof of Proposition 3.3 we obtain that $K(F,W) \times K(F,W) \subset 3W \subset V$. For any $i \in I$ there are $F_i \in \mathcal{F}_i$ and $G_i \in \mathcal{F}$ such that $F_i \times G_i \in W$. Hence $F_i \times (G_i \cap F) \in W$ and therefore $F_i \subset K(F,W)$. Consequently $K(F,W) \in \mathcal{F}_i$ for any $i \in I$. Then $K(F,W) \in \bigcap\limits_{i\in I} \mathcal{F}_i$ and moreover $K(F,W) \times K(F,W) \subset V$. It means that $\bigcap\limits_{i\in I} \mathcal{F}_i$ is a Cauchy filter on X. Since $K(F,W) \in \mathcal{F}$ $(F \subset K(F,W))$ we obtain that $\bigcap\limits_{i\in I} \mathcal{F}_i$ is equivalent to $\mathcal{F}$. ∎

Corollary 3.1. *If $\mathcal{F}$ is a Cauchy filter on X then $\bigcap\limits_{\mathcal{G}\in[\mathcal{F}]} \mathcal{G}$ is a Cauchy filter on X equivalent to $\mathcal{F}$. Since for any $\mathcal{F}_1 \in [\mathcal{F}]$ we have $\bigcap\limits_{\mathcal{G}\in[\mathcal{F}]} \mathcal{G} \subset \mathcal{F}_1$, we obtain $\bigcap\limits_{\mathcal{G}\in[\mathcal{F}]} \mathcal{G}$ is the minimal element of $[\mathcal{F}]$ with respect to the ordering relation $\subset$ on the family of all filters on a set X.* ∎

Definition 3.4. For any Cauchy filter $\mathcal{F}$ on X, the Cauchy filter $\bigcap\limits_{\mathcal{G}\in[\mathcal{F}]} \mathcal{G}$ is called *the minimal Cauchy filter on X defined by (smaller then) $\mathcal{F}$.*

Definition 3.5. A uniform space $(X,\mathcal{U})$ is said to be *complete* if each Cauchy filter on X is convergent in $\tau_{\mathcal{U}}$.

Theorem 3.1. *If $(X,\mathcal{U})$ is a complete uniform space and M is a closed subset of the topological space $(X,\tau_{\mathcal{U}})$ then a uniform space $(M,\mathcal{U}_M)$ is complete. Conversely, if $(M,\mathcal{U}_M)$ is a complete uniform subspace of some (not necessarily complete) uniform space $(X,\mathcal{U})$, then M is closed in X with respect to $\tau_{\mathcal{U}}$.*

For the proof see [1], [7] or [8].

It is well known that the uniform space of reals $(\mathbb{R},\mathcal{U})$ with the standard uniform structure $\mathcal{U} = \mathcal{U}_{\{id_{\mathbb{R}}\}}$ defined by the one element family of functions $\{id_{\mathbb{R}}\}$ (or by the standard metric) is complete. We have also more general

Proposition 3.4. *For any set I the uniform space $(\mathbb{R}^I,\mathcal{U}_{\mathcal{G}})$, where $\mathcal{G} = \{pr_i\}_{i\in I}$ is the set of all natural projections $pr_i : \mathbb{R}^I \to \mathbb{R}$,*

$$pr_i(x) = pr_i((x_j)_{j\in I}) = x_i, \qquad f \in \mathbb{R}^I,$$

for any $i \in I$, is complete.

Proof. Let $\mathcal{F}$ be a Cauchy filter on $\mathbb{R}^I$. Since for any $i \in I$ the map pr_i is uniform we obtain that the family $pr_i(\mathcal{F}) = \{pr_i(F) : F \in \mathcal{F}\}$ is a filtering base of some Cauchy filter on $\mathbb{R}$. Then the Cauchy filter corresponding to $pr_i(\mathcal{F})$ converges to some $y_i \in \mathbb{R}$. Putting $y(i) := y_i,\quad i \in I$ we obtain function $y \in \mathbb{R}^I$ such that $\mathcal{F} \to f$. ∎

Any uniform space can be treated as a uniform subspace of some complete uniform space. We have the following

Theorem 3.2. *For each uniform space $(X,\mathcal{U})$:*

(i) *there exists a complete uniform space $(\widetilde{X},\widetilde{\mathcal{U}})$ and a set $A \subset \widetilde{X}$ dense in $\widetilde{X}$ (with respect to the topology $\tau_{\widetilde{\mathcal{U}}}$) such that $(X,\mathcal{U})$ is uniformly homeomorphic to $(A,\widetilde{\mathcal{U}}_A)$;*

(ii) *if the complete uniform spaces $(\widetilde{X}_1,\widetilde{\mathcal{U}}_1)$ and $(\widetilde{X}_2,\widetilde{\mathcal{U}}_2)$ satisfies condition of the point* (i) *then they are uniformly homeomorphic.*

For the details of the proof see [1] or [8]. Here we only want to describe the construction of $(\widetilde{X},\widetilde{\mathcal{U}})$.

Let $\widetilde{X}$ be the set of all minimal Cauchy filters in X. For every $V \in \mathcal{U}$ we denote by $\widetilde{V}$ the set of all pairs $(\mathcal{F}_1,\mathcal{F}_2)$ of minimal Cauchy's filters, which have a common element being a small set of rank V. We define a family $\widetilde{\mathcal{U}}$ of subsets of set $\widetilde{X} \times \widetilde{X}$ as the smallest uniform structure on X containing all sets from the family $\{\widetilde{V} : V \in \mathcal{U}\}$.

Example 3.2. Let us consider two uniform structures $\mathcal{U}_{\{f\}}$ and $\mathcal{U}_{\{g\}}$ on the differential space $(\mathbb{R}, \mathcal{C}^\infty)$, where

$$f(x) = x, \quad g(x) = arctgx, \qquad x \in \mathbb{R}.$$

Then $(\mathbb{R}, \mathcal{U}_{\{f\}})$ is the complete space i.e. $\widetilde{\mathcal{U}_{\{f\}}} = \mathcal{U}_{\{f\}}$ while $(\mathbb{R}, \mathcal{U}_{\{g\}})$ is not complete and we have $\widetilde{\mathbb{R}} \simeq [-\frac{\pi}{2}; \frac{\pi}{2}]$. Consequently $\mathcal{U}_{\{f\}} \neq \mathcal{U}_{\{g\}}$.

Let N be a set, $M \subseteq N$, $M \neq \emptyset$, $\mathcal{C}$ be a differential structure on M.

Definition 3.6. The differential structure $\mathcal{D}$ on N is an *extension* of the differential structure $\mathcal{C}$ from the set M to the set N if $\mathcal{C} = \mathcal{D}_M$ (if we get the structure $\mathcal{C}$ by localization of the structure $\mathcal{D}$ to M).

For the sets N, M and the differential structure $\mathcal{C}$ on M we can construct many different extensions of the structure M to N.

Example 3.3. If for each function $f \in \mathcal{C}$ we assign $f_0 \in \mathbb{R}^N$ such that $f_{0|M} = f$ and $f_{0|N \setminus M} \equiv 0$, then the differential structure generated on N by the family of functions $\{f_0\}_{f \in \mathcal{C}}$ is an extension of $\mathcal{C}$ from M to N. Similarly, if for each function $f \in \mathcal{C}$ we assign the family $\mathcal{F}_f := \{g \in \mathbb{R}^N : g_{|M} = f\}$, then the differential structure on N generated the family of functions $\mathcal{F} := \bigcup_{f \in \mathcal{C}} \mathcal{F}_f$ is an extension of $\mathcal{C}$ from M to N. If the set $N \setminus M$ contain at least two elements, then the differential structures generated by the families $\{f_0\}_{f \in \mathcal{C}}$ and $\mathcal{F}$ are different.

Definition 3.7. If τ is a topology on the set N, then the extension $\mathcal{D}$ of the differential structure $\mathcal{C}$ from M to N is *continuous with respect to* τ if each function $f \in \mathcal{D}$ is continuous with respect to τ ($\tau_{\mathcal{D}} \subset \tau$).

If on the set N there exists a continuous with respect to some topology τ extension of the differential structure $\mathcal{C}$ from the set $M \subset N$, then the structure $\mathcal{C}$ is said to be *extendable from the set M to the topological space (N, τ)*.

Example 3.4. The differential structure $C^\infty(\mathbb{R})_{\mathbb{Q}}$ is extendable from the set of rationales to the set of reals. The continuous extensions are e.g. $C^\infty(\mathbb{R})$ and the structure $\mathcal{D}$ generated on $\mathbb{R}$ by the family of the functions $C^\infty(\mathbb{R}) \cup \{f\}$, where $f : \mathbb{R} \to \mathbb{R}$, $f(x) := |x - \sqrt{2}|$, $x \in \mathbb{R}$.

Proposition 3.5. *Let $M \neq \emptyset$, $(M, \mathcal{C})$ be a differential space and $\mathcal{G}$ be a family of generators of $\mathcal{C}$, i.e. $\mathcal{C} = gen(\mathcal{G})$. Let $(\widetilde{M}, \widetilde{\mathcal{U}_\mathcal{G}})$ be the completion of the uniform space $(M, \mathcal{U}_\mathcal{G})$. Then any function $g \in \mathcal{G}$ poses the continuous extension $\tilde{g} : \widetilde{M} \to \mathbb{R}$. If $\widetilde{\mathcal{G}}$ is the family of all continuous extensions of elements of $\mathcal{G}$ to $\widetilde{M}$ then the differential structure $\mathcal{D} = gen(\widetilde{\mathcal{G}})$ is an continuous extension of the differential structure $\mathcal{C}$ from the set M to the set $\widetilde{M}$. Moreover $\tau_{\mathcal{D}} = \tau_{\tilde{\mathcal{U}}_\mathcal{G}}$.*

Proof. Let $\phi_{\mathcal{G}}$ be the generator embedding of the differential space $(M, \mathcal{C})$ into the Cartesian space $(\mathbb{R}^{\mathcal{G}}, C^\infty(\mathbb{R}^{\mathcal{G}}))$ defined by $\mathcal{G}$. Then the closure $\overline{\phi_{\mathcal{G}}(M)}$ is a complete subspace of the complete uniform space $\mathbb{R}^{\mathcal{G}}$ (see Theorem 3.1 and Proposition 3.4). We know that $\phi_{\mathcal{G}} : (M, \mathcal{C}) \to (\phi_{\mathcal{G}}(M), C^\infty(\mathbb{R}^{\mathcal{G}})_{\phi_{\mathcal{G}}(M)})$ is a diffeomorphism and $pr_g \circ \phi_{\mathcal{G}} = g$ for any $g \in \mathcal{G}$. Moreover $\phi_{\mathcal{G}}(M)$ is dense in $\overline{\phi_{\mathcal{G}}(M)}$. Then identifying any $g \in \mathcal{G}$ with $pr_{g|\phi_{\mathcal{G}}(M)}$, $\mathcal{C}$ with $C^\infty(\mathbb{R}^{\mathcal{G}})_{\phi_{\mathcal{G}}(M)}$ and putting $\widetilde{M} := \overline{\phi_{\mathcal{G}}(M)}$ we obtain that $\tilde{g}$ should be identify with $pr_{g|\overline{\phi_{\mathcal{G}}(M)}}$ and $\mathcal{D} = C^\infty(\mathbb{R}^{\mathcal{G}})_{\overline{\phi_{\mathcal{G}}(M)}}$. We have also $\tau_{\mathcal{D}} = \tau_{\widetilde{M}} = \tau_{\widetilde{\mathcal{U}}_{\mathcal{G}}}$, where $\tau_{\widetilde{M}}$ is the topology of $\widetilde{M}$ as a topological subspace of $\mathbb{R}^{\mathcal{G}}$. ∎

Definition 3.8. The differential space $(\widetilde{M}, \mathcal{D})$ constructed in Proposition 3.7 will be called *the differential completion* of the differential space $(M, \mathcal{C})$ defined by the family of generators $\mathcal{G}$. The set $\widetilde{M}$ will be denoted by $compl_{\mathcal{G}}M$ and the differential structure $\mathcal{D}$ will be denoted by $compl_{\mathcal{G}}\mathcal{C}$.

4. The maximal differential completion

Let us consider two families $\mathcal{G}$ and $\mathcal{H}$ of generators of a differential structure $\mathcal{C}$ on a set $M \neq \emptyset$. If $\mathcal{G} \subset \mathcal{H}$ then for uniform structures $\mathcal{U}_{\mathcal{G}}$ and $\mathcal{U}_{\mathcal{H}}$ we have: $\mathcal{U}_{\mathcal{G}} \subset \mathcal{U}_{\mathcal{H}}$. Consequently, any Cauchy filter with respect to $\mathcal{U}_{\mathcal{H}}$ is a Cauchy filter with respect to $\mathcal{U}_{\mathcal{G}}$. In particular, any minimal Cauchy filter with respect to $\mathcal{U}_{\mathcal{H}}$ is a Cauchy (but not necessarily minimal Cauchy) filter with respect to $\mathcal{U}_{\mathcal{G}}$. This defines the natural map $\iota_{\mathcal{G}\mathcal{H}} : compl_{\mathcal{H}}M \to compl_{\mathcal{G}}M$ as follows: for any $\mathcal{F} \in compl_{\mathcal{H}}M$ the value $\iota_{\mathcal{G}\mathcal{H}}(\mathcal{F}) \in compl_{\mathcal{G}}M$ is the minimal Cauchy filter equivalent to $\mathcal{F}$ with respect to the uniform structure $\mathcal{U}_{\mathcal{G}}$.

Proposition 4.1. *For any two families $\mathcal{G}$ and $\mathcal{H}$ of generators of a differential structure $\mathcal{C}$ on a set $M \neq \emptyset$ such that $\mathcal{G} \subset \mathcal{H}$, the map $\iota_{\mathcal{G}\mathcal{H}} : compl_{\mathcal{H}}M \to compl_{\mathcal{G}}M$ defined above is smooth with respect to differential structures $compl_{\mathcal{H}}\mathcal{C}$ and $compl_{\mathcal{G}}\mathcal{C}$.*

Proof. For the smoothness of $\iota_{\mathcal{G}\mathcal{H}}$ it is enough to prove that for any $g \in \mathcal{G}$ the function $\tilde{g}_{\mathcal{G}} \circ \iota_{\mathcal{G}\mathcal{H}} \in compl_{\mathcal{H}}\mathcal{C}$, where $\tilde{g}_{\mathcal{G}}$ denotes the continuous extension of g onto $compl_{\mathcal{G}}M$. Since any $g \in \mathcal{G}$ is an element of $\mathcal{H}$ we have for each Cauchy filter $\mathcal{F} \in compl_{\mathcal{H}}M$

$$\tilde{g}_{\mathcal{G}} \circ \iota_{\mathcal{G}\mathcal{H}}(\mathcal{F}) = \lim g(\iota_{\mathcal{G}\mathcal{H}}(\mathcal{F})) = \lim g(\mathcal{F}) = \tilde{g}_{\mathcal{H}}(\mathcal{F}),$$

where $\tilde{g}_{\mathcal{H}}$ denotes the continuous extension of g onto $compl_{\mathcal{H}}M$. Hence $\tilde{g}_{\mathcal{G}} \circ \iota_{\mathcal{G}\mathcal{H}} = \tilde{g}_{\mathcal{H}} \in compl_{\mathcal{H}}\mathcal{C}$. ∎

In general the image $\iota_{\mathcal{GH}}(compl_{\mathcal{H}}M) \neq compl_{\mathcal{G}}M$ and it is not complete in $compl_{\mathcal{G}}M$.

Example 4.1. Let $M = \mathbb{Q}$, $\mathcal{C} = C^\infty(\mathbb{R})_{\mathbb{Q}}$, $\mathcal{G} = \{id\}$ and $\mathcal{H} = \{id, h\}$, where

$$h(x) := \begin{cases} x & for \;\; x < \sqrt{2}, \\ -x & for \;\; x > \sqrt{2}. \end{cases}$$

Then $\{\mathbb{Q}\cap(\sqrt{2}-\frac{1}{n}; \sqrt{2})\}_{n\in\mathbb{N}}$ and $\{\mathbb{Q}\cap(\sqrt{2}; \sqrt{2}+\frac{1}{n})\}_{n\in\mathbb{N}}$ are filtering bases of different Cauchy filters $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively in the uniform space $(\mathbb{Q}, \mathcal{U}_{\mathcal{H}})$. Since $\iota_{\mathcal{GH}}(\mathcal{F}_1) = \iota_{\mathcal{GH}}(\mathcal{F}_2)$ we obtain that $\iota_{\mathcal{GH}}$ is not an injective map.

Example 4.2. Let $M = (0; \frac{\pi}{2})$, $\mathcal{C} = C^\infty(M)$ and $\mathcal{G} = \{\alpha_1, \alpha_2\}$, $\mathcal{H} = \{\alpha_1, \alpha_2, \beta\}$, where

$$\alpha_1(x) = x\cos(\tan x), \quad \alpha_2(x) = x\sin(\tan x), \quad \beta(x) = \tan x, \qquad x \in M.$$

Since $id_M = \sqrt{\alpha_1^2 + \alpha_2^2}$ we have $C^\infty(M) = gen(\mathcal{G}) = gen(\mathcal{H})$. Moreover, $\mathcal{G} \subset \mathcal{H}$. Let $K((a,b), r)$ denotes the disc in $\mathbb{R}^2$ with the center at (a,b) and the radius $r > 0$. If $a^2 + b^2 = \frac{\pi^2}{4}$ then the family of sets $\{(\alpha_1, \alpha_2)^{-1}(K((a,b), \frac{1}{n}))\}_{n\in\mathbb{N}}$ is the filtering basis of the minimal Cauchy filter $\mathcal{F}$ in M with respect to the uniform structure $\mathcal{U}_{\mathcal{G}}$. But in the uniform space $(M, \mathcal{U}_{\mathcal{H}})$ there is no Cauchy filter $\mathcal{F}_0$ such that $\mathcal{F} = \iota_{\mathcal{GH}}(\mathcal{F}_0)$. Hence the map $\iota_{\mathcal{GH}}$ is not onto $compl_{\mathcal{G}}M$.

From the above consideration we obtain the following theorem.

Theorem 4.1. *For any differential space $(M, \mathcal{C})$ the differential completion $(compl_{\mathcal{C}}M$, $compl_{\mathcal{C}}\mathcal{C})$ has the following properties:*

(i) *for any differential completion $(N, \mathcal{D})$ of $(M, \mathcal{C})$ (where $M \subset N$) there exists a map*

$$\iota_{\mathcal{D}} : (compl_{\mathcal{C}}M, compl_{\mathcal{C}}\mathcal{C}) \to (N, \mathcal{D})$$

such that $\iota_{\mathcal{D}|M} = id_M$;

(ii) *for any function $g \in \mathcal{C}$ there exists uniquely defined extension $\tilde{g} \in compl_{\mathcal{C}}\mathcal{C}$.*

In the set of all differential completions of the space $(M, \mathcal{C})$ we can define an ordering relation $\preceq$ such that:

$$compl_{\mathcal{G}}M \preceq compl_{\mathcal{H}}M \Leftrightarrow \mathcal{G} \subset \mathcal{H},$$

where $\mathcal{G}$ and $\mathcal{H}$ are families of generators of the structure $\mathcal{C}$.

The above theorem says that $compl_{\mathcal{C}}M$ is the maximal with the respect to the order $\preceq$ completion of M which can be constructed using a set of generators of the differential structure $\mathcal{C}$ while $compl_{\mathcal{C}}\mathcal{C}$ is the maximal continuous extension of $\mathcal{C}$ from M to $compl_{\mathcal{C}}M$.

Definition 4.1. We will call the differential space $(compl_{\mathcal{C}}M, compl_{\mathcal{C}}\mathcal{C})$ *the maximal differential completion* of the differential space $(M, \mathcal{C})$.

Let us consider the situation when for some family of generators $\mathcal{G}$, the uniform space $(M, \mathcal{U}_{\mathcal{G}})$ is complete.

Theorem 4.2. *Let $(M, \mathcal{C})$ be a differential space and $\mathcal{G}$ be a family of generators of $\mathcal{C}$. If the uniform space $(M, \mathcal{U}_{\mathcal{G}})$ is complete then for any family $\mathcal{H}$ of generators of $\mathcal{C}$ such that $\mathcal{G} \subset \mathcal{H}$ we have*

$$compl_{\mathcal{H}}M = M \tag{3}$$

and

$$compl_{\mathcal{H}}\mathcal{C} = \mathcal{C}. \tag{4}$$

In particular $compl_{\mathcal{C}}M = M$ and $compl_{\mathcal{C}}\mathcal{C} = \mathcal{C}$.

Proof. Any element of $compl_{\mathcal{H}}M$ is represented by some filter $\mathcal{F}$ in M which is a Cauchy filter with respect to $\mathcal{U}_{\mathcal{H}}$. Then $\mathcal{F}$ is a Cauchy filter with respect to $\mathcal{U}_{\mathcal{G}}$ and therefore $\mathcal{F}$ can be identify with its limit in M. Then $compl_{\mathcal{H}}M \subset M$. On the other hand for any element $p \in M$ the filter $\mathcal{F}_p$ of all neighbourhoods of p is a Cauchy filter with respect to $\mathcal{U}_{\mathcal{H}}$. Hence we can write $M \subset compl_{\mathcal{H}}M$.

The equality (4) is an immediate consequence of the definition of $compl_{\mathcal{H}}\mathcal{C}$ and the equality (3). ∎

Let us consider the case when $\mathcal{G} = \mathcal{C}$.

Theorem 4.3. *Let $(M, \mathcal{C})$ be a differential space. If there exists a finite or countable family of generators of $\mathcal{C}$ then the uniform space $(M, \mathcal{U}_{\mathcal{C}})$ is complete.*

Proof. Suppose that $(M, \mathcal{U}_{\mathcal{C}})$ is not complete. Then there exists $x \in compl_{\mathcal{C}}M \setminus M$. Let $\chi : \mathcal{C} \to \mathbb{R}$ be a functional given by the formula

$$\chi(g) := \tilde{g}(x), \qquad g \in \mathcal{C}, \tag{5}$$

where $\tilde{g}$ is the continuous extension of g from M to $compl_{\mathcal{C}}M$. This functional is an element of the spectrum of the algebra $\mathcal{C}$, but it is not an evaluation functional on M (the algebra $compl_{\mathcal{C}}\mathcal{C}$ separates points of the space $compl_{\mathcal{C}}M$). Then $\mathcal{C}$ does not posses the spectral property. It is contradictory with Theorem 1 and Corollary 6 from the work [3] (see also Theorem 2.3 (Twierdzenie 2.3) and Corollary 2.6 (Wniosek 2.6) from [2]). ∎

Corollary 4.1. *Let X be a topological Hausdorff space. If the topology of X is given by a countable family $\mathcal{G}$ of real-valued functions as the weakest* *topology on X with respect to which all elements of $\mathcal{G}$ are continuous, then there exists a uniform structure $\mathcal{U}$ on X such that the uniform space $(X,\mathcal{U})$ is complete and the topology $\tau_{\mathcal{U}}$ coincides with the initial topology on X.* ∎

Example 4.3. Let $M = \mathbb{Q}$ be the set of rationales. Since the differential structure $C^\infty(\mathbb{R}_\mathbb{Q})$ is generated by the one element set $\{id_\mathbb{Q}\}$, we obtain that the uniform space $(\mathbb{Q}, \mathcal{U}_{C^\infty(\mathbb{R}_\mathbb{Q})})$ is complete. We will define the family $\mathcal{G} = \{f_r\}_{r\in I_0}$ of generators of $C^\infty(\mathbb{R}_\mathbb{Q})$ such that $I_0 \subset \mathbb{R}$ and the uniform structure $\mathcal{U}_\mathcal{G}$ is complete.

Let $I := \mathbb{R} \setminus \mathbb{Q}$, $I_0 := \{0\} \cup I$, $f_0 := id_\mathbb{Q}$ and for any $r \in I$

$$f_r(x) := \frac{1}{x-r}, \qquad x \in \mathbb{Q}.$$

For any $s \in I_0$ we have $f_s \in C^\infty(\mathbb{R}_\mathbb{Q})$ and $C^\infty(\mathbb{R}_\mathbb{Q}) = gen(\mathcal{G})$. Let $\phi_\mathcal{G} : \mathbb{Q} \to \mathbb{R}^{I_0}$ be the generator embedding. We will see that $M = \phi_\mathcal{G}$ ($\mathbb{Q}$ is a closed subset of $\mathbb{R}^{I_0}$). Let $p = (p_s)_{s\in I_0} \in \overline{M}$ ($\overline{M}$ is the closure of M in $\mathbb{R}^{I_0}$). Let $\mathcal{F}_p$ be a filter of all (not necessarily open) neighbourhoods of p in $\mathbb{R}^{I_0}$. Hence $\mathcal{F}_p$ is a Cauchy filter with respect to the uniform structure $\mathcal{U}_{\{pr_s\}_{s\in I_0}}$, where $\{pr_s\}_{s\in I_0}$ is the family of all coordinate projections on $\mathbb{R}^{I_0}$. It implies that the family of sets $\mathcal{F}_{p|M} = \{U \cap M : U \in \mathcal{F}_p\}$ is a Cauchy filter on M. Since for any $s \in I_0$ the function

$$pr_s((x_t)_{t\in I_0}) = x_s, \qquad (x_t)_{t\in I_0} \in \mathbb{R}^{I_0},$$

is uniform then $pr_s(\mathcal{F}_{p|M})$ is a base of some Cauchy filter in $\mathbb{R}$, which converges top_s. By the definition of M it follows that for any $s \in I$ we have

$$pr_s(\mathcal{F}_{p|M}) = f_s(pr_0(\mathcal{F}_{p|M})).$$

It means that $f_s(pr_0(\mathcal{F}_{p|M}))$ is a base of a filter which is convergent in $\mathbb{R}$.

Suppose now that $p_0 \in I$. Since $pr_0(\mathcal{F}_{p|M}) \to p_0$, we obtain (see the definition of f_{p_0}) that $f_{p_0}(pr_0(\mathcal{F}_{p|M}))$ is not convergent in $\mathbb{R}$. This leads to a contradiction. Hence $p_0 \in \mathbb{Q}$ and for any $s \in I_0$ we have: $p_s = f_s(p_0)$, which means that $p \in M$.

Since M is closed in $\mathbb{R}^{I_0}$, it is complete and by Theorem 4.2 we have

$$compl_{\mathcal{U}_\mathcal{G}}\mathbb{Q} = \mathbb{Q}, \quad compl_{\mathcal{U}_\mathcal{G}} C^\infty(\mathbb{R}_\mathbb{Q}) = C^\infty(\mathbb{R}_\mathbb{Q}).$$

5. Compactification of a differential space

Let $(M, \mathcal{C})$ be a differential space such that $\mathcal{C} = gen(\mathcal{G})$. Let $f \in \mathcal{C}$, $m \in M$. Then there exist: neighbourhood U of m, number $n \in \mathbb{N}$, functions $\alpha_1, \ldots, \alpha_n \in \mathcal{G}$ and $\omega \in C^\infty(\mathbb{R}^n)$ such that $f|_U = \omega \circ (\alpha_1, \ldots, \alpha_n)|_U$. We denote $y_0 := (\alpha_1(m), \ldots, \alpha_n(m))$. Let us take cubes:

$P :=$

$(\alpha_1(m)-1, \alpha_1(m)+1)\times(\alpha_2(m)-1, \alpha_2(m)+1)\times\ldots\times(\alpha_n(m)-1, \alpha_n(m)+1)$,

$P' :=$

$(\alpha_1(m)-2, \alpha_1(m)+2)\times(\alpha_2(m)-2, \alpha_2(m)+2)\times\ldots\times(\alpha_n(m)-2, \alpha_n(m)+2)$.

Let $\eta \in C^\infty(\mathbb{R}^n)$, such that: $\eta|_P \equiv 1$, $\eta|_{\mathbb{R}^n\setminus P'} \equiv 0$ and $|\eta| \leq 1$. We mark: $\alpha := (\alpha_1, \ldots, \alpha_n)$, $\beta_i := \alpha_i \cdot \eta(\alpha_1, \ldots, \alpha_n)$, $\beta := (\beta_1, \ldots, \beta_n)$. Let $V := \alpha^{-1}(P)$, $V' := \alpha^{-1}(P')$. Then $V \subset V' \subset M$ and $m \in U \cap V$. For any $x \in U \cap V$ we have: $f(x) = \omega(\alpha_1(x), \ldots, \alpha_n(x)) = \omega(\beta_1(x), \ldots, \beta_n(x))$. We observe that $\forall i \in \{1, \ldots, n\}$ $|\beta_i(x)| \leq max\{|\alpha_1(m)+2|, |\alpha_1(m)-2|\} =: \mu_i$. Then $f(x) = \omega(\mu_1\frac{\beta_1(x)}{\mu_1}, \mu_2\frac{\beta_2(x)}{\mu_2}, \ldots, \mu_n\frac{\beta_n(x)}{\mu_n}) = \omega(\mu_1\gamma_1(x), \ldots, \mu_n\gamma_n(x)) = \omega_1(\gamma_1, \ldots, \gamma_n)(x)$, where: $\forall 1 \leq i \leq n, \forall x \in M$ $\gamma_i(x) = \frac{\beta_i(x)}{\mu_i}$ and $|\gamma_i(x)| < 1|$. Hence we get the following theorem:

Theorem 5.1. *For any differential space $(M, \mathcal{C})$ there exists the family of bounded generators, in particular $\mathcal{C} = gen(\mathcal{G}_1)$, where $\mathcal{G}_1 = \{\gamma_i\}_{|i\in I}$ such that $|\gamma_i| \leq 1$ $\forall i \in I$.*

For any Hausdorff differential space we consider the generators embedding of that space using the family of generators described in Theorem 5.1 (it takes values in the cube $[-1,1]^I$). So we have the embedding of the differential space into the compact space and we close the image. Hence we get the compact set $\overline{M}_{\mathcal{G}}$.

If we mark $J := [-1,1]$, then $\overline{M}_{\mathcal{G}} \subset J^I$. On J^I there exists the natural differential structure $C^\infty(J^I) = C^\infty(\mathbb{R}^I)|_{J^I}$ generated by the family of projections $\{pr_{i|_J}\}_{i\in I}$, where $pr_i : \mathbb{R}^I \to \mathbb{R}$ is the projection onto i-th coordinate. By localization of that structure to the set $\overline{M}_{\mathcal{G}}$ we get the differential space $(\overline{M}_{\mathcal{G}}, C^\infty(J^I)_{\overline{M}_{\mathcal{G}}})$ which is a differential subspace of the space $(J^I, C^\infty(J^I))$. We call that differential space *the (differential) compactification of the differential space $(M, \mathcal{C})$ by the family of generators $\mathcal{G}$* and we denote it by $(compt_{\mathcal{G}}M, compt_{\mathcal{G}}\mathcal{C})$. We see that $compt_{\mathcal{G}}M = compl_{\mathcal{G}}M$ and $compt_{\mathcal{G}}\mathcal{C} = compl_{\mathcal{G}}\mathcal{C}$. So for the compactification of the differential space we have analogous theorems like for the completion.

Remark 5.1. *Let $\mathcal{G}$ and $\mathcal{H}$ be the families of bounded generators of the differential structure $\mathcal{C}$ on the set $M \neq \emptyset$. If $\mathcal{G} \subset \mathcal{H}$, that there exists smooth function $\iota_{\mathcal{G}\mathcal{H}} : (compt_{\mathcal{H}}M, compt_{\mathcal{H}}\mathcal{C}) \to (compt_{\mathcal{G}}M, compt_{\mathcal{G}}\mathcal{C})$ such that $\iota_{\mathcal{G}\mathcal{H}|M} = id_M$.*

Let consider the differential space $(M, \mathcal{C})$ and the family $\mathcal{C}_0$ of all smooth functions on that space which takes values in $[-1,1]$. Using the procedure of the compactification, described earlier, we get the differential space that we mark by $(compt_{\mathcal{C}}M, compt_{\mathcal{C}}\mathcal{C})$ it means $compt_{\mathcal{C}}M := compt_{\mathcal{C}_0}M$, $compt_{\mathcal{C}}\mathcal{C} := compt_{\mathcal{C}_0}\mathcal{C}$.

Definition 5.1. The differential space $(compt_{\mathcal{C}}M, compt_{\mathcal{C}}\mathcal{C})$ is called the *maximal differential compactification* of the space $(M, \mathcal{C})$.

Let us assume that the topological space $(M, \tau_{\mathcal{C}})$ is compact. Then all the functions from $\mathcal{C}$ are limited and by the normalization of each function $\alpha \in \mathcal{C} \setminus \{0\}$ according the formula:

$$N\alpha(p) = \frac{1}{sup_{q\in M}|\alpha(q)|}\alpha(p), p \in M$$

we get the family $N\mathcal{C} = \{N\alpha : \alpha \in \mathcal{C}\}$ of the generators of the structure $\mathcal{C}$. Then the generator embedding given by $N\mathcal{C}$ converts diffeomorphically $(M, \mathcal{C})$ onto the compact subspace of $(J^{N\mathcal{C}}, C^\infty(J^{N\mathcal{C}}))$. Similarly, if $\mathcal{G}$ is any family of generators of the structure $\mathcal{C}$, then $N\mathcal{G} = \{N\alpha : \alpha \in \mathcal{G}\}$ is the family of generators of $\mathcal{C}$ too, and an appropriate generator embedding is a diffeomorphism of $(M, \mathcal{C})$ onto a compact subspace of $(J^{N\mathcal{G}}, C^\infty(J^{N\mathcal{G}}))$. We have

$$compt_{\mathcal{C}}M = M, \quad compt_{\mathcal{G}}\mathcal{C} = \mathcal{C},$$

where the equality is the identification of the diffeomorphic spaces and structures.

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D. Dziewa-Dawidczyk
DEPARTMENT OF APPLIED MATHEMATICS
FACULTY OF APPLIED INFORMATICS AND MATHEMATICS
WARSAW UNIVERSITY OF LIFE SCIENCES – SGGW
Nowoursynowska 159
02-776 WARSAW, POLAND
E-mail: ddziewa@mini.pw.edu.pl
Z. Pasternak-Winiarski
FACULTY OF MATHEMATICS AND INFORMATION SCIENCE
WARSAW UNIVERSITY OF TECHNOLOGY
Pl. Politechniki 1
00-661 WARSAW, POLAND
INSTITUTE OF MATHEMATICS
UNIVERSITY OF BIAŁYSTOK
Akademicka 2
15-267 BIAŁYSTOK, POLAND
E-mail: Z.Pasternak-Winiarski@mini.pw.edu.pl

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