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J. K. Kohli, Jeetendra Aggarwal

CLOSEDNESS OF CERTAIN CLASSES OF FUNCTIONS IN THE TOPOLOGY OF UNIFORM CONVERGENCE

Abstract. In this paper, closedness of certain classes of functions in Y^X in the topology of uniform convergence is observed. In particular, we show that the function spaces $SC(X,Y)$ of quasi continuous (≡semi-continuous) functions, $C_\alpha(X,Y)$ of α-continuous functions and $L(X,Y)$ of cl-supercontinuous functions are closed in Y^X in the topology of uniform convergence.

1. Introduction

It is well known that the function space $C(X,Y)$ of continuous functions from a topological space X into a uniform space Y is closed in Y^X in the topology of uniform convergence but not necessarily closed in the topology of pointwise convergence. Moreover, it is of fundamental importance in topology, analysis and other branches of mathematics to know that whether a given function space is closed in Y^X in the topology of pointwise convergence/the topology of uniform convergence. So the question arises: are there other classes of functions besides continuous functions which are closed in Y^X in the topology of pointwise convergence or in the topology of uniform convergence? Sierpinski [19] showed that uniform limit of a sequence of connected (Darboux) functions need not be a connected function. So, in general, the set of all connected functions from a space X into a uniform space Y is not necessarily closed in Y^X in the topology of uniform convergence. In contrast, Naimpally [14] proved that the set of all connectivity functions from a space X into a uniform space Y is closed in Y^X in the topology of uniform convergence. Furthermore, in [15] Naimpally introduced the notion of graph topology Γ for a function space and showed that the set of all almost continuous functions (in the sense of Stallings [21]) is not only closed in Y^X

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in the graph topology but indeed represents the closure of $C(X,Y)$ in the graph topology. In the same vein, Hoyle [6] showed that the set $SW(X,Y)$ of all somewhat continuous functions from a space X into a uniform space Y is closed in Y^X in the topology of uniform convergence.

Moreover, in [16] Naimpally showed that if X is locally connected and Y is Hausdorff, then the function space $S(X,Y)$ of all strongly continuous functions from X into Y is closed in Y^X in the topology of pointwise convergence. Recently, Kohli and Singh [10] extended Naimpally's result to a larger framework wherein it is shown that if X is sum connected [9] (in particular, X is connected or locally connected) and Y is Hausdorff, then the function space $P(X,Y)$ of all perfectly continuous functions as well as the function space $L(X,Y)$ of all cl-supercontinuous functions is closed in Y^X in the topology of pointwise convergence. In this paper, we restrict ourselves to the study of function spaces which are closed in Y^X in the topology of uniform convergence. In particular, we show that the function spaces $SC(X,Y)$ of quasi continuous ($\equiv$semi-continuous) functions, $C_\alpha(X,Y)$ of α-continuous functions, and $L(X,Y)$ of cl-supercontinuous functions are closed in Y^X in the topology of uniform convergence.

2. Preliminaries and basic definitions

2.1. Definitions. A subset A in a topological space X is called

(i) **semi-open** [12] ($\equiv$**quasi open** [8]) if there exists an open set O such that $O \subset A \subset \operatorname{cl} O$, equivalently, A is semi-open if and only if $A \subset \operatorname{cl}(\operatorname{int} A)$.

(ii) α**-open** [17] if $A \subset \operatorname{int}(\operatorname{cl}(\operatorname{int} A))$.

(iii) α**-neighborhood** [13] of a point $x \in X$ if there exists an α-open set U in X such that $x \in U \subset A$.

(iv) **cl-open** [20] ($\equiv$**co-open** [3]) if for each $x \in A$ there exists a clopen set H such that $x \in H \subset A$, or equivalently, A is expressible as a union of clopen sets.

2.2. Definitions. A function $f : X \to Y$ from a topological space X into a topological space Y is said to be

(i) **semi-continuous** [12] ($\equiv$**quasi continuous** [8]) if $f^{-1}(V)$ is semi-open in X for each open set V in Y. Equivalently, for each point x of X and for each open set V in Y containing $f(x)$, there exists a semi-open set A in X such that $x \in A$ and $f(A) \subset V$.

(ii) α**-continuous** [13] if $f^{-1}(V)$ is α-open in X for each open set V in Y. Equivalently, for each point x of X and for each neighborhood V of $f(x)$, there exists an α-neighborhood U of x such that $f(U) \subset V$.

(iii) **somewhat continuous** [5] if for each open set V in Y such that $f^{-1}(V) \neq \varnothing$, there exists a nonempty open set U in X such that $U \subset f^{-1}(V)$.

(iv) **cl-supercontinuous** [20] ($\equiv$**clopen continuous** [18]) if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists a clopen set U in X containing x such that $f(U) \subset V$.

(v) **almost cl-supercontinuous** [11] ($\equiv$**almost clopen continuous** [2]) if for each $x \in X$ and each regular open set V in Y containing $f(x)$, there exists a clopen set U in X containing x such that $f(U) \subset V$.

2.3. Remark. Somewhat continuous functions have also been referred to in the literature as **feebly continuous** (see [1], [4]). However, Frolík [4] requires mappings to be surjective.

The following implications are immediate from definitions.

$$\begin{array}{ccccccc}
\textbf{cl-supercontinuous} & \rightarrow & \textbf{almost cl-supercontinuous} & & & & \\
(\equiv\textbf{clopen continuous}) & & (\equiv\textbf{almost clopen continuous}) & & & & \\
\downarrow & & & & & & \\
\textbf{continuous} & \rightarrow & \alpha\textbf{-continuous} & \rightarrow & \textbf{semi-continuous} & \rightarrow & \textbf{somewhat continuous}
\end{array}$$

However, none of the above implication is reversible.

We now recall the notions of the topology of pointwise convergence and that of the topology of uniform convergence. For details (see [7]).

Let $Y^X = \{f : X \to Y \text{ is a function}\}$ be the set of all functions from X to Y, where X is a set and Y is a topological space. A subcollection $F \subset Y^X$ has the topology $\mathcal{P}$ **of pointwise convergence** (or **pointwise topology**) if it has the relativized product topology. A subbase for $\mathcal{P}$ is the family of all the subsets of the form $\{f : f(x) \in U\}$, where x is a point of X and U is open in Y. A family $F \subset Y^X$ is **pointwise closed** if and only if F is a closed subset of the product space Y^X. Further, let Y be a uniform space with uniformity $\mathcal{V}$ and let $F \subset Y^X$. Then a basis for the uniformity of uniform convergence $\mathcal{U}$ for F is the collection $\{W(V) : V \in \mathcal{V}\}$, where $W(V) = \{(f, g) \in F \times F : (f(x), g(x)) \in V \text{ for all } x \in X\}$. The topology of $\mathcal{U}$ is known as the **topology of uniform convergence**.

2.4. Definition. [7] A uniform space $(Y, \mathcal{V})$ is said to be **complete** if and only if every Cauchy net in the space converges to a point of the space.

2.5. Theorem. [7, p. 194] *The product of uniform spaces is complete if and only if each coordinate space is complete.*

3. Function spaces closed in the topology of uniform convergence

Throughout the section we assume that X is a topological space and Y is a uniform space and the function space Y^X of all functions from a space X into a space Y is endowed with the topology of uniform convergence. In this section, we shall show that function spaces $SC(X, Y)$ of semi-continuous functions, $C_\alpha(X, Y)$ of α-continuous functions and $L(X, Y)$ of cl-supercontinuous functions are closed in the topology of uniform convergence.

3.1. Theorem. *Let X be topological space and let $(Y, \mathcal{V})$ be a uniform space. Then the set $SC(X, Y)$ of all semi-continuous functions from X into Y is closed in Y^X in the topology of uniform convergence. Moreover, if Y is complete, so is $SC(X, Y)$ in the topology of uniform convergence.*

Proof. Let f be an element of Y^X which is a limit point of $SC(X, Y)$. Suppose $f(x) \in W$, where W is an open set in the uniform topology of Y. Then there exists $V \in \mathcal{V}$, such that $f(x) \in V[f(x)] \subset W$. Choose a symmetric member $U \in \mathcal{V}$, such that $U \circ U \circ U \subset V$. Since f is a limit point of $SC(X, Y)$, there exists a semi-continuous function g in $SC(X, Y)$ such that $g(y) \in U[f(y)]$ for all $y \in X$. In particular, $g(x) \in U[f(x)]$. Now, since g is a semi-continuous function, there exists a semi-open set A containing x such that $g(A) \subset U[f(x)]$. To show that f is a semi-continuous function, we shall prove that $f(A) \subset W$. Let $z \in A$. By symmetry of U, $f(z) \in U[g(z)] \subset U \circ U[f(x)] \subset V[f(x)] \subset W$. Therefore $f(A) \subset V[f(x)] \subset W$ and so f is semi-continuous. The last assertion is immediate in view of the fact that a closed subspace of a complete uniform space is complete [7, Theorem 22, p. 192]. ∎

3.2. Theorem. *Let X be a topological space and let $(Y, \mathcal{V})$ be a uniform space. Then the set $C_\alpha(X, Y)$ of all α-continuous functions from X into Y is closed in Y^X in the topology of uniform convergence. Further, if Y is complete, so is the function space $C_\alpha(X, Y)$ in the topology of uniform convergence.*

3.3. Remark. Proof of Theorem 3.2 is similar to that of Theorem 3.1 and hence omitted.

In [10] it is shown that if X is a sum connected space and Y is a Hausdorff space, then the set $L(X, Y)$ of all cl-supercontinuous functions from X into Y is closed in the topology of pointwise convergence. We now prove that the set $L(X, Y)$ of all cl-supercontinuous functions from X into Y is also closed in the topology of uniform convergence where X is a topological space and Y is a uniform space.

3.4. Theorem. *Let X be topological space and let $(Y, \mathcal{V})$ be a uniform space. Then the set $L(X, Y)$ of all cl-supercontinuous functions from X into Y is closed in Y^X in the topology of uniform convergence. Further, if*

Y is complete, so is the function space $L(X,Y)$ in the topology of uniform convergence.

Proof. Let $f \in Y^X$ be a limit point of $L(X,Y)$ which is not cl-supercontinuous at a point $x \in X$. Then there exists $V \in \mathcal{V}$ such that $f^{-1}(V[f(x)])$ does not contain any clopen set containing x. Choose a symmetric member W of $\mathcal{V}$ such that $W \circ W \circ W \subset V$. Since f is a limit point of $L(X,Y)$, there exists $g \in L(X,Y)$ such that $g(y) \in W[f(y)]$ for all $y \in X$. Then $g \subset W \circ f$ and $g^{-1} \subset f^{-1} \circ W^{-1} = f^{-1} \circ W$ and hence $g^{-1} \circ W \circ g \subset f^{-1} \circ W \circ W \circ W \circ f \subset f^{-1} \circ V \circ f$. Therefore, $g^{-1}[W(g(x)] \subset f^{-1}[V(f(x)]$. Since $f^{-1}[V(f(x)]$ does not contain any clopen set containing x, neither does $g^{-1}[W(g(x)]$, which contradicts cl-supercontinuity of g. Therefore, $f \in L(X,Y)$. The last assertion is immediate, since a closed subspace of a complete uniform space is complete. ■

3.5. Remark. In view of the above discussion, the following inclusions are immediate/well known.

$$L(X,Y) \subset C(X,Y) \subset C_\alpha(X,Y) \subset SC(X,Y) \subset SW(X,Y) \subset Y^X .$$

Since in the topology of uniform convergence each of the above function spaces is a closed subspace of its succeeding one, the completeness of any one of them implies that of its predecessor. In particular, if Y is complete, then each of the above function spaces is complete.

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J. K. Kohli
DEPARTMENT OF MATHEMATICS
HINDU COLLEGE
UNIVERSITY OF DELHI
DELHI 110007, INDIA
E-mail: jk_kohli@yahoo.co.in
Jeetendra Aggarwal:
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF DELHI
DELHI 110007, INDIA
E-mail: jitenaggarwal@gmail.com

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