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ON SUP-MEASURABILITY OF MULTIFUNCTIONS WITH SOME DENSITY PROPERTIES

Abstract. The paper is concerned with sup-measurability of a multifunction F defined on the product $X \times Y$ of metric spaces with some differentiation bases. We introduce the lower D_α property and the upper D_α property of multifunction, where $\alpha \in (0,1) \subset \mathbb{R}$, and we prove sup-measurabilty of F when it has the upper D_α property at (x,y), and $F(x,\cdot)$ has the lower D_α property at y for every $(x,y) \in X \times Y$. Some application of this theorem to the existence of solutions of differential inclusions $f'(x) \in F(x, f(x))$ is given.

1. Introduction

Sup-measurability, roughly speaking, means measurability of the Carathéodory superposition $H(x) = F(x, G(x)) = \bigcup_{y \in G(x)} F(x,y)$, where F is a multifunction form $X \times Y$ to Z and G is a measurable multifunction from X to Y. In the single valued version, the problem of sup-measurability have been studied extensively (an overview of some papers in this field can be found in [10]). Far less is known, however, in the multivalued case, although in various fields of mathematics and its applications, the superposition $F(x, G(x))$ occurs frequently (see for instance [1], [2], [12], [13] and [16]).

The problem of sup-measurability was for the first time considered by Carathéodory in his book [6]. He formulated a sufficient condition for sup-measurability of a function $f : \mathbb{R}^2 \to \mathbb{R}$, namely, measurability of $f(\cdot, y)$ and continuity of $f(x, \cdot)$. It is known that the continuity of $f(x,\cdot)$ in the Carathéodory theorem cannot be replaced by the approximate continuity. But if we suppose approximate continuity of $f(x,\cdot)$ and measurability of f instead of the measurability of $f(\cdot,y)$, then f is sup-measurable [10, Théorème 25].

The purpose of this paper is to prove a new sup-measurability result concerning multifunctuions. We begin with notations, terminology and

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facts known in the literature (Section 2). Here, we also consider multifunctions defined on a measurable metric space with some differentiation basis. Given $\alpha \in (0,1) \subset \mathbb{R}$, we define some density properties of a multifunction, called the lower D_α property and the upper D_α property with respect to the differentiotion basis, more general than the approximate semicontinuity.

In Section 3, we consider multifunctions defined on the product $X \times Y$ of measurable metric spaces with differentiation bases. We show sup-measurability of a multifunction F when it has the upper D_α property at (x,y) and $F(x,\cdot)$ has the lower D_α property at y for every $(x,y) \in X \times Y$.

Many problems of applied mathematics lead us to the study of dynamical systems having velocities not uniquely determined by the state of the systems, but depending only loosely upon it. In these cases, the classical equation of the form $f'(x) = g(x, f(x))$, describing the dynamics of the system, is replaced by a relation of the form $f'(x) \in F(x, f(x))$, where F is a multifunction. Such a "set valued differential equation" is called a differential inclusion.

In Section 4, we give some application of our theorem on sup-measurability to the existence of solutions of differential inclusions. We consider the initial value problem

$$f'(x) \in F(x, f(x)) \quad \text{and} \quad f(x_0) = y_0. \tag{1}$$

Here we deal with multifunctions in more specjal spaces; F stands for a multifunction from $I \times Y$ to Y, where $I \subset \mathbb{R}$ is an interval and Y is the k-dimensional Euclidean space. If the initial state $(x_0, y_0) \in I \times Y$, then by a solution of (1) we mean any absolutely continuous function $f : [x_0, b] \to Y$, $b \in I$, such that $f(x_0) = y_0$ and $f'(x) \in F(x, f(x))$ for almost all $x \in [x_0, b]$.

The existence of solutions of (1) may be shown in many ways. The conditions to be imposed on the multifunction F in order that to have solutions of (1) are mainly of two kinds: continuity or semicontinuity of F and some conditions type of Lipschitz or integrable boundness of F. This was extended to the Carathéodory case, i.e. $F(x,\cdot)$ is continuous and $F(\cdot, y)$ is measurable (see [2], [8], [9], [14], [15]). We obtain a solution of (1) when F has the upper D_α property and $F(x,\cdot)$ is continuous.

2. Preliminaries

As a rule we will denote by $\mathbb{N}$ and $\mathbb{R}^k$, $k \in \mathbb{N}$, the set of positive integers and the k-dimensional Euclidean space, respectively.

Let S and Z be nonempty sets and let Φ be a mapping which associates to each point $s \in S$ a nonempty set $\Phi(s) \subset Z$. Such a mapping is called a multifunction from S to Z and we write $\Phi : S \rightsquigarrow Z$.

Any function $\phi : S \to Z$ such that $\phi(s) \in \Phi(s)$ for every $s \in S$ is called a selection of the multifunction $\Phi : S \rightsquigarrow Z$.

If $\Phi : S \rightsquigarrow Z$ is a multifunction and $G \subset Z$, then two inverse images of G under Φ are defined as follows:

$$\Phi^+(G) = \{s \in S : \Phi(s) \subset G\} \quad \text{and} \quad \Phi^-(G) = \{s \in S : \Phi(s) \cap G \neq \emptyset\}.$$

A function $f : S \to Z$ may be considered as a multifunction assigning to $s \in S$ the singleton $\{f(s)\}$. In this case we have $f^+(G) = f^-(G) = f^{-1}(G)$ for $G \subset Z$.

Let $(S, \mathcal{T}(S))$ be a topological space and $s \in S$. We will use $\mathcal{B}(s)$ to denote the filterbase of open neighbourhoods of s. Moreover, $\mathcal{B}(S)$ will denote the σ-field of Borel subsets of S. The closure of $A \subset S$ will be denoted by $\mathrm{Cl}(A)$ and the interior of A by $\mathrm{Int}(A)$.

Let (Z, d) be a metric space and let $\mathcal{P}_0(Z)$ be the family of all nonempty subsets of Z. If $z \in Z$ and $r > 0$, then, as usual, $B(z, r)$ will denote the open ball centred at z and radius r.

A multifunction $\Phi : S \rightsquigarrow Z$ is called *h-lower* (resp. *h-upper*) *semicontinuous* at a point $s_0 \in S$ if, for each $\varepsilon > 0$ there exists $U(s_0) \in \mathcal{B}(s_0)$ such that $h_l(\Phi(s), \Phi(s_0)) < \varepsilon)$ (resp. $h_u(\Phi(s), \Phi(s_0)) < \varepsilon)$ for each $s \in U(s_0)$, where h_l and h_u are, respectively, the lower and the upper hemimetrics in the space $\mathcal{P}_0(Z)$ defined by

$$h_l(A, B) = \sup\{d(x, B) : x \in A\} \quad \text{and} \quad h_u(A, B) = \sup\{d(x, A) : x \in B\}.$$

We say that Φ is *h-lower* (resp. *h-upper*) *semicontinuous* if it is h-lower (resp. h-upper) semicontinuous at each point $s_0 \in S$; Φ is called *h-continuous* if it is both h-lower and h-upper semicontinuous.

Let $\mathcal{C}_b(Z)$ be the family of all nonempty closed and bounded subsets of Z. The pseudometric h in $\mathcal{P}_0(Z)$ defined by

$$h(A, B) = \max(h_l(A, B), h_u(A, B))$$

is a metric in $\mathcal{C}_b(Z)$. This metric is called the Hausdorff metric generated by the metric d. We will denote it by d_H.

Let us note that

(2) If a multifunction $\Phi : S \rightsquigarrow Z$ is h-lower semicontinuous, then for each open set $G \subset Y$, the set $\Phi^-(G)$ is open.

Indeed, if $s_0 \in \Phi^-(G)$, then $\Phi(s_0) \cap G \neq \emptyset$. Let $z_0 \in \Phi(s_0) \cap G$ and let $\varepsilon > 0$ be such that $B(z_0, \varepsilon) \subset G$. By h-lower semicontinuity of Φ at s_0, there is $U(s_0)$ such that $h_l(\Phi(s), \Phi(s_0)) < \varepsilon$ for $s \in U(s_0)$. Then $\Phi(s) \cap B(z, \varepsilon) \neq \emptyset$ for any $z \in \Phi(s_0)$. Thus $\Phi(s) \cap G \neq \emptyset$ for $s \in U(s_0)$ and $\Phi^-(G)$ is open.

Let $\Phi : S \rightsquigarrow Z$ be a multifunction. For a fixed point $z \in Z$ we define the function $g_z : S \to \mathbb{R}$ by

$$g_z(s) = d(z, \Phi(s)).$$

It is known that

(3) If Φ is h-upper semicontinuous, then for every $z \in Z$ the function g_z is lower semicontinuous [13, Ch. 1, Prop. 2.64].

(4) If Φ is h-lower semicontinuous, then for every $z \in Z$ the function g_z is upper semicontinuous [13, Ch. 1, Prop. 2.66 and 2.26].

Let $(S, \mathcal{M}(S))$ be a measurable space and $(Z, \mathcal{T}(Z))$ a topological space. A multifunction $\Phi : S \rightsquigarrow Z$ is called $\mathcal{M}(S)$-*measurable* if $\Phi^-(G) \in \mathcal{M}(S)$ for each $G \in \mathcal{T}(Z)$.

If $S = \mathbb{R}^k$, then by $\mathcal{M}(S)$ we will understood the σ-field of Lebesgue measurable sets and we will say, simply, Φ is measurable.

Let $\Phi : S \rightsquigarrow Z$ be a multifunction. Consider the following properties:

(a) The function g_z is $\mathcal{M}(S)$-measurable for each $z \in Z$;

(b) Φ admits a sequence of $\mathcal{M}(S)$-measurable selections $(\phi_n)_{n\in\mathbb{N}}$ such that $\Phi(s) = \mathrm{Cl}(\{\phi_n(s) : n \in \mathbb{N}\})$ for each $s \in S$.

If (Z, d) is separable, then

(5) (i) $\mathcal{M}(S)$-measurability of Φ is equivalent to (a) [7, Th. III.2].
(ii) If in addition Φ is complete valued, then $\mathcal{M}(S)$-measurability of Φ is equivalent to (b) [7, Th. III.9].

Let $(S, \varrho, \mathcal{M}(S), \mu)$ be a separable metric space with metric ϱ, where $\mathcal{M}(S)$ is a σ-field of subsets of S containing $\mathcal{B}(S)$ and μ is a σ-finite regular and complete measure on $\mathcal{M}(S)$; μ^* will denote the outer measure generated by μ.

(6) Let $\mathcal{F}(S) \subset \mathcal{M}(S)$ be a countable family of μ-measurable sets with nonempty interiors of a positive and finite measure μ, the boundaries of which are of μ-measure zero.
Let $\{I_n\}_{n\in N} \subset \mathcal{F}(S)$ and $s \in S$. We write $I_n \to s$ if $s \in \mathrm{Int}(I_n)$ for each $n \in N$ and the diameter of I_n converges to zero as $n \to \infty$. We assume that for every $s \in S$, there exists a sequence $(I_n)_{n\in N}$ of sets from $\mathcal{F}(S)$ such that $I_n \to s$. The pair $(\mathcal{F}(S), \to)$ then forms a differentiation basis for the space $(S, \varrho, \mathcal{M}(S), \mu)$ in Bruckner's terminology [5, p. 30].

Now we assume that $(\mathcal{F}(S), \to)$ is a differentiation basis for the space $(S, \varrho, \mathcal{M}(S), \mu)$. If $A \subset S$ and $s \in S$, then the lower outer density of A at s with respect to $\mathcal{F}(S)$ is defined by

$$\mathrm{liminf}_{I_n \to s} \frac{\mu^*(A \cap I_n)}{\mu(I_n)}.$$

Replacing liminf by limsup we obtain the upper outer density of A at s with respect to $\mathcal{F}(S)$. These densities will be denoted by $D^*{}_l(A,s)$ and $D^*{}_u(A,s)$, respectively. If they are equal, their common value will be called the outer density of A at s with respect to $\mathcal{F}(S)$ and denoted by $D^*(A,s)$. If $A \in \mathcal{M}(S)$, then the outer densities of A at $s \in S$ with respect to $\mathcal{F}(S)$ will be called the densities of A at s with respect to $\mathcal{F}(S)$ and denoted with no asterisk.

A point $s \in S$ will be called a density point of a set $A \subset S$ with respect to $\mathcal{F}(S)$ if there exists a $B \in \mathcal{M}(S)$ such that $B \subset A$ and the density of B at s with respect to $\mathcal{F}(S)$ is equal to 1. We will write $D(A,s) = 1$.

We assume that

(7) $\mathcal{F}(S)$ has the density property, i.e., $\mu(\{s \in A : D^*{}_l(A,s) < 1\}) = 0$ for every set $A \subset S$.

Definition 1. A multifunction $\Phi : S \rightsquigarrow Z$ is called approximately h-lower (resp. h-upper) semicontinuous with respect to $\mathcal{F}(S)$ at a point $s \in S$, if there exists a set $E_s \in \mathcal{M}(S)$ with $s \in E$ such that $D(E,s) = 1$ and the restriction $\Phi|_{E_s}$ is h-lower (resp. h-upper) semicontinuous at s.

If Φ is approximately h-lower (resp. h-upper) semicontinuous with respect to $\mathcal{F}(S)$ at any point $s \in S$, then it is called approximately h-lower (resp. h-upper) semicontinuous with respect to $\mathcal{F}(S)$.

Definition 2. Let $\Phi : S \rightsquigarrow Z$ be a multifunction, $s \in S$ and $\alpha \in (0,1) \subset \mathbb{R}$. We say that Φ has the lower (resp. upper) D_α property with respect to $\mathcal{F}(S)$ at s, if there is a set $A_s \in \mathcal{M}(S)$ with $s \in A_s$ such that $D_l(A_s,s) > 1-\alpha$ (resp. $D_l(A_s,s) > \alpha$) and the restriction $\Phi|_{A_s}$ is h-lower (resp. h-upper) semicontinuous at s; Φ has the lower (resp. upper) D_α property with respect to $\mathcal{F}(S)$, if it has this property at each point $s \in S$.

It is clear that

(8) If a multifunction $\Phi : S \rightsquigarrow Z$ is approximalely h-lower (resp. h-upper) semicontinuous with respect to $\mathcal{F}(S)$ at a point $s \in S$ and $\alpha \in (0,1)$, then Φ has the lower (resp. upper) D_α property with respect to $\mathcal{F}(S)$ at the point s.

In the case $S = \mathbb{R}^k$, μ will denote the Lebesgue measure and by $\mathcal{F}(S)$ we will understood the family of balls with rational radius and centred at a point with rational coordinates. In this case we will say simply that Φ is approximately semicontinuous or Φ has the lower (resp. upper) D_α property.

3. Main results

Let X and Y be nonempty sets and let $F : X \times Y \rightsquigarrow Z$ be a multifunction. If $x \in X$ is fixed, then the multifunction $F_x : Y \rightsquigarrow Z$ given by $F_x(y) =$

$F(x,y)$ is called the x-cection of F. Analogously, for fixed $y \in Y$, the y-section F^y of F is defined.

Now let $(X, \mathcal{M}(X))$ be a measurable space and let $(Y, \mathcal{T}(Y))$ and $(Z, \mathcal{T}(Z))$ be topological spaces. A multifunction $F : X \times Y \rightsquigarrow Z$ is called $\mathcal{M}(X)$-*sup-measurable* if for each $\mathcal{M}(X)$-measurable closed valued multifunction $G : X \rightsquigarrow Y$ the Carathéodory superposition $H : X \rightsquigarrow Z$, given by

$$H(x) = F(x, G(x)) = \bigcup_{y \in G(x)} F(x,y),$$

is an $\mathcal{M}(X)$-measurable multifunction.

From now on we assume $(X, \varrho, (X), \mu)$ and $(Y, \rho, \mathcal{M}(Y), \nu)$ to be separable metric spaces with the differentiation bases $(\mathcal{F}(X), \to)$ and $(\mathcal{F}(Y), \to)$, respectively, defined as in (6). We suppose that $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ have the density property (see (7)).

Let $\mathcal{M}(X) \otimes \mathcal{M}(Y)$ be the σ-field in $X \times Y$ generated by the family of sets $A \times B$, where $A \in \mathcal{M}(X)$ and $B \in \mathcal{M}(Y)$, and let $\mu \times \nu$ be the product measure on $\mathcal{M}(X) \otimes \mathcal{M}(Y)$. Then the family

$$\mathcal{F}(X) \times \mathcal{F}(Y) = \{I \times J : I \in \mathcal{F}(X) \wedge J \in \mathcal{F}(Y)\}$$

has the density property, because $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ it have [5, pp. 5 and 34].

We begin with some result on sup-measurability of real functions.

Theorem 1. *Let $\alpha \in (0,1)$ and let $f : X \times Y \to \mathbb{R}$ be a function. If for every $(x,y) \in X \times Y$ there are the sets $A(x,y) \subset Y$ and $B(x,y) \in \mathcal{M}(X) \otimes \mathcal{M}(Y)$ such that*

(i) *$y \in A(x,y)$ and $D_l^*(A(x,y), y) > 1 - \alpha$,*
(ii) *the restriction $(f_x)|_{A(x,y)}$ is upper semicontinuous at y,*
(iii) *$(x,y) \in B(x,y)$ and $D_l(B(x,y), (x,y)) > \alpha$,*
(iv) *the restriction $f|_{B(x,y)}$ is lower semicontinuous at (x,y),*

then f is $\mathcal{M}(X)$-sup-measurable.

Proof. Suppose that, on the contrary, f is not $\mathcal{M}(X)$-sup-measurable. Then there is an $\mathcal{M}(X)$-measurable function $g : X \to Y$ such that the Carathéodory's superposition $h : X \to \mathbb{R}$, given by $h(x) = f(x, g(x))$, is not $\mathcal{M}(X)$-measurable. Then there is $a \in \mathbb{R}$ such that $h^{-1}((-\infty, a)) \notin \mathcal{M}(X)$. Let $\mathcal{M}_+(X) = \{A \in \mathcal{M}(X) : \mu(A) > 0\}$ and let $\mathcal{X}$ be the family of all sets $W \in \mathcal{M}_+(X)$ such that

$$\mu(h^{-1}((-\infty, a)) \cap W) = 0 \quad \text{or} \quad \mu(h^{-1}([a, \infty)) \cap W) = 0.$$

Let $\mathcal{Y} = \mathcal{M}_+(X) \setminus \mathcal{X}$. Then $\mathcal{Y} \neq \emptyset$. Indeed, if it were not true, we could find a countable family $\{I_{n_k}\}_{k \in \mathbb{N}} \subset \mathcal{F}(X)$ such that $\mu(I_{n_k} \cap h^{-1}((-\infty, a))) = 0$

for each $k \in \mathbb{N}$. Then we would have $\mu(h^{-1}((-\infty, a))) = 0$, in contradiction with $h^{-1}((-\infty, a)) \notin \mathcal{M}(X)$.

Of course

$$\mu^*(h^{-1}((-\infty, a)) \cap P) > 0 \quad \text{and} \quad \mu^*(h^{-1}([a, \infty)) \cap P) > 0$$

for each set $P \in \mathcal{Y}$. Let $T = \bigcup_{A \in \mathcal{Y}} A$. Note that there is $n \in \mathbb{N}$ such that

$$\mu^*\left(h^{-1}\left(\left(-\infty, a - \frac{1}{n}\right)\right) \cap T\right) > 0.$$

Let $M \in \mathcal{M}_+(X)$ be a set such that

$$(9) \qquad P \cap h^{-1}\left(\left(-\infty, a - \frac{1}{n}\right)\right) \neq \emptyset \quad \text{and} \quad P \cap h^{-1}([a, \infty)) \neq \emptyset$$

for each set $P \in \mathcal{M}_+(X)$ with $P \subset M$.

Let $A = M \cap h^{-1}((-\infty, a - \frac{1}{n}))$ and $x \in A$. Then $f(x, g(x)) < a - \frac{1}{n}$. By (i) and (ii), for the point $(x, g(x))$ there is a set $A(x, g(x)) \subset Y$ including $g(x)$ such that

(10) $D_l^*(A(x, g(x)), g(x)) > 1 - \alpha$, and

(11) the restriction $f_x|_{A(x,g(x))}$ is upper semicontinuous at $g(x)$, i.e. there is $O(g(x)) \in \mathcal{B}(g(x))$ such that

$$f(x, y) < a - \frac{1}{n} \quad \text{for} \quad y \in O(g(x)) \cap A(x, g(x)).$$

By (10), there is $U(x, g(x)) \in \mathcal{F}(Y)$ with $g(x) \in \operatorname{Int}(U(x, g(x)))$ such that

(12) $\nu^*(A(x, g(x)) \cap K) > (1 - \alpha + \beta(x))\nu(K)$ for each $K \in \mathcal{F}(Y)$ with $g(x) \in \operatorname{Int}(K) \subset K \subset U(x, g(x))$, where $\beta(x) > 0$.

Since the family $\mathcal{F}(Y)$ is countable, it follows that there is $U \in \mathcal{F}(Y)$ such that the set

$$B = \{x \in A : U(x, g(x)) = U\}$$

is of positive outer measure μ^*.

Let $C = M \cap g^{-1}(\operatorname{Int}(U))$. By $\mathcal{M}(X)$-measurability of g, $C \in \mathcal{M}(X)$. Furthermore $C \subset M$ and $\mu(C) > 0$. Thus, by (9) we have

$$C \cap h^{-1}([a, \infty)) \neq \emptyset.$$

Let $D = C \cap h^{-1}([a, \infty))$ and $x' \in D$. Then $f(x', g(x')) \geq a$. Similarly to that above, there is $U' \in \mathcal{F}(Y)$ such that $U' \subset U$ and the set

$$E = \{x \in D : U(x, g(x)) = U'\}$$

is of positive outer measure μ^*.

Let $x_0 \in E$ and $D(M, x_0) = 1$. Then $f(x_0, g(x_0)) \geq a > a - \frac{1}{2n}$. For the point $(x_0, g(x_0))$, by (iii) and (iv), there is a set $B(x_0, g(x_0)) \in \mathcal{M}(X) \otimes \mathcal{M}(Y)$ including $(x_0, g(x_0))$ such that

(13) $D_l(B(x_0, g(x_0)), (x_0, g(x_0))) > \alpha$, and

(14) $f|_{B(x_0,g(x_0))}$ is lower semicontinuous at $(x_0, g(x_0))$, i.e. there is a set $O(x_0, g(x_0)) \in \mathcal{B}(x_0, g(x_0))$ such that

$$f(x,y) > a - \frac{1}{2n} \quad \text{for} \quad (x,y) \in B(x_0, g(x_0)) \cap O(x_0, g(x_0)).$$

By (13), there are the sets $I \in \mathcal{F}(X)$ and $J \in \mathcal{F}(Y)$ such that $x_0 \in \mathrm{Int}(I)$, $g(x_0) \in \mathrm{Int}(J) \subset J \subset U'$ and

(15) $\quad \mu \times \nu(B(x_0, g(x_0)) \cap (I \times J)) > (\alpha + \gamma(x_0))\mu(I)\nu(J)$, where $\gamma(x_0) > 0$.

Moreover, x_0 is a density point of M. Hence, we can select the set I in such a way that $\mu(M \cap I) > (1-\eta)\mu(I)$, $\eta > 0$, and $x_0 \in \mathrm{Int}(I)$. Then, by (9), $M \cap I \cap h^{-1}((-\infty, a - \frac{1}{n})) \neq \emptyset$. Let $x_1 \in A \cap I$. By (12) and (15), there is $y_1 \in Y$ such that $(x_1, y_1) \in [A \times (A(x_1, g(x_1)) \cap O(g(x_1)))] \cap B(x_0, g(x_0)) \cap O(x_0, g(x_0)) \cap (I \times J)$. Then, by (11), $f(x_1, y_1) < a - \frac{1}{n}$, which contradicts (14) and the proof of Theorem 1 is finished. ∎

Now we can prove the main result of this section. We suppose that (Y, ρ) is a Polish space and (Z, d) is a separable metric space.

Theorem 2. *Let $\alpha \in (0,1)$. If $F : X \times Y \rightsquigarrow Z$ is a closed valued multifunction such that for each $(x,y) \in X \times Y$, F has the upper D_α property with respect to $\mathcal{F}(X) \times \mathcal{F}(Y)$ at (x,y) and F_x has the lower D_α property with respect to $\mathcal{F}(Y)$ at y, then F is $\mathcal{M}(X)$-sup-measurable.*

Proof. Let $(x,y) \in X \times Y$, $z \in Z$ and let $g_z(x,y) = d(z, F(x,y))$. By the upper D_α property of F at (x,y), there is a set $B(x,y) \in \mathcal{M}(X) \otimes \mathcal{M}(Y)$ including (x,y) such that $D_l(B(x,y), (x,y)) > \alpha$ and the restriction $F|_{B(x,y)}$ is h-upper semicontinuous at (x,y). Then, by (3),

(16) $D_l(B(x,y), (x,y)) > \alpha$ and the function $g_z|_{B(x,y)}$ is lower semicontinuos at (x,y).

By the lower D_α property of F_x with respect to $\mathcal{F}(Y)$ at y, there is an $A(x,y) \in \mathcal{M}(Y)$ including y such that $D_l(A(x,y), y) > 1 - \alpha$ and $F_x|_{A(x,y)}$ is h-lower semicontinuous at y. So that, by (4),

(17) $D_l(A(x,y), y) > 1 - \alpha$ and the function $(g_z)_x|_{A(x,y)}$ is upper semicontinuous at y.

Thus, by (16), (17) and Theorem 1, the function g_z is $\mathcal{M}(X)$-sup-measurable. Therefore the function $h : X \to \mathbb{R}$, given by $h(x) = g_z(x, g(x))$, is $\mathcal{M}(X)$-measurable for every $\mathcal{M}(X)$-measurable function $g : X \to Y$. Thus, by (5)(i),

(18) a multifunction $\Phi : X \rightsquigarrow Z$ given by $\Phi(x) = F(x, g(x))$ is $\mathcal{M}(X)$-measurable for every $\mathcal{M}(X)$-measurable function $g : X \to Y$.

Let $G : X \rightsquigarrow Y$ be an $\mathcal{M}(X)$-measurable multifunction with closed values. The task is now to show that the multifunction $H : X \rightsquigarrow Z$ given by $H(x) = F(x, G(x))$ is $\mathcal{M}(X)$-measurable.

By (5)(ii), for the multifunction G we can select a sequence $(g_n)_{n\in\mathbb{N}}$ of $\mathcal{M}(X)$-measurable functions $g_n : X \to Y$ such that $G(x) = \mathrm{Cl}(\{g_n(x)\}_{n\in\mathbb{N}})$ for any $x \in X$.

Let $U \subset Z$ be an open set and $n \in \mathbb{N}$. Let us define

$$B_n = \{x \in X : F(x, g_n(x)) \cap U \neq \emptyset\}.$$

Since all functions g_n are $\mathcal{M}(X)$-measurable, by (18) we have

$$B_n \in \mathcal{M}(X) \quad \text{for any } n \in \mathbb{N}. \tag{19}$$

Furthermore $F_x|_{A(x,y)}$ is lower semicontinuous at y and $D(A(x,y), y) > 1-\alpha$. Therefore, we have

$$\begin{aligned}\{x \in X : \mathrm{Cl}(\{g_n(x)\}_{n\in\mathbb{N}}) \cap F_x^-(U) \neq \emptyset\}&\\ = \{x \in X : \{g_n(x)\}_{n\in\mathbb{N}} \cap F_x^-(U) \neq \emptyset\},&\end{aligned}$$

(see (2)). Observe that

$$\begin{aligned} H^-(U) &= \{x \in X : F(x, G(x)) \cap U \neq \emptyset\}\\ &= \{x \in X : \bigcup_{y\in G(x)} F(x,y) \cap U \neq \emptyset\}\\ &= \Big\{x \in X : \exists y \in G(x) \wedge F(x,y) \cap U \neq \emptyset\Big\}\\ &= \{x \in X : G(x) \cap F_x^-(U) \neq \emptyset\}\\ &= \{x \in X : \mathrm{Cl}(\{g_n(x)\}_{n\in\mathbb{N}}) \cap F_x^-(U) \neq \emptyset\}\\ &= \{x \in X : \{g_n(x)\}_{n\in\mathbb{N}} \cap F_x^-(U) \neq \emptyset\}.\end{aligned}$$

Therefore,

$$H^-(U) = \bigcup_{n\in\mathbb{N}} \{x \in X : F(x, g_n(x)) \cap U \neq \emptyset\} = \bigcup_{n\in\mathbb{N}} B_n,$$

and, by (19), $H^-(U) \in \mathcal{M}(X)$, which finishes the proof of Theorem 2. ∎

4. An existence theorem

From now on we suppose that $I = [a, b] \subset \mathbb{R}$ and $Y = \mathbb{R}^k$. Let $L_1(I, \mu, Y)$ be the space of all μ integrable functions $\phi : I \to Y$. Let $\Phi : I \rightsquigarrow Y$ be a multifunction and let

$$\mathcal{F}_I(\Phi) = \{\phi \in L_1(I, \mu, Y) : \phi(x) \in \Phi(x) \text{ a.e. in } I\}.$$

The set

$$\int_I \Phi(x)\,dx = \Big\{\int_I \phi(x)\,dx : \phi \in \mathcal{F}_I(\Phi)\Big\}$$

is called the *Aumann integral* of Φ (briefly, the *integral* of Φ). Φ is called *integrable* if $\int_I \Phi(x)\,dx \neq \emptyset$ [3].

A multifunction $\Phi : I \rightsquigarrow Y$ is called *integrably bounded* if there exists a Lebesgue integrable function $g : I \to \mathbb{R}$ such that $||y|| \leq g(x)$ for all $x \in I$ and y such that $y \in \Phi(x)$.

If Φ is measurable and closed valued, then, by (5)(ii), we can select a measurable selection of Φ. Thus

(20) If a closed valued multifunction $\Phi : I \rightsquigarrow Y$ is measurable and integrably bounded, then Φ is integrable.

Let $\mathcal{S}_I(\Phi) = \{\psi_\phi : \phi \in \mathcal{F}_I(\Phi)\}$, where $\psi_\phi(x) = \int_a^x \phi(t)\,dt$ for $x \in I$. The set $\mathcal{S}_I(\Phi)$ may be considered as a subset of the metric space of all absolutely continuous functions $f : I \to Y$ vanishing at the left point of I with the norm $< f >= \int_I ||f'(t)||\,dt$. Note that $\mathcal{S}_I(\Phi)$ is an equicontinuous set. If Φ is integrably bounded, then $\mathcal{S}_I(\Phi)$ is uniformly bounded.

Let $\mathcal{I}$ be a set of indices. A family $\{\Phi_i\}_{i\in\mathcal{I}}$ of multifunctions from I to Y is called *uniformly integrably bounded* if there is a Lebesgue integrable function $g : I \to \mathbb{R}$ such that $||y|| \leq g(x)$ for all $x \in I$, $i \in \mathcal{I}$ and y such that $y \in \Phi_i(x)$.

Let ϱ_H be the Hausdorff metric in the space of nonempty compact subsets of the space $\mathcal{C}(I,Y)$ of continuous functions $f : I \to Y$, generated by the supremum norm in $\mathcal{C}(I,Y)$. Then the following is true (see [4, Th. 3.2]):

(21) Let $\Phi, \Phi_n : I \rightsquigarrow Y$, $n \in \mathbb{N}$, be compact and convex valued multifunctions with $\lim_{n\to\infty} d_H(\Phi_n(x), \Phi(x)) = 0$ for $x \in I$. If $\{\Phi_n\}_{n\in\mathbb{N}}$ is uniformly integrably bounded and all multifunctions Φ_n are measurable, then the sets $\mathcal{S}_I(\Phi_n)$ and $\mathcal{S}_I(\Phi)$ are compact and convex in $\mathcal{C}(I,Y)$, and $\lim_{n\to\infty} \varrho_H(\mathcal{S}_I(\Phi_n), \mathcal{S}_I(\Phi)) = 0$.

Theorem 3. *Let $(x_0, y_0) \in I \times Y$ and $\alpha \in (0,1)$. Let $F : I \times Y \rightsquigarrow Y$ be a compact and convex valued multifunction which has the upper D_α property at (x,y) and F_x is h-continuous at y for each $(x,y) \in I \times Y$. If the family $\{G_f\}_{f\in\mathcal{C}(I,Y)}$ is uniformly integrably bounded, where G_f is the Carathéodory superposition of F and f, then there exists an absolutely continuous function $f : [x_0, b] \to Y$ such that*

$$f(x) = y_0 + \int_{x_0}^{x} \phi(t)\,dt,$$

where ϕ is an integrable selection of G_f.

Proof. Let $\eta > 0$ be such that $[x_0, x_0 + \eta] \subset I$ and let g be a uniform integrable bound of $\{G_f\}_{f\in\mathcal{C}(I,Y)}$. Idea of the proof is based on that of Hartman [11, Th. 2.1]. First let us take a multifunction $G_{y_0}(x) = F(x, y_0)$

for $x \in [x_0, x_0 + \eta]$. Then G_{y_0} is measurable, by Theorem 2. It is also integrably bounded. Thus, by (20), G_{y_0} is integrable. Let ϕ_1 be an itegrable selection of G_{y_0} in $[x_0, x_0+\eta]$, and let $\psi_1(x) = \int_{x_0}^{x} \phi_1(t)\,dt$ for $x \in [x_0, x_0+\eta]$. Then $\psi_1 \in \mathcal{S}_{[x_0,x_0+\eta]}(G_{y_0})$. Let us put $f_\eta(x) = y_0 + \psi_1(x)$, i.e.

$$f_\eta(x) = y_0 + \int_{x_0}^{x} \phi_1(t)\,dt \quad \text{for} \quad x \in [x_0, x_0 + \eta].$$

Then $f_\eta \in \{y_0\} + \mathcal{S}_{[x_0,x_0+\eta]}(G_{y_0})$. Moreover, by integrable boundness of G_{y_0},

$$||f_\eta(x)|| \le ||y_0|| + \int_{x_0}^{x} g(t)\,dt \qquad \text{for } x \in [x_0, x_0 + \eta] \quad \text{and} \tag{22}$$

$$||f_\eta(x_1) - f_\eta(x_2)|| \le \left|\int_{x_1}^{x_2} g(t)\,dt\right| \qquad \text{for } x_1, x_2 \in [x_0, x_0 + \eta].$$

If $x_0 + 2\eta < b$, then we put

$$G_{f_\eta}(x) = \begin{cases} F(x, y_0), & \text{if } x \in [x_0, x_0 + \eta], \\ F(x, f_\eta(x - \eta)), & \text{if } x \in (x_0 + \eta, x_0 + 2\eta]. \end{cases}$$

By the continuity of f_η and Theorem 2, G_{f_η} is measurable. It is also integrably bounded. Hence, by (20), G_{f_η} is integrable. Let ϕ_2 be an integrable selection of G_{f_η}, and let $\psi_2(x) = \int_{x_0+\eta}^{x} \phi_2(t)\,dt$ for $x \in [x_0 + \eta, x_0 + 2\eta]$. Then $\psi_2 \in \mathcal{S}_{[x_0+\eta,x_0+2\eta]}F(\cdot, f_\eta(\cdot - \eta))$. We can extend f_η to the interval $[x_0 + \eta, x_0 + 2\eta]$ putting $f_\eta(x) = f_\eta(x_0 + \eta) + \psi_2(x)$ for $x \in [x_0 + \eta, x_0 + 2\eta]$ and we have

$$f_\eta(x) = \begin{cases} y_0 + \int_{x_0}^{x} \phi_1(t)\,dt, & \text{if } x \in [x_0, x_0 + \eta], \\ f_\eta(x_0 + \eta) + \int_{x_0+\eta}^{x} \phi_2(t)\,dt, & \text{if } x \in (x_0 + \eta, x_0 + 2\eta], \end{cases}$$

and $f_\eta \in \{y_0\} + \mathcal{S}_{[x_0,x_0+2\eta]}(G_{f_\eta})$.

Note that the extended function f_η fulfil (22) for $x \in [x_0, x_0 + 2\eta]$.

If $x_0 + 3\eta < b$, the process can be continued. Finally at most in a finite meny steps f_η can be extended to $[x_0, b]$ such that $f_\eta \in \{y_0\} + \mathcal{S}_{[x_0,b]}(G_{f_\eta})$, where $G_{f_\eta} : [x_0, b] \rightsquigarrow Y$ is given by

$$G_{f_\eta}(x) = \begin{cases} F(x, y_0), & \text{if } x \in [x_0, x_0 + \eta], \\ F(x, f_\eta(x - \eta)), & \text{if } x \in (x_0 + \eta, b], \end{cases}$$

and (22) holds true for $x \in [x_0, b]$.

Now let $(\eta_n)_{n\in\mathbb{N}}$ be a decreasing sequence of numbers from $[x_0, b]$ converging to 0. Then

(23) the family $\{G_{f_{\eta_n}}\}_{n\in\mathbb{N}}$ is uniformly integrably bounded on $[x_0, b]$ and $G_{f_{\eta_n}}$ is measurable for each $n \in \mathbb{N}$.

Moreover,

$$\{f_{\eta_n}\}_{n\in\mathbb{N}} \subset \{y_0\} + \mathcal{S}_{[x_0,b]}(G_{f_{\eta_n}}) \quad \text{and} \quad \{f_{\eta_n}\}_{n\in\mathbb{N}}$$

is uniformly bounded and equicontinuous. By the Arzelà-Ascoli theorem, $(f_{\eta_n})_{n\in\mathbb{N}}$ contains a subsequence (let us assume that it is the original) converging uniformly on $[x_0, b]$ to a continuous limit function f.

Note that

$$\begin{aligned}||f(x_1) - f(x_2)|| &\le ||f(x_1) - f_{\eta_n}(x_1)|| + ||f(x_2) - f_{\eta_n}(x_2)|| \\ &\quad + ||f_{\eta_n}(x_1) - f_{\eta_n}(x_2)|| \\ &< \varepsilon + \Big|\int_{x_1}^{x_2} g(x)\,dx\Big|\end{aligned}$$

for $\varepsilon > 0$, $x_1, x_2 \in [x_0, b]$ and n sufficiently large. Hence, f is absolutely continuous on $[x_0, b]$.

By equicontinuity of $\{f_{\eta_n}\}_{n\in\mathbb{N}}$, $\lim_{n\to\infty} f_{\eta_n}(x-\eta_n) = f(x)$ for $x \in [x_0, b]$. Thereby, h-continuity of $F(x,\cdot)$ implies

$$\lim_{n\to\infty} d_H(G_{f_{\eta_n}}(x), G_f(x)) = 0 \quad \text{for } x \in [x_0, b], \tag{24}$$

where $G_f(x) = F(x, f(x))$ for $x \in [x_0, b]$. Then by (23), (24) and (21), $\mathcal{S}_{[x_0,b]}(G_f)$ is a compact subset of $\mathcal{C}([x_0, b], Y)$ and

$$\lim_{n\to\infty} \varrho_H(\mathcal{S}_{[x_0,b]}(G_{f_{\eta_n}}), \mathcal{S}_{[x_0,b]}(G_f)) = 0.$$

Therefore, $f \in \{y_0\} + \mathcal{S}_{[x_0,b]}(G_f)$. Moreover, f is absolutely continuous. Thus, the proof of Theorem 3 is finished. ∎

Corollary 1. *Let $(x_0, y_0) \in I \times Y$ and $\alpha \in (0, 1)$. Let $F : I \times Y \rightsquigarrow Y$ be a compact and convex valued multifunction with the upper D_α property and such that F_x is h-continuous for each $x \in I$. If the family $\{G_f\}_{f\in\mathcal{C}(I,Y)}$ is uniformly integrably bounded, where G_f is the Carathéodory superposition of F and f, then there is a solution of* (1).

References

[1] J. Appel, E. De Pascale, N. H. Thái, P. P. Zabrejko, *Multivalued superpositions*, Dissertationes Math. 345 (1995).

[2] J. P. Aubin, A. Cellina, *Differential Inclusions. Set-Valued Maps and Viability Theory*, Springer, Berlin, 1984.

[3] R. J. Aumann, *Integrals of set-valued functions*, J. Math. Anal. Appl. 12 (1965), 1–12.

[4] F. T. Bridgland, *Trajectory integrals of set valued functions*, Pacific J. Math. 33 (1970), 43–68.

[5] A. M. Bruckner, *Differentiation of integrals*, Amer. Math. Monthly 78(9) (1971), P II, 1–54.

[6] C. Carathéodory, *Vorlesungen über Reelle Funktionen*, Leipzig–Berlin, 1927.
[7] C. Castaing, M. Valadier, *Convex Analysis and Mesurable Multifunctions*, Lecture Notes in Mathematics 580, Springer–Verlag, Berlin–New York, 1977.
[8] K. Deimling, *Multivalued Differential Equations*, de Gruyter Ser. Nonlinear Anal. Appl. 1, de Gruyter, Berlin, 1992.
[9] A. F. Filippov, *The existence of solutions of generalized differential equations*, Math. Notes 10 (1971), 608–611.
[10] Z. Grande, *La mesurabilité des fonctions de deux variables et de la superposition* $F(x, f(x))$, Dissertationes Math. 159 (1978), 1–49.
[11] P. Hartman, *Ordinary Differential Equations*, J. Willey Sons, New York, 1964.
[12] W. Hildenbrand, *Core and Equilibria of a Large Economy*, with an appendix to Chapter 2 by K. Hildenbrand, Princeton Stud. Math. Economics, Princeton Univ. Press, Princeton, NJ, 1974.
[13] S. Hu, N. S. Papageorgiou, *Handbook of Multivalued Analysis*, vol. I *Theory*, *Mathematics and its Applications*, 419, Kluwer Academic Publishers, Dordrecht, 1997.
[14] A. Nowak, *Random differential inclusions: measurable selection approach*, Ann. Polon. Math. 49 (1989), 291–296.
[15] H. Kaczyński, C. Olech, *Existence of solutions of orientor fields with nonconvex right-hand side*, Ann. Polon Math. 29 (1974), 61–66.
[16] T. Ważewski, *On an optimal control problem*, in: *Differential Equations and their applications*, Proc. Conf., Prague 1962, 229–242.

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