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SUBALGEBRA LATTICES OF A PARTIAL UNARY ALGEBRA

Abstract. Necessary and sufficient conditions will be found for quadruples of lattices to be isomorphic to lattices of weak, relative, strong subalgebras and initial segments, respectively, of one partial unary algebra. To this purpose we will start with a characterization of pairs of lattices that are weak and strong subalgebra lattices of one partial unary algebra, respectively. Next, we will describe the initial segment lattice of a partial unary algebra. Applying this result, pairs of lattices of strong subalgebras and initial segments will be characterized. Further, we will characterize pairs of lattices of relative and strong subalgebras and also other pairs of subalgebra lattices of one partial unary algebra.

1.

The subalgebra lattice and its relations with the algebra itself are quite important both in universal algebra and in classical algebra. For instance, many results describe algebras or varieties of algebras with special subalgebra lattices, e.g., distributive, modular, etc. (see e.g., [7], [20], [21]). Some papers investigate subalgebra lattices for algebras which belong to a given variety or a given type (see e.g., [10]). Note also important results on connections between groups and their subgroup lattices (see [19]).

Theory of partial algebras provides additional tools for such investigations, because at least four different structures may be considered in this case (see e.g., [4] or [5] or [11]). In this paper we consider the following four kinds of subalgebras: weak, relative, strong subalgebras and initial segments. The ordinary subalgebras will be called strong here to distinguish them from other kinds of partial subalgebras. Consequently, we have four kinds of subalgebra lattices for a given partial algebra. It seems that these

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four structures yield a lot of interesting information on an algebra, also total. For instance, it is proved in [13] that for a locally finite total unary algebra of finite type, its weak subalgebra lattice uniquely determines its strong subalgebra lattice. This result has been generalized in [15] for some kind of non-unary algebras.

In [11] we have formulated some useful connections between partial unary algebras and graphs. For example, using this graph-algebraic language we have found in [12] necessary and sufficient conditions for an unary algebra $\mathbf{A}$ and a lattice $\mathbf{L}$ such that the strong subalgebra lattice of $\mathbf{A}$ is isomorphic to $\mathbf{L}$. In the same paper, necessary and sufficient conditions are also found for unary algebras to have isomorphic strong subalgebra lattices. Analogous two results for weak subalgebra lattices are solved in [11].

In the present paper we use results from [11] and [12] to describe connections between four kinds of subalgebra lattices of one partial unary algebra. More precisely, we first characterize pairs of lattices that are weak and strong subalgebra lattices of one partial unary algebra, respectively. To this purpose we generalize notions from [16], where such problem is investigated, but only for some special classes of lattices and unary algebras. Secondly, we show that for a given partial unary algebra, its initial segment lattice is the dual of its strong subalgebra lattice. Applying this result we characterize the initial segment lattice and also we are able to describe pairs of lattices of strong subalgebras and initial segments of one partial unary algebra. Thirdly, we characterize pairs of lattices to be isomorphic to lattices of relative and strong subalgebras of one partial unary algebra, respectively. We solve also analogous problems for other pairs of subalgebra lattices. Fourthly, having all these results, necessary and sufficient conditions are found for quadruples of lattices to be isomorphic to weak, relative, strong subalgebra and initial segment lattices, respectively, of one partial unary algebra.

At the end of this paper we will show that results from [12] may be translated into the initial segment lattice of a partial unary algebra. In particular, we first obtain necessary and sufficient conditions for a partial unary algebra $\mathbf{A}$ and a lattice $\mathbf{L}$ such that the initial segment lattice of $\mathbf{A}$ is isomorphic to $\mathbf{L}$. Secondly, we describe pairs of unary algebras with isomorphic initial segment lattices.

2.

A partial unary algebra is a pair $\mathbf{A} = \langle A, (k^{\mathbf{A}})_{k\in K}\rangle$, where A is the carrier of $\mathbf{A}$, K is the set of unary operation symbols of $\mathbf{A}$ and for each $k \in K$, $k^{\mathbf{A}}$ is a partial unary operation in A, i.e., $k^{\mathbf{A}}$ is defined on a subset of A. The set K will be called a type of $\mathbf{A}$ here. For details about the theory of partial algebras see e.g., [4], [5].

In the present paper we consider the following four kinds of partial subalgebras

Definition 2.1. Let $\mathbf{A} = \langle A, (k^{\mathbf{A}})_{k\in K}\rangle$ and $\mathbf{B} = \langle B, (k^{\mathbf{B}})_{k\in K}\rangle$ be partial unary algebras of type K. Then

(a) $\mathbf{B}$ is a weak subalgebra of $\mathbf{A}$ ($\mathbf{B} \leq_w \mathbf{A}$) if $B \subseteq A$ and $k^{\mathbf{B}} \subseteq k^{\mathbf{A}}$ for all $k \in K$.
(b) $\mathbf{B}$ is a relative subalgebra of $\mathbf{A}$ ($\mathbf{B} \leq_r \mathbf{A}$) if $B \subseteq A$ and $k^{\mathbf{B}} = k^{\mathbf{A}} \cap (B \times B)$ for all $k \in K$.
(c) $\mathbf{B}$ is a strong subalgebra of $\mathbf{A}$ ($\mathbf{B} \leq_s \mathbf{A}$) if $B \subseteq A$ and $k^{\mathbf{B}} = k^{\mathbf{A}} \cap (B \times A)$ for all $k \in K$.
(d) $\mathbf{B}$ is an initial segment of $\mathbf{A}$ ($\mathbf{B} \leq_i \mathbf{A}$) if $B \subseteq A$ and $k^{\mathbf{B}} = k^{\mathbf{A}} \cap (A \times B)$ for all $k \in K$.

The notion of strong subalgebra is a direct translation of the classical definition of subalgebra into the partial case. But other three kinds of subalgebras are strictly connected with partiality.

Similarly as in the total case, it is easy to show (see e.g., [4] for details) that families $S_w(\mathbf{A})$, $S_r(\mathbf{A})$, $S_s(\mathbf{A})$ and $S_i(\mathbf{A})$ of all the weak, relative, strong subalgebras and initial segments of $\mathbf{A}$ with subalgebra inclusions $\leq_w$, $\leq_r$, $\leq_s$ and $\leq_i$, respectively, form complete lattices $\mathbf{S}_w(\mathbf{A}) = \langle S_w(\mathbf{A}), \leq_w\rangle$, $\mathbf{S}_r(\mathbf{A}) = \langle S_r(\mathbf{A}), \leq_r\rangle$, $\mathbf{S}_s(\mathbf{A}) = \langle S_s(\mathbf{A}), \leq_s\rangle$ and $\mathbf{S}_i(\mathbf{A}) = \langle S_i(\mathbf{A}), \leq_i\rangle$.

Let us now recall also some notions from lattice theory. First (see [6] or [10]), a non-zero element l of a complete lattice $\mathbf{L} = \langle L, \leq_{\mathbf{L}}\rangle$ is completely join-irreducible if for each subset $K \subseteq L$, $l = \bigvee K$ implies $l \in K$ (where $\bigvee X$ denotes the supremum of a set X in $\mathbf{L}$). It is join-irreducible if the condition is satisfied for all two-element subsets. The set of all the completely join-irreducible elements of $\mathbf{L}$ will be denoted here by $\mathcal{CI}(\mathbf{L})$.

Secondly, if $\mathbf{L} = \langle L, \leq_{\mathbf{L}}\rangle$ has the least element 0, then its element a is called an atom if for all $l \in L$, $0 \leq_{\mathbf{L}} l \leq_{\mathbf{L}} a$ implies that $l = 0$ or $l = a$. The set of all the atoms of $\mathbf{L}$ will be denoted here by $\mathcal{A}(\mathbf{L})$.

Thirdly, a complete lattice $\mathcal{L}$ is algebraic if each of its elements is a join of compact elements. An element $c \in L$ is compact if for each set $S \subseteq L$, $c \leq \bigvee S$ implies $c \leq \bigvee S_0$ for some finite subset $S_0 \subseteq S$.

The following well-known result (see [10] or [9]) describe the strong subalgebra lattice

Theorem 2.2. *A lattice* $\mathbf{L}$ *is isomorphic to the strong subalgebra lattice of some (partial or total) unary algebra if and only if* $\mathbf{L}$ *is algebraic, distributive and*

(S) *every element of* $\mathbf{L}$ *is a join of completely join-irreducible elements.*

Note that this result is proved for total unary algebras, but it is also true in the partial case. Formally, a partial unary algebra $\mathbf{A} = \langle A, (k^{\mathbf{A}})_{k\in K}\rangle$ of type K can be completed to a total algebra $\mathbf{B}$ of the same type K by setting $k^{\mathbf{A}}(a) = a$ if $k^{\mathbf{A}}$ is not defined on a, for each $k \in K$ and $a \in A$. This construction preserves relative, strong subalgebras and initial segments. In particular, $\mathbf{A}$ and $\mathbf{B}$ have the same lattices of relative, strong subalgebras and initial segments, respectively.

Further, [1] gives the following complete characterization of the weak subalgebra lattice

Theorem 2.3. *A lattice $\mathbf{L} = \langle L, \leq_{\mathbf{L}}\rangle$ is isomorphic to the weak subalgebra lattice of a partial unary algebra if and only if $\mathbf{L}$ is algebraic, distributive and*

(W.1) *each element of $\mathbf{L}$ is a join of join-irreducible elements,*
(W.2) *each non-zero and non-atomic join-irreducible element contains exactly one or exactly two atoms,*
(W.3) *the set of all non-zero and non-atomic join-irreducible elements is an antichain with respect to the lattice order $\leq_{\mathbf{L}}$.*

It is well-known (see [6]) that a lattice $\mathbf{L}$ is isomorphic to the lattice of all the subsets of a set if and only if $\mathbf{L}$ is a complete and atomic Boolean lattice. Moreover, a complete and atomic Boolean lattice $\mathbf{L}$ is isomorphic to the lattice of all the subsets of the set $\mathcal{A}(\mathbf{L})$ of all the atoms of $\mathbf{L}$.

It is also well-known (see e.g., [4]) that the relative subalgebra lattice of any partial algebra $\mathbf{A} = \langle A, (k^{\mathbf{A}})_{k\in K}\rangle$ of an arbitrary (not only unary) type is isomorphic to the lattice of all the subsets of the set A. It follows from the obvious fact that each relative subalgebra of $\mathbf{A}$ is uniquely determined by its carrier, an arbitrary subset of A. Thus

Theorem 2.4. *A lattice $\mathbf{L}$ is isomorphic to the relative subalgebra lattice of a partial algebra if and only if $\mathbf{L}$ is a complete and atomic Boolean lattice.*

Lattice isomorphisms preserve atoms, so we obtain also (where $|X|$ denotes the cardinality of a set X)

Theorem 2.5. *Let $\mathbf{A}$ be a partial algebra and let $\mathbf{L}$ be a complete and atomic Boolean lattice. Then*

$$\mathbf{S}_r(\mathbf{A}) \simeq \mathbf{L} \quad \textit{iff} \quad |A| = |\mathcal{A}(\mathbf{L})|.$$

Because bijective sets have isomorphic powerset lattices.

We will also use definitions and facts from [11] and [12]. To simplify reading the paper we now shortly recall needed results.

Since we use digraphs and graphs to represent partial unary algebras, vertex and edge sets may have arbitrary cardinalities, and also multiple

edges and isolated vertices are admitted. Therefore a digraph $\mathbf{D}$ is represented here by a triple $\langle V^{\mathbf{D}}, E^{\mathbf{D}}, I^{\mathbf{D}}\rangle$, where $V^{\mathbf{D}}$, $E^{\mathbf{D}}$ are sets of vertices and edges, respectively, and $I^{\mathbf{D}} = \langle I_1^{\mathbf{D}}, I_2^{\mathbf{D}}\rangle$ is the incidence mapping from $E^{\mathbf{D}}$ to $V^{\mathbf{D}} \times V^{\mathbf{D}}$. Analogously, a graph $\mathbf{G}$ is a triple $\langle V^{\mathbf{G}}, E^{\mathbf{G}}, I^{\mathbf{G}}\rangle$, where $V^{\mathbf{G}}$, $E^{\mathbf{G}}$ are sets of vertices and edges, respectively, and $I^{\mathbf{G}}$ is the incidence mapping from $E^{\mathbf{G}}$ to the family of all the one- and two-element subsets of $V^{\mathbf{G}}$.

For a digraph $\mathbf{D}$ let $\mathbf{D}^*$ be a graph obtained from $\mathbf{D}$ by omitting the orientation of all the edges, i.e.,

$$V^{\mathbf{D}^*} = V^{\mathbf{D}},\ E^{\mathbf{D}^*} = E^{\mathbf{D}} \text{ and } I^{\mathbf{D}^*}(e) = \{I_1^{\mathbf{D}}(e), I_2^{\mathbf{D}}(e)\} \text{ for each } e \in E^{\mathbf{D}^*}.$$

Next, unary algebras are represented by digraphs in the following natural way (see [11])

Definition 2.6. Let $\mathbf{A} = \langle A, (k^{\mathbf{A}})_{k\in K}\rangle$ be a partial unary algebra. Then $\mathbf{D}(\mathbf{A})$ is the digraph such that

$$V^{\mathbf{D}(\mathbf{A})} = A, \quad E^{\mathbf{D}(\mathbf{A})} = \{\langle a, k, b\rangle \colon \langle a, b\rangle \in k^{\mathbf{A}}\}$$

and for each $\langle a, k, b\rangle \in E^{\mathbf{D}(\mathbf{A})}$, $I^{\mathbf{D}(\mathbf{A})}(\langle a, k, b\rangle) = \langle a, b\rangle$.

Further, four kinds of subdigraphs are defined in [11], which correspond to weak, relative, strong subalgebras and initial segments. The ordinary subdigraphs will be called weak here as opposed to the other kinds of subdigraphs. More formally,

Definition 2.7. Let $\mathbf{D} = \langle V^{\mathbf{D}}, E^{\mathbf{D}}, I^{\mathbf{D}}\rangle$ and $\mathbf{H} = \langle V^{\mathbf{H}}, E^{\mathbf{H}}, I^{\mathbf{H}}\rangle$ be digraphs. Then

(a) $\mathbf{H}$ is a weak subdigraph of $\mathbf{D}$ (which will be denoted by $\mathbf{H} \leq_w \mathbf{D}$) if $V^{\mathbf{H}} \subseteq V^{\mathbf{D}}$, $E^{\mathbf{H}} \subseteq E^{\mathbf{D}}$ and $I^{\mathbf{H}}(e) = I^{\mathbf{D}}(e)$ for all $e \in E^{\mathbf{H}}$.

(b) A weak subdigraph $\mathbf{H}$ of $\mathbf{D}$ is said to be relative ($\mathbf{H} \leq_r \mathbf{D}$) if for each edge e, if all endpoints of e belong to $\mathbf{H}$, then $\mathbf{H}$ contains e (in other words, $I_1^{\mathbf{D}}(e), I_2^{\mathbf{D}}(e) \in V^{\mathbf{H}}$ implies that $e \in V^{\mathbf{H}}$).

(c) $\mathbf{H}$ is a strong subdigraph of $\mathbf{D}$ ($\mathbf{H} \leq_s \mathbf{D}$) if $\mathbf{H}$ is a relative subdigraph of $\mathbf{D}$ and for each edge e, $I_1^{\mathbf{D}}(e) \in V^{\mathbf{H}}$ implies that $I_2^{\mathbf{D}}(e) \in V^{\mathbf{H}}$ (so also $e \in E^{\mathbf{H}}$).

(d) $\mathbf{H}$ is a dually-strong subdigraph of $\mathbf{D}$ ($\mathbf{H} \leq_d \mathbf{D}$) if $\mathbf{H}$ is a relative subdigraph of $\mathbf{D}$ and for each edge e, $I_2^{\mathbf{D}}(e) \in V^{\mathbf{H}}$ implies that $I_1^{\mathbf{D}}(e) \in V^{\mathbf{H}}$ (so also $e \in E^{\mathbf{H}}$).

For graphs, we have two kinds of subgraphs, weak and relative, which are defined analogously.

For each subset W of vertices of a digraph (graph) $\mathbf{D}$, the exactly one relative subdigraph (subgraph) induced by W is denoted by $[W]_{\mathbf{D}}$, i.e., $[W]_{\mathbf{D}}$ consists of all the vertices from W and of all the edges with endpoints in W.

It is proved in [11] that sets of all the weak, relative, strong and dually-strong subdigraphs of a digraph $\mathbf{D}$ form complete lattices

$$\begin{aligned}\mathbf{S}_w(\mathbf{D}) &= \langle S_w(\mathbf{D}), \leq_w\rangle,\\ \mathbf{S}_r(\mathbf{D}) &= \langle S_r(\mathbf{D}), \leq_r\rangle,\\ \mathbf{S}_s(\mathbf{D}) &= \langle S_s(\mathbf{D}), \leq_s\rangle,\\ \mathbf{S}_d(\mathbf{D}) &= \langle S_d(\mathbf{D}), \leq_d\rangle,\end{aligned}$$

respectively, with (weak, relative, strong and dually-strong) inclusions $\leq_w$, $\leq_r$, $\leq_s$ and $\leq_d$.

Analogously, sets of all the weak and relative subgraphs of a graph $\mathbf{G}$ form complete lattices $\mathbf{S}_w(\mathbf{G}) = \langle S_w(\mathbf{G}), \leq_w\rangle$ and $\mathbf{S}_r(\mathbf{G}) = \langle S_r(\mathbf{G}), \leq_r\rangle$.

In the same paper, the following two simple results are given

Proposition 2.8. *For each digraph* $\mathbf{D}$ *there is a partial unary algebra* $\mathbf{A}$ *such that* $\mathbf{D}(\mathbf{A}) \simeq \mathbf{D}$.

Theorem 2.9. *For every partial unary algebra* $\mathbf{A} = \langle A, (k^{\mathbf{A}})_{k\in K}\rangle$,

$$\begin{aligned}\mathbf{S}_w(\mathbf{A}) &\simeq \mathbf{S}_w(\mathbf{D}(\mathbf{A})),\\ \mathbf{S}_r(\mathbf{A}) &\simeq \mathbf{S}_r(\mathbf{D}(\mathbf{A})),\\ \mathbf{S}_s(\mathbf{A}) &\simeq \mathbf{S}_s(\mathbf{D}(\mathbf{A})),\\ \mathbf{S}_i(\mathbf{A}) &\simeq \mathbf{S}_d(\mathbf{D}(\mathbf{A})).\end{aligned}$$

Theorem 2.9 is obtained in [11] by standard verification that functions which assign its digraph to each weak, relative, strong subalgebra or initial segment, respectively, are suitable isomorphisms.

Having the above two results we can translate Theorems 2.2, 2.3, 2.4 and 2.5 to obtain analogous characterizations of lattices of weak, relative, strong and dually-strong subdigraphs.

Recall (see [11]) that with an algebraic and distributive lattice satisfying **(W.1)**–**(W.3)** of Theorem 2.3 we can associate an (undirected) graph. Formally,

Definition 2.10. Let an algebraic and distributive lattice $\mathbf{L}$ satisfy **(W.1)**–**(W.3)** of Theorem 2.3. Then $\mathbf{G}(\mathbf{L})$ is the graph such that

(i) the vertex set $V^{\mathbf{G}(\mathbf{L})}$ consists of all the atoms of $\mathbf{L}$, i.e., $V^{\mathbf{G}(\mathbf{L})} = \mathcal{A}(\mathbf{L})$,
(ii) the edge set $E^{\mathbf{G}(\mathbf{L})}$ consists of all non-zero and non-atomic join-irreducible elements of $\mathbf{L}$,
(iii) for each edge $e \in E^{\mathbf{G}(\mathbf{L})}$, $I^{\mathbf{G}(\mathbf{L})}(e) = \{a \in V^{\mathbf{G}(\mathbf{L})} \colon a \leq_{\mathbf{L}} e\}$ (i.e., atoms contained in e are terminal vertices of e).

Then (see also [11])

Theorem 2.11. *Let* $\mathbf{D}$ *be a digraph, and* $\mathbf{L}$ *be an algebraic and distributive lattice satisfying* **(W.1)**–**(W.3)** *of Theorem* 2.3. *Then*

$$\mathbf{S}_w(\mathbf{D}) \simeq \mathbf{L} \quad \textit{iff} \quad \mathbf{D}^* \simeq \mathbf{G}(\mathbf{L}).$$

Hence, a partial unary algebra $\mathbf{A}$ has a weak subalgebra lattice isomorphic to $\mathbf{L}$ iff $\mathbf{D}(\mathbf{A})^* \simeq \mathbf{G}(\mathbf{L})$. But we do not need this fact in the sequel.

We will use slightly other notions of paths and cycles than usual (see e.g., [3]), because such definitions will be more useful to formulate and prove some results in the sequel.

A finite and non-empty sequence of pairwise distinct vertices $(v_1, v_2, \ldots, v_n)$ of a digraph $\mathbf{D}$ is said to be a directed path, if for each $i = 1, 2, \ldots, n-1$ there is an edge e_i such that $I_1^{\mathbf{D}}(e_i) = v_i$ and $I_2^{\mathbf{D}}(e_i) = v_{i+1}$.

A directed path $(v_1, v_2, \ldots, v_n)$ of $\mathbf{D}$ is called a directed cycle, if there is an edge e_n such that $I_1^{\mathbf{D}}(e_n) = v_n$ and $I_2^{\mathbf{D}}(e_n) = v_1$.

Undirected paths and undirected cycles in graphs are defined similarly (see also [3]). More precisely, a non-empty sequence of pairwise distinct vertices $(v_1, v_2, \ldots, v_n)$ of a graph $\mathbf{G}$ is an undirected path, if there are edges $e_1, e_2, \ldots, e_{n-1}$ such that $I^{\mathbf{G}}(e_i) = \{v_i, v_{i+1}\}$ for each $i = 1, 2, \ldots, n-1$.

A non-empty sequence of pairwise distinct vertices $(v_1, v_2, \ldots, v_n)$ of $\mathbf{G}$ is an undirected cycle, if there are edges $e_1, e_2, \ldots, e_n$ (for $n = 2$ we additionally assume that these edges are pairwise distinct) such that $I^{\mathbf{G}}(e_i) = \{v_i, v_{i+1}\}$ for each $i = 1, 2, \ldots, n$, where $v_{n+1} = v_1$.

Note that a vertex v forms a cycle if there is a loop in v. Recall that an edge e of a digraph (graph) $\mathbf{G}$ is a loop if $I_1^{\mathbf{G}}(e) = I_2^{\mathbf{G}}(e)$ ($I^{\mathbf{G}}(e)$ is a one-element set). A path (cycle) is non-trivial if contains at least two distinct vertices.

We say that an edge e of a digraph (graph) $\mathbf{G}$ lies on a directed (undirected) cycle $(v_1, v_2, \ldots, v_n)$ if $I_1^{\mathbf{G}}(e) = v_i$ and $I_2^{\mathbf{G}}(e) = v_{i+1}$ ($I^{\mathbf{G}}(e) = \{v_i, v_{i+1}\}$) for some $i = 1, 2, \ldots, n$, where $v_{n+1} = v_1$.

A digraph is strongly connected (see [3]) if for each two distinct vertices v and w, there is a directed path going from v to w (equivalently, there is a directed cycle containing v and w). Recall also that a graph is connected if each two distinct vertices are connected by an undirected path.

Theorem 2.12. (Robbins, H. E.) *All the edges of a graph* $\mathbf{G}$ *can be directed to form a strongly connected digraph if and only if* $\mathbf{G}$ *is connected and each of its edges lies on an undirected cycle.*

H. E. Robbins proved this theorem for finite case only (see [18] or [3], Chapter 9, Theorem 10). The proof of general case when vertex and edge sets

are of arbitrary cardinalities (which is needed in this paper) is technically more complicated and is given in [16].

We will need also the notion of quotient digraph and graph introduced in [12]).

Definition 2.13. Let $\mathbf{D}$ be a digraph (graph) and let θ be an equivalence relation on the vertex set $V^{\mathbf{D}}$. Then the quotient digraph (graph) $\mathbf{D}/\theta$ is defined in the following way:

$$V^{\mathbf{D}/\theta} = V^{\mathbf{D}}/\theta,$$

i.e., its vertex set is the family of all the equivalence classes of θ,

$$E^{\mathbf{D}/\theta} = E^{\mathbf{D}}$$

and for each edge $e \in E^{\mathbf{D}/\theta}$,

$$I_i^{\mathbf{D}/\theta}(e) = I_i^{\mathbf{D}}(e)/\theta \quad \text{for} \quad i = 1, 2, \qquad (I^{\mathbf{D}/\theta}(e) = I^{\mathbf{D}}(e)/\theta),$$

i.e., $I_i^{\mathbf{D}/\theta}(e)$ is the equivalence class of $I_i^{\mathbf{D}}(e)$ (i.e., $I^{\mathbf{D}/\theta}(e)$ is the family of equivalence classes of elements from $I^{\mathbf{D}}(e)$).

Let $\mathbf{D}$ be a digraph. Then we have (see [17]) the following reflexive and transitive relation $\leq_{\mathbf{D}}$ on the vertex set $V^{\mathbf{D}}$:

$v \leq_{\mathbf{D}} w$ if and only if $v = w$ or there is a directed path going from w to v.

Clearly this relation is antisymmetric if and only if $\mathbf{D}$ has no non-trivial directed cycles.

Having $\leq_{\mathbf{D}}$ we define the following special equivalence relation $\Theta(\mathbf{D})$ on $V^{\mathbf{D}}$: For all $v, w \in V^{\mathbf{D}}$, $v\Theta(\mathbf{D})w$ if and only if $v = w$ or there is a directed path going from v to w and there is a directed path going from w to v (equivalently, there is a directed cycle containing v, w).

It is easy to show (see [12]) that the quotient digraph $\mathbf{D}/\Theta(\mathbf{D})$ has no non-trivial directed cycles, so $\leq_{\mathbf{D}/\Theta(\mathbf{D})}$ is a partial order on $V^{\mathbf{D}/\theta} = V^{\mathbf{D}}/\theta$.

Further, the following result is proved in [12] (Theorem 3.9)

Theorem 2.14. *Let* $\mathbf{D}$ *be a digraph and let* $\mathbf{L} = \langle L, \leq_{\mathbf{L}} \rangle$ *be an algebraic and distributive lattice satisfying* **(S)** *of Theorem* 2.2. *Then*

$$\mathbf{S}_s(\mathbf{D}) \simeq \mathbf{L} \quad \text{iff} \quad \langle \mathcal{CI}(\mathbf{L}), \leq_{\mathbf{L}} \rangle \simeq \langle V^{\mathbf{D}/\Theta(\mathbf{D})}, \leq_{\mathbf{D}/\Theta(\mathbf{D})} \rangle,$$

recall that $\mathcal{CI}(\mathbf{L})$ *is the family of all the completely join-irreducible elements of* $\mathbf{L}$.

The analogous result can be formulated for partial unary algebras $\mathbf{A}$, too. It is sufficient to replace $\mathbf{D}$ by $\mathbf{D}(\mathbf{A})$. But we will not need this fact in the sequel.

3.

In this section we characterize pairs of weak and strong subalgebra lattices of one partial unary algebra. To this purpose we first prove the following result (which solve our problem for digraphs).

Theorem 3.1. *Let* $\mathbf{G}$ *be a graph and let* $\mathbf{L} = \langle L, \leq_{\mathbf{L}} \rangle$ *be an algebraic and distributive lattice satisfying* **(S)** *of Theorem* 2.2. *Then all the edges of* $\mathbf{G}$ *can be directed to form a digraph* $\mathbf{D}$ *such that* $\mathbf{S}_s(\mathbf{D}) \simeq \mathbf{L}$ *if and only if*

(SW) *There is an equivalence relation* θ *on* $V^{\mathbf{G}}$ *and a bijection* $\varphi\colon V^{\mathbf{G}/\theta} \longrightarrow \mathcal{CI}(\mathbf{L})$ *such that*

- **(1)** *for each equivalence class* $W \in V^{\mathbf{G}/\theta}$, *the relative subgraph* $[W]_{\mathbf{G}}$ *(induced by* W*) is connected and each of its edges lies on an undirected cycle,*
- **(2)** *for each edge* e, $\varphi(I^{\mathbf{G}/\theta}(e))$ *is a chain in* $\langle \mathcal{CI}(\mathbf{L}), \leq_{\mathbf{L}} \rangle$,
- **(3)** *for each two distinct elements* $k, l \in \mathcal{CI}(\mathbf{L})$, *if* $k \leq_{\mathbf{L}} l$, *then there is an undirected path* $(u_1, \ldots, u_n)$ *in* $\mathbf{G}/\theta$ *such that*

$$k = \varphi(u_n) \leq_{\mathbf{L}} \varphi(u_{n-1}) \leq_{\mathbf{L}} \ldots \leq_{\mathbf{L}} \varphi(u_1) = l.$$

Proof. $\Longrightarrow$: Direct all the edges of $\mathbf{G}$ to obtain a digraph $\mathbf{D}$ such that

$$\mathbf{S}_s(\mathbf{D}) \simeq \mathbf{L}.$$

Next, take the equivalence relation $\Theta = \Theta(\mathbf{D})$ on $V^{\mathbf{D}} = V^{\mathbf{G}}$. Then for each equivalence class $W \in V^{\mathbf{G}}/\Theta$ we easily obtain

$$([W]_{\mathbf{D}})^* = [W]_{\mathbf{D}^*} = [W]_{\mathbf{G}}.$$

Next, $[W]_{\mathbf{D}}$ is a strongly connected digraph by the definition of Θ. Thus by Theorem 2.12, Θ satisfies **(SW.1)**.

To prove **(SW.2)** and **(SW.3)** observe first that

$$(\mathbf{D}/\Theta)^* = \mathbf{D}^*/\Theta = \mathbf{G}/\Theta,$$

which easily follows from the definition of quotient digraph and of *.

Secondly, by Theorem 2.14, there is an order isomorphism φ between $\langle V^{\mathbf{D}/\Theta}, \leq_{\mathbf{D}/\Theta} \rangle$ and $\langle \mathcal{CI}(\mathbf{L}), \leq_{\mathbf{L}} \rangle$.

Hence and by the definition of $\leq_{\mathbf{D}/\Theta}$ we obtain that φ is a bijection from $V^{\mathbf{G}/\Theta}$ onto $\mathcal{CI}(\mathbf{L})$ satisfying **(SW.2)** and **(SW.3)**, since every directed path in $\mathbf{D}/\Theta$ is an undirected path in $\mathbf{G}/\Theta$.

$\Longleftarrow$: Take an equivalence relation θ and a bijection φ from **(SW)**. By **(SW.2)**, for each $e \in E^{\mathbf{G}/\theta} = E^{\mathbf{G}}$, the set $\varphi(I^{\mathbf{G}}(e))$ has the greatest element, say $p_1(e)$, and the least element, say $p_2(e)$. In this way we have two functions p_1 and p_2 from $E^{\mathbf{G}/\theta}$ into $\mathcal{CI}(\mathbf{L})$.

Let $\mathbf{C}$ be the following digraph

$$V^{\mathbf{C}} = V^{\mathbf{G}/\theta}, \quad E^{\mathbf{C}} = E^{\mathbf{G}/\theta} \quad \text{and} \quad I_i^{\mathbf{C}} = \varphi^{-1} \circ p_i \quad \text{for} \ \ i = 1, 2.$$

For each $e \in E^{\mathbf{C}}$ we have

$$\varphi(I^{\mathbf{G}/\theta}(e)) = \{p_1(e), p_2(e)\},$$

so

$$I^{\mathbf{G}/\theta}(e) = \{\varphi^{-1}(p_1(e)), \varphi^{-1}\big(p_2(e)\big)\} = \{I_1^{\mathbf{C}}(e), I_2^{\mathbf{C}}(e)\} = I^{\mathbf{C}^*}(e).$$

Thus

$$\mathbf{C}^* = \mathbf{G}/\theta. \tag{1}$$

Take $v, w \in V^{\mathbf{C}}$ such that $v \leq_{\mathbf{C}} w$ and assume $v \neq w$. Then there is a directed path $(u_1, \ldots, u_n)$ in $\mathbf{C}$ going from w to v, i.e., $u_1 = w$ and $u_n = v$.

By definitions of p_1, p_2 and $\mathbf{C}$ we have that $\varphi(I_2^{\mathbf{C}}(e)) \leq_{\mathbf{L}} \varphi(I_1^{\mathbf{C}}(e))$ for each $e \in E^{\mathbf{C}} = E^{\mathbf{G}/\theta}$. Thus

$$\varphi(v) = \varphi(u_n) \leq_{\mathbf{L}} \varphi(u_{n-1}) \leq_{\mathbf{L}} \ldots \leq_{\mathbf{L}} \varphi(u_2) \leq_{\mathbf{L}} \varphi(u_1) = \varphi(w).$$

Now, let $v, w \in V^{\mathbf{G}/\theta}$ be vertices such that $\varphi(v) \leq_{\mathbf{L}} \varphi(w)$ and assume $v \neq w$. Then by **(SW.3)**, there is an undirected path $(u_1, \ldots, u_n)$ in $\mathbf{G}/\theta$ such that

$$\varphi(v) = \varphi(u_n) \leq_{\mathbf{L}} \ldots \leq_{\mathbf{L}} \varphi(u_1) = \varphi(w).$$

Let $f_1, f_2, \ldots, f_{n-1}$ be edges of $\mathbf{G}/\theta$ such that $I^{\mathbf{G}/\theta}(f_i) = \{u_i, u_{i+1}\}$ for $i = 1, 2, \ldots, n-1$. Then we have $I^{\mathbf{C}}(f_i) = \langle u_i, u_{i+1} \rangle$. Moreover, $u_1 = w$ and $u_{n+1} = v$, because φ is bijective. Thus $(u_1, \ldots, u_n)$ is a directed path in $\mathbf{C}$ going from w to v. Hence,

$$v \leq_{\mathbf{C}} w.$$

Since φ is bijective, the above two facts provide that φ is an isomorphism between relational systems $\langle V^{\mathbf{C}}, \leq_{\mathbf{C}} \rangle$ and $\langle \mathcal{CI}(\mathbf{L}), \leq_{\mathbf{L}} \rangle$. In particular, $\leq_{\mathbf{C}}$ is a partial order, so $\mathbf{C}$ does not contain non-trivial directed cycles. Hence, the equivalence relation $\Theta(\mathbf{C})$ is the identity relation (Proposition 3.3 in [12]), so

$$\mathbf{C}/\Theta(\mathbf{C}) = \mathbf{C}.$$

By all these facts and Theorem 2.14 we deduce

$$\mathbf{S}_s(\mathbf{C}) \simeq \mathbf{L}. \tag{2}$$

Take an equivalence class $W \in V^{\mathbf{G}}/\theta$. Then by **(SW.1)** and Theorem 2.12, all the edges of the relative subgraph $[W]_{\mathbf{G}}$ of $\mathbf{G}$ can be directed to form a strongly connected digraph $\mathbf{H}_W$.

Take an arbitrary edge e of $\mathbf{G}$. Then exactly one from the following two conditions holds:

$$I^{\mathbf{G}}(e) \subseteq W \quad \text{for some } W \in V^{\mathbf{G}}/\theta$$

or

$$I^{\mathbf{G}}(e) \not\subseteq W \quad \text{for each } W \in V^{\mathbf{G}}/\theta.$$

In the first case, e belongs to $[W]_{\mathbf{G}}$, so also to $\mathbf{H}_W$, because $[W]_{\mathbf{G}}$ is a relative subgraph of $\mathbf{G}$. Then we can direct e in the same way as in $\mathbf{H}_W$.

In the second case, there are $U_1, U_2 \in V^{\mathbf{G}}/\theta$ such that one terminal vertex of e belongs to U_1 and the other belongs to U_2. Then $\left|I^{\mathbf{G}}(e)/\theta\right| = 2$ and by (1), $I^{\mathbf{G}}(e)/\theta = I^{\mathbf{G}/\theta}(e) = I^{\mathbf{C}^*}(e) = \{I_1^{\mathbf{C}}(e), I_2^{\mathbf{C}}(e)\}$, so we can import the orientation of e from the digraph $\mathbf{C}$. More precisely, we take the initial vertex v of e to be such that $v \in I^{\mathbf{G}}(e)$ and $v/\theta = I_1^{\mathbf{C}}(e)$, and the final vertex w of e is assumed to satisfy $w \in I^{\mathbf{G}}(e)$ and $w/\theta = I_2^{\mathbf{C}}(e)$.

In this way, since e was arbitrarily chosen, we construct a digraph $\mathbf{D}$ such that

$$\mathbf{D}^* = \mathbf{G}, \tag{3}$$

$$\mathbf{D}/\theta = \mathbf{C}. \tag{4}$$

Now take an equivalence class $W \in V^{\mathbf{G}}/\theta = V^{\mathbf{D}}/\theta$. Then first, $\mathbf{H}_W^* = [W]_{\mathbf{G}}$ is a relative subgraph of $\mathbf{D}^* = \mathbf{G}$. Secondly, $\mathbf{H}_W$ is a weak subdigraph of $\mathbf{D}$, by the construction of $\mathbf{D}$. By these two facts we obtain that $\mathbf{H}_W$ is a relative subdigraph of $\mathbf{D}$.

Since $\mathbf{H}_W$ and $[W]_{\mathbf{D}}$ are relative subdigraphs of $\mathbf{D}$ and they have the same vertex set, we infer that

$$\mathbf{H}_W = [W]_{\mathbf{D}}.$$

Hence, for each $W \in V^{\mathbf{D}}/\theta$, $[W]_{\mathbf{D}}$ is strongly connected. In particular, $\mathbf{D}$ and θ satisfies the following condition: for each $v, w \in V^{\mathbf{D}}$ with $v \neq w$, if $v\theta w$, then there is a directed path from v to w. We have shown in [12] (Theorem 2.6) that this condition implies

$$\mathbf{S}_s(\mathbf{D}) \simeq \mathbf{S}_s(\mathbf{D}/\theta).$$

Thus (2) and (4) entail

$$\mathbf{S}_s(\mathbf{D}) \simeq \mathbf{L}. \tag{5}$$

This completes the proof of the implication $\Longleftarrow$. ∎

Theorem 3.1 and its proof imply the following

Corollary 3.2. *Let* $\mathbf{G}$ *be a graph and let* $\mathbf{L} = \langle L, \leq_{\mathbf{L}} \rangle$ *be an algebraic and distributive lattice satisfying* **(S)** *of Theorem* 2.2. *Then all the edges of* $\mathbf{G}$

can be directed to form a digraph $\mathbf{D}$ *without non-trivial directed cycles such that* $\mathbf{S}_s(\mathbf{D}) \simeq \mathbf{L}$ *if and only if*

(PSW) *there is a bijection* $\varphi\colon V^{\mathbf{G}} \longrightarrow \mathcal{CI}(\mathbf{L})$ *satisfying* **(SW.2)** *and* **(SW.3)** *of Theorem* 3.1*,*

where $\mathbf{G}/\theta$ *is replaced by* $\mathbf{G}$*.*

Proof. The implication $\Longrightarrow$ follows from the fact (see [12], Proposition 3.3) that, if $\mathbf{D}$ does not contain non-trivial cycles, then $\Theta(\mathbf{D})$ is the identity relation.

Observe that if an equivalence relation θ from the proof of the implication $\Longleftarrow$ of Theorem 3.1 is the identity relation, then $\mathbf{G}/\theta = \mathbf{G}$, so $\mathbf{C}^* = \mathbf{G}$. Thus, by this proof, we obtain the implication $\Longleftarrow$. ∎

By Theorem 3.1 we obtain our first algebraic result.

Theorem 3.3. *Let* $\mathbf{K} = \langle K, \leq_{\mathbf{K}} \rangle$ *and* $\mathbf{L} = \langle L, \leq_{\mathbf{L}} \rangle$ *be algebraic and distributive lattices such that*

(∗) $\mathbf{K}$ *satisfies* **(W.1)**–**(W.3)** *of Theorem* 2.3*,*
(∗∗) $\mathbf{L}$ *satisfies* **(S)** *of Theorem* 2.2*.*

Then the following two conditions are equivalent:

(a) *There is a partial unary algebra* $\mathbf{A}$ *such that* $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{K}$ *and* $\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{L}$ *(i.e.,* $\mathbf{K}$ *and* $\mathbf{L}$ *are isomorphic to lattices of weak and strong subalgebras of* $\mathbf{A}$*, respectively).*
(b) $\mathbf{G}(\mathbf{K})$ *and* $\mathbf{L}$ *satisfy* **(SW)** *of Theorem* 3.1*.*

Proof. First, we know (see Theorem 2.9 and Proposition 2.8) that (a) holds if and only if $\mathbf{K}$ and $\mathbf{L}$ are isomorphic to lattices $\mathbf{S}_w(\mathbf{D})$ and $\mathbf{S}_s(\mathbf{D})$ of weak and strong subdigraphs, respectively, of one digraph $\mathbf{D}$. Secondly, by Theorem 2.11, lattices $\mathbf{S}_w(\mathbf{D})$ and $\mathbf{K}$ are isomorphic if and only if graphs $\mathbf{D}^*$ and $\mathbf{G}(\mathbf{K})$ are isomorphic, in other words, all the edges of $\mathbf{G}(\mathbf{K})$ can be directed to form a digraph isomorphic to $\mathbf{D}$. Now, it remains to use Theorem 3.1. ∎

It is shown in [12] (the proof of Corollary 3.13) that for every partial unary algebra $\mathbf{A}$, the digraph $\mathbf{D}(\mathbf{A})$ does not contain non-trivial directed cycles if and only if each two distinct elements of $\mathbf{A}$ generate two distinct strong subalgebras. Hence and by Corollary 3.2 we obtain, in a similar way as above, the following result.

Corollary 3.4. *Let* $\mathbf{K} = \langle K, \leq_{\mathbf{K}} \rangle$ *and* $\mathbf{L} = \langle L, \leq_{\mathbf{L}} \rangle$ *be algebraic and distributive lattices satisfying* (∗) *and* (∗∗) *of Theorem* 3.3*. Then the following conditions are equivalent:*

(a) *There is a partial unary algebra* $\mathbf{A}$ *such that*
  (1) *each two distinct elements of* $\mathbf{A}$ *generate two distinct strong subalgebras,*
  (2) $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{K}$ *and* $\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{L}$.
(b) $\mathbf{G}(\mathbf{K})$ *and* $\mathbf{L}$ *satisfy* **(PSW)** *of Corollary* 3.2.

Finally note that the following fact is an immediate consequence of Theorems 3.1 and 3.3

Corollary 3.5. *Let lattices* $\mathbf{K}$ *and* $\mathbf{L}$ *be isomorphic to lattices of weak and strong subalgebras, respectively, of one partial unary algebra. Then* $|\mathcal{A}(\mathbf{K})| \geq |\mathcal{CI}(\mathbf{L})|$.

4.

We start this section with a simple fact that for a given digraph, its dually-strong subdigraph lattice is the dual of its strong subdigraph lattice. By this result we obtain a characterization of the initial segment lattice of a partial unary algebra. Moreover, having this fact we are able to describe pairs of lattices of strong subalgebras and initial segments of one partial unary algebra.

Next, necessary and sufficient conditions are found for pairs of lattices to be isomorphic to lattices of strong and relative subalgebras of one partial unary algebra. We describe also other pairs of subalgebra lattices.

Having these results we can characterize quadruples of lattices to be isomorphic to lattices of weak, relative, strong subalgebras and initial segments of one partial unary algebra.

We will frequently use the following simple fact (see [11])

Lemma 4.1. *Let* $\mathbf{D}$ *be a digraph and let* $\mathbf{H}_1, \mathbf{H}_2 \leq_d \mathbf{D}$ *(*$\mathbf{H}_1, \mathbf{H}_2 \leq_s \mathbf{D}$*) be dually-strong (strong) subdigraphs of* $\mathbf{D}$. *Then*

$$\mathbf{H}_1 \leq_d \mathbf{H}_2 \ \ (\mathbf{H}_1 \leq_s \mathbf{H}_2) \quad \textit{iff} \quad V^{\mathbf{H}_1} \subseteq V^{\mathbf{H}_2},$$
$$\mathbf{H}_1 = \mathbf{H}_2 \quad \textit{iff} \quad V^{\mathbf{H}_1} = V^{\mathbf{H}_2}.$$

Let $\mathbf{P} = \langle P, \leq_P \rangle$ be a partially ordered set. Then $\overline{\mathbf{P}} = \overline{\langle P, \leq_P \rangle}$ denotes the dual of $\langle P, \leq_P \rangle$, i.e., $\overline{\mathbf{P}} = \langle P, \leq_P^{-1} \rangle$, where $\leq_P^{-1}$ is the inverse relation to $\leq_P$. It is well-know that the dual of each lattice is also a lattice.

Now we show our first result

Proposition 4.2.

(a) *For each digraph* $\mathbf{D}$, $\mathbf{S}_d(\mathbf{D}) \simeq \overline{\mathbf{S}_s(\mathbf{D})}$.
(b) *For each partial unary algebra* $\mathbf{A}$, $\mathbf{S}_i(\mathbf{A}) \simeq \overline{\mathbf{S}_s(\mathbf{A})}$.

Proof. Note first that (b) follows from (a) and Theorem 2.9.

(a): Take a weak subdigraph $\mathbf{H} \leq_w \mathbf{D}$ and observe that the following facts are easily shown:

(1) if $\mathbf{H}$ is a strong subdigraph of $\mathbf{D}$, then $[V^{\mathbf{H}}]_{\mathbf{D}} = \mathbf{H}$,
(2) if $\mathbf{H}$ is a strong subdigraph of $\mathbf{D}$, then $[V^{\mathbf{D}} \setminus V^{\mathbf{H}}]_{\mathbf{D}}$ is a dually-strong subdigraph of $\mathbf{D}$,
(3) if $\mathbf{H}$ is a dually-strong subdigraph of $\mathbf{D}$, then $[V^{\mathbf{D}} \setminus V^{\mathbf{H}}]_{\mathbf{D}}$ is a strong subdigraph of $\mathbf{D}$.

Having (3) we can take the following function $\varphi\colon S_d(\mathbf{D}) \longrightarrow S_s(\mathbf{D})$,

$$\varphi(\mathbf{H}) = [V^{\mathbf{D}} \setminus V^{\mathbf{H}}]_{\mathbf{D}} \quad \text{for each} \quad \mathbf{H} \leq_d \mathbf{D}.$$

Take an arbitrary strong subdigraph $\mathbf{K}$ of $\mathbf{D}$ and let $\mathbf{H} = [V^{\mathbf{D}} \setminus V^{\mathbf{K}}]_{\mathbf{D}}$. By (2), $\mathbf{H}$ is a dually-strong subdigraph of $\mathbf{D}$. Further,

$$\varphi(\mathbf{H}) = [V^{\mathbf{D}} \setminus V^{\mathbf{H}}]_{\mathbf{D}} = [V^{\mathbf{D}} \setminus (V^{\mathbf{D}} \setminus V^{\mathbf{K}})]_{\mathbf{D}} = [V^{\mathbf{K}}]_{\mathbf{D}},$$

so

$$\varphi(\mathbf{H}) = \mathbf{K}$$

by (1). Thus φ is a surjection.

Take two dually-strong subdigraphs $\mathbf{H}_1, \mathbf{H}_2 \leq_d \mathbf{D}$ such that $\varphi(\mathbf{H}_1) = \varphi(\mathbf{H}_2)$. Then

$$[V^{\mathbf{D}} \setminus V^{\mathbf{H}_1}]_{\mathbf{D}} = [V^{\mathbf{D}} \setminus V^{\mathbf{H}_2}]_{\mathbf{D}}.$$

Hence,

$$V^{\mathbf{H}_1} = V^{\mathbf{H}_2},$$

which implies

$$\mathbf{H}_1 = \mathbf{H}_2.$$

Thus φ is also an injection.

Now it is sufficient to show that φ and its inverse φ^{-1} preserve lattice orders of $\mathbf{S}_d(\mathbf{D})$ and $\overline{\mathbf{S}_s(\mathbf{D})}$. To see it take $\mathbf{H}_1, \mathbf{H}_2 \leq_d \mathbf{D}$. Then by (3) we conclude

$$\mathbf{H}_1 \leq_d \mathbf{H}_2 \text{ iff } V^{\mathbf{H}_1} \subseteq V^{\mathbf{H}_2} \text{ iff } V^{\mathbf{D}} \setminus V^{\mathbf{H}_2} \subseteq V^{\mathbf{D}} \setminus V^{\mathbf{H}_1} \text{ iff } \varphi(\mathbf{H}_2) \leq_s \varphi(\mathbf{H}_1).$$

Thus φ is indeed the desired lattice isomorphism from $\mathbf{S}_d(\mathbf{D})$ onto $\overline{\mathbf{S}_s(\mathbf{D})}$. ∎

The following two facts are simple consequences of Proposition 4.2 and Theorem 2.2 (see also notes following Theorem 2.2)

Theorem 4.3. *A lattice* $\mathbf{L}$ *is isomorphic to the initial segment lattice of a (partial or total) unary algebra if and only if* $\mathbf{L}$ *is algebraic, distributive and satisfies* **(S)** *of Theorem* 2.2.

Proof. First, it is well-known (see [6], p. 83) that if a lattice $\mathbf{L}$ is algebraic, distributive and satisfies **(S)**, then its dual $\overline{\mathbf{L}}$ is also algebraic, distributive and satisfies **(S)**.

Assume that $\mathbf{L} \simeq \mathbf{S}_d(\mathbf{A})$ for some unary algebra $\mathbf{A}$. Then $\overline{\mathbf{L}} \simeq \overline{\mathbf{S}_d(\mathbf{A})} \simeq \mathbf{S}_s(\mathbf{A})$, by Proposition 4.2. Hence, $\overline{\mathbf{L}}$ is algebraic, distributive and satisfies **(S)**. But $\mathbf{L} = \overline{\overline{\mathbf{L}}}$, so $\mathbf{L}$ also has these properties.

Let $\mathbf{L}$ be an algebraic and distributive lattice satisfying **(S)**. Then $\overline{\mathbf{L}}$ has the same properties, so $\overline{\mathbf{L}} \simeq \mathbf{S}_s(\mathbf{A})$ for some unary algebra $\mathbf{A}$. Hence it follows that $\mathbf{L} \simeq \overline{\mathbf{S}_s(\mathbf{A})}$. Thus $\mathbf{L}$ is isomorphic to $\mathbf{S}_d(\mathbf{A})$, by Proposition 4.2. ∎

Theorem 4.4. *Let* $\mathbf{K}$ *and* $\mathbf{L}$ *be algebraic and distributive lattices satisfying* **(S)** *of Theorem* 2.2. *Then the following two conditions are equivalent:*

(a) *There is a (total) unary algebra* $\mathbf{A}$ *such that* $\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{K}$ *and* $\mathbf{S}_i(\mathbf{A}) \simeq \mathbf{L}$ *(i.e.,* $\mathbf{K}$ *and* $\mathbf{L}$ *are isomorphic to lattices of strong subalgebras and initial segments of* $\mathbf{A}$*, respectively).*

(b) $\mathbf{L} \simeq \overline{\mathbf{K}}$.

Proof. It follows directly from Theorem 4.3 and Proposition 4.2. ∎

Now we describe pairs of lattices of strong and relative subalgebras of one (total) unary algebra (see Theorems 2.4 and 2.5).

Theorem 4.5. *Let* $\mathbf{K}$ *be an algebraic and distributive lattice satisfying* **(S)** *of Theorem* 2.2 *and let* $\mathbf{L}$ *be a complete and atomic Boolean lattice. Then the following two conditions are equivalent:*

(a) *There is a (total) unary algebra* $\mathbf{A}$ *such that* $\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{K}$ *and* $\mathbf{S}_r(\mathbf{A}) \simeq \mathbf{L}$ *(i.e.,* $\mathbf{K}$ *and* $\mathbf{L}$ *are isomorphic to lattices of strong and relative subalgebras of* $\mathbf{A}$*, respectively).*

(b) $|\mathcal{CI}(\mathbf{K})| \leq |\mathcal{A}(\mathbf{L})|$.

Proof. (a)$\Longrightarrow$(b): Take an unary algebra $\mathbf{A} = \langle A, (k^{\mathbf{A}})_{k\in K}\rangle$ such that

$$\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{K} \quad \text{and} \quad \mathbf{S}_r(\mathbf{A}) \simeq \mathbf{L}.$$

Then first, we know (see [10])

$$\mathcal{CI}(\mathbf{S}_s(\mathbf{A})) = \{\langle a\rangle_{\mathbf{A}} \colon\ a \in A\},$$

where $\langle a\rangle_{\mathbf{A}}$ is the strong subalgebra generated by a.

Secondly, by Theorem 2.5,

$$|\mathcal{A}(\mathbf{L})| = |A|.$$

These two facts imply

$$|\mathcal{CI}(\mathbf{K})| = |\mathcal{CI}(\mathbf{S}_s(\mathbf{A}))| \leq |A| = |\mathcal{A}(\mathbf{L})|.$$

(b)$\Longrightarrow$(a): Having Theorem 2.5 and Proposition 2.8 (see also notes following Theorem 2.9) it remains to show that there is a digraph $\mathbf{D}$ such that

$$\mathbf{S}_s(\mathbf{D}) \simeq \mathbf{L} \quad \text{and} \quad |V^{\mathbf{D}}| = |\mathcal{A}(\mathbf{L})|.$$

For this purpose we first take a digraph $\mathbf{H}$ such that

$$V^{\mathbf{H}} = \mathcal{CI}(\mathbf{K}) \quad \text{and} \quad E^{\mathbf{H}} = \{\langle i, j\rangle \in \mathcal{CI}(\mathbf{K}) \times \mathcal{CI}(\mathbf{K}) \colon j \lneq_{\mathbf{K}} i\},$$

(obviously for an edge $\langle i, j\rangle$, i is its initial vertex and j is its final vertex).

By this definition we infer

$$\langle V^{\mathbf{H}}, \leq_{\mathbf{H}}\rangle \simeq \langle \mathcal{CI}(\mathbf{K}), \leq_{\mathbf{K}}\rangle. \tag{1}$$

Since $|\mathcal{CI}(\mathbf{K})| \leq |\mathcal{A}(\mathbf{L})|$, a set W may be taken such that

$$W \cap \mathcal{CI}(\mathbf{K}) = \emptyset \quad \text{and} \quad |W \cup \mathcal{CI}(\mathbf{K})| = |\mathcal{A}(\mathbf{L})|.$$

Take also an arbitrary element $j_0 \in \mathcal{CI}(\mathbf{K})$ and let $\mathbf{D}$ be the digraph obtained from $\mathbf{H}$ by adding elements from W as new vertices and all the pairs $\langle v, w\rangle$ of distinct elements from $W \cup \{j_0\}$ as new directed edges.

Now we show that for each vertices $v, w \in V^{\mathbf{D}}$,

$$v\Theta(\mathbf{D})w \quad \text{iff} \quad v = w \quad \text{or} \quad v, w \in W \cup \{j_0\}. \tag{2}$$

The implication $\Longleftarrow$ is trivial.

$\Longrightarrow$. Take two vertices $v, w \in V^{\mathbf{D}}$ such that $v\Theta(\mathbf{D})w$ and $v \neq w$. Then there are paths p_1 and p_2 in $\mathbf{D}$ going from v to w and from w to v, respectively. Assume additionally that one of these vertices does not belong to $W \cup \{j_0\}$, for instance, $v \notin W \cup \{j_0\}$.

The construction of $\mathbf{D}$ implies that if an edge e of $\mathbf{D}$ starts from (ends in) some vertex in $W \cup \{j_0\}$ and its final (initial) vertex is outside $W \cup \{j_0\}$, then e must start from (end in) j_0 and e belongs to $\mathbf{H}$. This fact entails that if there is a directed path in $\mathbf{D}$ connecting a vertex u outside $W \cup \{j_0\}$ with some vertex in this set, then there is a path in $\mathbf{H}$ connecting u with j_0. Further, if p is a directed path with endpoints outside $W \cup \{j_0\}$, then j_0 is the unique vertex from $W \cup \{j_0\}$ which may belong to p, because a path does not encounter the same vertex twice.

Now, if $w \notin W \cup \{j_0\}$, then the above facts provide that p_1 and p_2 are also directed paths in $\mathbf{H}$. Hence, $w \leq_{\mathbf{H}} v$ and $v \leq_{\mathbf{H}} w$. By (1) we have that $\leq_{\mathbf{H}}$ is a partial order. Thus $v = w$, which contradicts our assumptions.

If $w \in W \cup \{j_0\}$, then p_1 and p_2 contain j_0. Hence and by the above facts, there are directed paths p_1' and p_2' in $\mathbf{H}$ going from v to j_0 and from j_0 to v, respectively. Thus $j_0 \leq_{\mathbf{H}} v$ and $v \leq_{\mathbf{H}} j_0$, so $v = j_0 \in W \cup \{j_0\}$, which contradicts our assumptions, again.

Having (2) we deduce that the digraph $\mathbf{D}/\Theta(\mathbf{D})$ is isomorphic (up to some loops in the vertex j_0) to $\mathbf{H}$, because all the edges of $\mathbf{D}$ that are not in $\mathbf{H}$ form loops in $\mathbf{D}/\Theta(\mathbf{D})$.

Hence and by (1)

$$\langle V^{\mathbf{D}/\Theta(\mathbf{D})}, \leq_{\mathbf{D}/\Theta(\mathbf{D})}\rangle \simeq \langle \mathcal{CI}(\mathbf{K}), \leq_{\mathbf{K}}\rangle,$$

so by Theorem 2.14,

$$\mathbf{S}_s(\mathbf{D}) \simeq \mathbf{K},$$

which completes our proof, since

$$|V^{\mathbf{D}}| = |W \cup \mathcal{CI}(\mathbf{K})| = |\mathcal{A}(\mathbf{L})|. \blacksquare$$

Theorem 4.6. *Let* $\mathbf{K}$ *be an algebraic and distributive lattice satisfying* **(W.1)**–**(W.3)** *of Theorem* 2.3 *and let* $\mathbf{L}$ *be a complete and atomic Boolean lattice. Then the following two conditions are equivalent:*

(a) *There is a partial unary algebra* $\mathbf{A}$ *such that* $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{K}$ *and* $\mathbf{S}_r(\mathbf{A}) \simeq \mathbf{L}$ *(i.e.,* $\mathbf{K}$ *and* $\mathbf{L}$ *are isomorphic to lattices of weak and relative subalgebras of* $\mathbf{A}$*, respectively).*

(b) $|\mathcal{A}(\mathbf{K})| = |\mathcal{A}(\mathbf{L})|$.

Proof. (a)$\Longrightarrow$(b): Take a digraph $\mathbf{D}$ such that $\mathbf{S}_w(\mathbf{D}) \simeq \mathbf{K}$ and $\mathbf{S}_r(\mathbf{D}) \simeq \mathbf{L}$. Then by Theorems 2.5 and 2.11 (see also notes following Theorem 2.9) we obtain

$$\mathbf{D}^* \simeq \mathbf{G}(\mathbf{K}) \quad \text{and} \quad |V^{\mathbf{D}}| = |\mathcal{A}(\mathbf{L})|.$$

Hence,

$$|\mathcal{A}(\mathbf{K})| = |V^{\mathbf{G}(\mathbf{K})}| = |V^{\mathbf{D}^*}| = |V^{\mathbf{D}}| = |\mathcal{A}(\mathbf{L})|,$$

since $\mathcal{A}(\mathbf{K})$ is the set of all the vertices of $\mathbf{G}(\mathbf{K})$.

(b)$\Longrightarrow$(a): We direct (using the axiom of choice if necessary) all the edges of the graph $\mathbf{G}(\mathbf{K})$ to form a digraph $\mathbf{D}$. Then first,

$$\mathbf{S}_w(\mathbf{D}) \simeq \mathbf{K},$$

by Theorem 2.11. Secondly,

$$|V^{\mathbf{D}}| = |V^{\mathbf{G}(\mathbf{K})}| = |\mathcal{A}(\mathbf{K})| = |\mathcal{A}(\mathbf{L})|,$$

so also

$$\mathbf{S}_r(\mathbf{D}) \simeq \mathbf{L}. \blacksquare$$

Theorem 4.7. *Let* $\mathbf{L}_1$ *be an algebraic and distributive lattice satisfying* **(W.1)**–**(W.3)** *of Theorem* 2.3*, let* $\mathbf{L}_2$ *be a complete and atomic Boolean lattice and let* $\mathbf{L}_3$ *and* $\mathbf{L}_4$ *be algebraic and distributive lattices satisfying* **(S)** *of Theorem* 2.2*. Then the following two conditions are equivalent:*

(a) *There is a partial unary algebra* $\mathbf{A}$ *such that*

$$\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}_1, \ \ \mathbf{S}_r(\mathbf{A}) \simeq \mathbf{L}_2, \ \ \mathbf{S}_s(\mathbf{A}) \simeq \mathbf{L}_3, \ \ \mathbf{S}_i(\mathbf{A}) \simeq \mathbf{L}_4,$$

(i.e., $\mathbf{L}_1$*,* $\mathbf{L}_2$*,* $\mathbf{L}_3$ *and* $\mathbf{L}_4$ *are isomorphic to lattices of weak, relative, strong subalgebras and initial segments of* $\mathbf{A}$*, respectively).*

(b) $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$ *and* $\mathbf{L}_4$ *satisfy the following three conditions*
(1) $|\mathcal{A}(\mathbf{L}_1)| = |\mathcal{A}(\mathbf{L}_2)|$,
(2) $\mathbf{L}_4 \simeq \overline{\mathbf{L}}_3$,
(3) *the graph* $\mathbf{G}(\mathbf{L}_1)$ *and the lattice* $\mathbf{L}_3$ *satisfy* **(SW)** *of Theorem* 3.1.

Proof. The implication (a)$\Longrightarrow$(b) follows from Theorems 3.3, 4.4 and 4.6.

(b)$\Longrightarrow$(a): By Theorem 3.3 and the condition (3), there is a partial unary algebra $\mathbf{A}$ such that

$$\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{L}_1 \quad \text{and} \quad \mathbf{S}_s(\mathbf{A}) \simeq \mathbf{L}_3.$$

Then

$$\mathbf{S}_d(\mathbf{A}) \simeq \overline{\mathbf{S}_s(\mathbf{A})} \simeq \overline{\mathbf{L}}_3 \simeq \mathbf{L}_4,$$

by (2) and Proposition 4.2.

By Theorem 2.11 we have also $\mathbf{D}(\mathbf{A})^* \simeq \mathbf{G}(\mathbf{L}_1)$. In particular, $|A| = |V^{\mathbf{D}(\mathbf{A})}| = |\mathcal{A}(\mathbf{L}_1)|$. Thus by (1) and Theorem 2.5 we obtain

$$\mathbf{S}_r(\mathbf{A}) \simeq \mathbf{L}_2. \ \blacksquare$$

Note that Theorems 4.3, 4.4, 4.5, 4.6 and 4.7 can be easily translated into digraphs and their weak, relative, strong and dually-strong subdigraph lattices.

5.

We start this section with the following simple construction which is useful in investigations of the initial segment lattice of a partial unary algebra.

Definition 5.1. Let $\mathbf{D}$ be a digraph. Then $\overline{\mathbf{D}}$ (called the dual of $\mathbf{D}$) is the digraph obtained from $\mathbf{D}$ by inverting the orientation of all the edges.

Clearly, the following facts hold:

Lemma 5.2. *Let* $\mathbf{D}$ *be a digraph and* θ *be an equivalence relation on* $V^{\mathbf{D}}$. *Then*

(1) $\overline{\overline{\mathbf{D}}} = \mathbf{D}$,
(2) $\overline{\mathbf{D}}^* = \mathbf{D}^*$,
(3) $\overline{\mathbf{D}/\theta} = \overline{\mathbf{D}}/\theta$,
(4) $\Theta(\overline{\mathbf{D}}) = \Theta(\mathbf{D})$,
(5) $\langle V^{\overline{\mathbf{D}}}, \leq_{\overline{\mathbf{D}}} \rangle \simeq \overline{\langle V^{\mathbf{D}}, \leq_{\mathbf{D}} \rangle}$.

Now we describe subdigraph lattices of the dual of a given digraph.

Proposition 5.3. *Let* $\mathbf{D}$ *be a digraph. Then*

(1) $\mathbf{S}_w(\mathbf{D}) \simeq \mathbf{S}_w(\overline{\mathbf{D}})$,
(2) $\mathbf{S}_r(\mathbf{D}) \simeq \mathbf{S}_r(\overline{\mathbf{D}})$,
(3) $\mathbf{S}_s(\mathbf{D}) \simeq \mathbf{S}_d(\overline{\mathbf{D}})$,

(4) $\mathbf{S}_d(\mathbf{D}) \simeq \mathbf{S}_s(\overline{\mathbf{D}})$,
(5) $\mathbf{S}_s(\overline{\mathbf{D}}) \simeq \mathbf{S}_s(\mathbf{D})$.

Proof. First, (5) follows from (4) and Proposition 4.2. Secondly, (2) follows from the fact that $\mathbf{S}_r(\mathbf{D})$ and $\mathbf{S}_r(\overline{\mathbf{D}})$ are isomorphic to the powerset lattice of $V^{\mathbf{D}} = V^{\overline{\mathbf{D}}}$ (see Theorem 2.5). Note also that this point may be proved in a similar way as (1) (see below).

(1): Take a weak subdigraph $\mathbf{H}$ of $\mathbf{D}$ and observe that $\overline{\mathbf{H}}$ is a weak subdigraph of $\overline{\mathbf{D}}$. Hence, $\varphi\colon S_w(\mathbf{D}) \longrightarrow S_w(\overline{\mathbf{D}})$ such that

$$\varphi(\mathbf{H}) = \overline{\mathbf{H}} \quad \text{for each } \mathbf{H} \leq_w \mathbf{D},$$

is a well-defined function.

For each weak subdigraph $\mathbf{K}$ of $\overline{\mathbf{D}}$, we have that $\overline{\mathbf{K}}$ is a weak subdigraph of $\mathbf{D}$, and also

$$\varphi(\overline{\mathbf{K}}) = \overline{\overline{\mathbf{K}}} = \mathbf{K}.$$

Hence, φ is surjective.

Next, assume that $\varphi(\mathbf{H}_1) = \varphi(\mathbf{H}_2)$ for some weak subdigraphs $\mathbf{H}_1, \mathbf{H}_2 \leq_w \mathbf{D}$. Then $\mathbf{H}_1$ and $\mathbf{H}_2$ have the same vertex and edge sets, so they are equal. Thus φ is also injective.

Having these two facts it remains to show that φ and its inverse φ^{-1} preserve lattices orders. But this follows from the following obvious fact

$$\mathbf{H}_1 \leq_w \mathbf{H}_2 \quad \text{iff} \quad \overline{\mathbf{H}}_1 \leq_w \overline{\mathbf{H}}_2,$$

for all weak subdigraphs $\mathbf{H}_1$, $\mathbf{H}_2$ of $\mathbf{D}$.

(3): It is easily seen that if $\mathbf{H} \leq_s \mathbf{D}$ is a strong subdigraph of $\mathbf{D}$, then $\overline{\mathbf{H}}$ is a dually-strong subdigraph of $\overline{\mathbf{D}}$. And conversely, if $\mathbf{K}$ is a dually-strong subdigraph of $\overline{\mathbf{D}}$, then $\overline{\mathbf{K}}$ is a strong subdigraph of $\mathbf{D}$. By these facts we obtain that the restriction of φ to the set of all the strong subdigraphs of $\mathbf{D}$ is a bijection from $S_s(\mathbf{D})$ onto $S_d(\overline{\mathbf{D}})$.

Moreover, by Lemma 4.1 we obtain that for each strong subdigraphs $\mathbf{H}_1, \mathbf{H}_2$ of $\mathbf{D}$,

$$\mathbf{H}_1 \leq_s \mathbf{H}_2 \quad \text{iff} \quad V^{\mathbf{H}_1} \subseteq V^{\mathbf{H}_2} \quad \text{iff} \quad V^{\overline{\mathbf{H}}_1} \subseteq V^{\overline{\mathbf{H}}_2} \quad \text{iff} \quad \overline{\mathbf{H}}_1 \leq_d \overline{\mathbf{H}}_2.$$

The proof of (4) is analogous. ∎

Remark 1. Let $\mathbf{L}$ be an algebraic and distributive lattice satisfying **(S)** of Theorem 2.2. Then there is a digraph $\mathbf{D}$ such that $\mathbf{S}_s(\mathbf{D}) \simeq \mathbf{L}$. Hence and by Proposition 5.3(5), $\mathbf{S}_s(\overline{\mathbf{D}}) \simeq \overline{\mathbf{L}}$. Thus we obtain another proof of the following lattice fact (see [6], p. 83), used in the proof of Theorem 4.3:

If a lattice $\mathbf{L}$ is algebraic, distributive and satisfies **(S)**, then its dual $\overline{\mathbf{L}}$ is also algebraic, distributive and satisfies **(S)**.

Remark 2. By Proposition 5.3 we also obtain another proof of Theorem 4.3. Observe first (see Proposition 2.8 and Theorem 2.9) that this result can be formulated in the following way: a lattice $\mathbf{L}$ is isomorphic to the dually-strong subdigraph lattice $\mathbf{S}_d(\mathbf{D})$ of some digraph $\mathbf{D}$ if and only if $\mathbf{L} \simeq \mathbf{S}_s(\mathbf{H})$ for some digraph $\mathbf{H}$.

By Proposition 5.3, it is sufficient to take $\mathbf{H} = \overline{\mathbf{D}}$ to see the implication $\Longrightarrow$ and $\mathbf{D} = \overline{\mathbf{H}}$ in the proof of $\Longleftarrow$.

Having Lemma 5.2 and Proposition 5.3 we obtain by Theorems 3.3 and 4.5, the following characterizations of pairs of lattices of weak subalgebras and initial segments and of pairs of lattices of relative subalgebras and initial segments (analogous two results hold also for digraphs and their subdigraph lattices)

Corollary 5.4. *Let* $\mathbf{K}$ *be an algebraic and distributive lattice satisfying* **(W.1)**–**(W.3)** *of Theorem* 2.3 *and let* $\mathbf{L}$ *be an algebraic and distributive lattice satisfying* **(S)** *of Theorem* 2.2. *Then the following two conditions are equivalent:*

(a) *There is a partial unary algebra* $\mathbf{A}$ *such that* $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{K}$ *and* $\mathbf{S}_i(\mathbf{A}) \simeq \mathbf{L}$ *(i.e.,* $\mathbf{K}$ *and* $\mathbf{L}$ *are isomorphic to lattices of weak subalgebras and initial segments of* $\mathbf{A}$*, respectively).*
(b) *The graph* $\mathbf{G}(\mathbf{K})$ *and* $\mathbf{L}$ *satisfy* **(SW)** *of Theorem* 3.1.

Corollary 5.5. *Let* $\mathbf{K}$ *be a complete and atomic Boolean lattice and let* $\mathbf{L}$ *be an algebraic and distributive lattice satisfying* **(S)** *of Theorem* 2.2. *Then the following two conditions are equivalent:*

(a) *There is a (total) unary algebra* $\mathbf{A}$ *such that* $\mathbf{S}_r(\mathbf{A}) \simeq \mathbf{K}$ *and* $\mathbf{S}_i(\mathbf{A}) \simeq \mathbf{L}$ *(i.e.,* $\mathbf{K}$ *and* $\mathbf{L}$ *are isomorphic to lattices of relative subalgebras and initial segments of* $\mathbf{A}$*, respectively).*
(b) $|\mathcal{CI}(\mathbf{L})| \leq |\mathcal{A}(\mathbf{K})|$.

Having Proposition 5.3(4) we can translate results from [12] (about strong subdigraph lattices and strong subalgebra lattices) for dually-strong subdigraph lattices and next, for the case of partial unary algebras and their initial segment lattices. For example, we obtain the following analogue of Theorem 2.6 from [12]

Corollary 5.6. *Let* $\mathbf{D}$ *be a digraph and* θ *be an equivalence relation on* $V^{\mathbf{D}}$ *such that*

(∗) *for all* $v, w \in V^{\mathbf{D}}$*, if* $v\theta w$ *and* $v \neq w$*, then there is a (directed) path going from* v *to* w.

Then $\mathbf{S}_d(\mathbf{D}/\theta) \simeq \mathbf{S}_d(\mathbf{D})$.

Because, if $(u_1, \dots, u_n)$ is a path going from v to w in $\mathbf{D}$, then $(u_n, \dots, u_1)$ is a path in $\overline{\mathbf{D}}$ going from w to v.

Further, by Theorems 3.9 and 3.11 from [12] (see also Theorem 2.14) we obtain

Corollary 5.7. *Let* $\mathbf{L}$ *be an algebraic and distributive lattice satisfying* **(S)** *of Theorem* 2.2 *and let* $\mathbf{D}$ *be a digraph. Then the following three conditions are equivalent:*

(a) $\mathbf{S}_d(\mathbf{D}) \simeq \mathbf{L}$,
(b) $\langle V^{\mathbf{D}/\Theta(\mathbf{D})}, \leq_{\mathbf{D}/\Theta(\mathbf{D})}\rangle \simeq \overline{\langle \mathcal{CI}(\mathbf{L}), \leq_{\mathbf{L}}\rangle}$,
(c) $\langle V^{\mathbf{D}/\Theta(\mathbf{D})}, \leq_{\mathbf{D}/\Theta(\mathbf{D})}\rangle \simeq \overline{\langle \mathcal{CMI}(\mathbf{L}), \leq_{\mathbf{L}}\rangle}$,

where $\mathcal{CMI}(\mathbf{L})$ *is the family of all the completely meet-irreducible elements of* $\mathbf{L}$.

Recall that an element l of a complete lattice $\mathbf{L} = \langle L, \leq_{\mathbf{L}}\rangle$ is completely meet-irreducible if for each subset $K \subseteq L$, $l = \bigwedge K$ implies $l \in K$ (where $\bigwedge X$ denotes the infimum of a set X in $\mathbf{L}$).

Proof. (a)$\Longleftrightarrow$(b) By Theorem 2.14, Proposition 5.3(4) and Lemma 5.2(3), (4), (5) we obtain that

$$\begin{aligned}
&\mathbf{S}_d(\mathbf{D}) \simeq \mathbf{L} \ \text{ iff } \ \mathbf{S}_s(\overline{\mathbf{D}}) \simeq \mathbf{L} \ \text{ iff}\\
&\langle V^{\overline{\mathbf{D}}/\Theta(\overline{\mathbf{D}})}, \leq_{\overline{\mathbf{D}}/\Theta(\overline{\mathbf{D}})}\rangle \simeq \langle \mathcal{CI}(\mathbf{L}), \leq_{\mathbf{L}}\rangle \ \text{ iff}\\
&\langle V^{\overline{\mathbf{D}/\Theta(\mathbf{D})}}, \leq_{\overline{\mathbf{D}/\Theta(\mathbf{D})}}\rangle \simeq \langle \mathcal{CI}(\mathbf{L}), \leq_{\mathbf{L}}\rangle \ \text{ iff}\\
&\overline{\langle V^{\mathbf{D}/\Theta(\mathbf{D})}, \leq_{\mathbf{D}/\Theta(\mathbf{D})}\rangle} \simeq \langle \mathcal{CI}(\mathbf{L}), \leq_{\mathbf{L}}\rangle \ \text{ iff}\\
&\langle V^{\mathbf{D}/\Theta(\mathbf{D})}, \leq_{\mathbf{D}/\Theta(\mathbf{D})}\rangle \simeq \overline{\langle \mathcal{CI}(\mathbf{L}), \leq_{\mathbf{L}}\rangle}.
\end{aligned}$$

(a)$\Longleftrightarrow$(c) By Theorem 2.14 and Proposition 4.2 we obtain that

$$\begin{aligned}
&\mathbf{S}_d(\mathbf{D}) \simeq \mathbf{L} \ \text{ iff } \ \overline{\mathbf{S}_d(\mathbf{D})} \simeq \overline{\mathbf{L}} \ \text{ iff}\\
&\mathbf{S}_s(\mathbf{D}) \simeq \overline{\mathbf{L}} \ \text{ iff } \ \langle V^{\mathbf{D}/\Theta(\mathbf{D})}, \leq_{\mathbf{D}/\Theta(\mathbf{D})}\rangle \simeq \langle \mathcal{CI}(\overline{\mathbf{L}}), \leq_{\overline{\mathbf{L}}}\rangle.
\end{aligned}$$

Next, it is easy to see that $\mathcal{CI}(\overline{\mathbf{L}}) = \mathcal{CMI}(\mathbf{L})$ and

$$\langle \mathcal{CI}(\overline{\mathbf{L}}), \leq_{\overline{\mathbf{L}}}\rangle = \overline{\langle \mathcal{CMI}(\mathbf{L}), \leq_{\mathbf{L}}\rangle}. \ \blacksquare$$

By Theorem 2.2 and the equivalence (b)$\Longleftrightarrow$(c) of Corollary 5.7 we have that

Remark 3. Let $\mathbf{L}$ be an algebraic and distributive lattice satisfying **(S)** of Theorem 2.2. Then

$$\langle \mathcal{CMI}(\mathbf{L}), \leq_{\mathbf{L}}\rangle \simeq \langle \mathcal{CI}(\mathbf{L}), \leq_{\mathbf{L}}\rangle.$$

Clearly, the sets $\mathcal{CMI}(\mathbf{L})$ and $\mathcal{CI}(\mathbf{L})$ are distinct in general (and even disjoint).

Corollary 5.8. *Let* $\mathbf{D}$, $\mathbf{H}$ *be arbitrary digraphs. Then*

$$\mathbf{S}_d(\mathbf{D}) \simeq \mathbf{S}_d(\mathbf{H}) \quad \textit{iff} \quad \langle V^{\mathbf{D}/\Theta(\mathbf{D})}, \leq_{\mathbf{D}/\Theta(\mathbf{D})}\rangle \simeq \langle V^{\mathbf{H}/\Theta(\mathbf{H})}, \leq_{\mathbf{H}/\Theta(\mathbf{H})}\rangle.$$

Proof. First, by Proposition 5.3(4),

$$\mathbf{S}_d(\mathbf{D}) \simeq \mathbf{S}_d(\mathbf{H}) \quad \text{iff} \quad \mathbf{S}_s(\overline{\mathbf{D}}) \simeq \mathbf{S}_s(\overline{\mathbf{H}}).$$

Secondly, by Theorem 3.11 from [12],

$$\mathbf{S}_s(\overline{\mathbf{D}}) \simeq \mathbf{S}_s(\overline{\mathbf{H}}) \quad \text{iff} \quad \langle V^{\overline{\mathbf{D}}/\Theta(\overline{\mathbf{D}})}, \leq_{\overline{\mathbf{D}}/\Theta(\overline{\mathbf{D}})}\rangle \simeq \langle V^{\overline{\mathbf{H}}/\Theta(\overline{\mathbf{H}})}, \leq_{\overline{\mathbf{H}}/\Theta(\overline{\mathbf{H}})}\rangle.$$

Thirdly, by Lemma 5.2(3),(4),(5),

$$\langle V^{\overline{\mathbf{D}}/\Theta(\overline{\mathbf{D}})}, \leq_{\overline{\mathbf{D}}/\Theta(\overline{\mathbf{D}})}\rangle \simeq \langle V^{\overline{\mathbf{D}/\Theta(\mathbf{D})}}, \leq_{\overline{\mathbf{D}/\Theta(\mathbf{D})}}\rangle \simeq \overline{\langle V^{\mathbf{D}/\Theta(\mathbf{D})}, \leq_{\mathbf{D}/\Theta(\mathbf{D})}\rangle}$$

and

$$\langle V^{\overline{\mathbf{H}}/\Theta(\overline{\mathbf{H}})}, \leq_{\overline{\mathbf{H}}/\Theta(\overline{\mathbf{H}})}\rangle \simeq \langle V^{\overline{\mathbf{H}/\Theta(\mathbf{H})}}, \leq_{\overline{\mathbf{H}/\Theta(\mathbf{H})}}\rangle \simeq \overline{\langle V^{\mathbf{H}/\Theta(\mathbf{H})}, \leq_{\mathbf{H}/\Theta(\mathbf{H})}\rangle}.$$

These facts complete the proof, because partially ordered sets are isomorphic if and only if their dual are isomorphic. ∎

By Corollaries 5.7 and 5.8 we obtain

Corollary 5.9. *Let* $\mathbf{L}$ *be an algebraic and distributive lattice satisfying* **(S)** *of Theorem* 2.2 *and let* $\mathbf{A}$ *be a partial unary algebra. Then the following three conditions are equivalent:*

(a) $\mathbf{S}_d(\mathbf{A}) \simeq \mathbf{L}$,
(b) $\langle V^{\mathbf{D}(\mathbf{A})/\Theta(\mathbf{D}(\mathbf{A}))}, \leq_{\mathbf{D}(\mathbf{A})/\Theta(\mathbf{D}(\mathbf{A}))}\rangle \simeq \overline{\langle \mathcal{CI}(\mathbf{L}), \leq_{\mathbf{L}}\rangle}$,
(c) $\langle V^{\mathbf{D}(\mathbf{A})/\Theta(\mathbf{D}(\mathbf{A}))}, \leq_{\mathbf{D}(\mathbf{A})/\Theta(\mathbf{D}(\mathbf{A}))}\rangle \simeq \overline{\langle \mathcal{CMI}(\mathbf{L}), \leq_{\mathbf{L}}\rangle}$.

Corollary 5.10. *Let* $\mathbf{A}$, $\mathbf{B}$ *be arbitrary partial unary algebras (possible of different types). Then the following two conditions are equivalent:*

(a) $\mathbf{S}_d(\mathbf{A}) \simeq \mathbf{S}_d(\mathbf{B})$,
(b) $\langle V^{\mathbf{D}(\mathbf{A})/\Theta(\mathbf{D}(\mathbf{A}))}, \leq_{\mathbf{D}(\mathbf{A})/\Theta(\mathbf{D}(\mathbf{A}))}\rangle \simeq \langle V^{\mathbf{D}(\mathbf{B})/\Theta(\mathbf{D}(\mathbf{B}))}, \leq_{\mathbf{D}(\mathbf{B})/\Theta(\mathbf{D}(\mathbf{B}))}\rangle$.

Note that other results from [12] can be similarly translated for digraphs and their dually-strong subdigraph lattices and next, for partial unary algebras and their initial segment lattices.

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