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DENSITY TOPOLOGY GENERATED BY THE CONVERGENCE EVERYWHERE EXCEPT FOR A FINITE SET

Abstract. In this paper we shall study a density-type topology generated by the convergence everywhere except for a finite set similarly as the classical density topology is generated by the convergence in measure. Among others it is shown that the set of finite density points of a measurable set need not be measurable.

1. Introduction

Throughout the paper $\mathcal{L}$ will denote the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$ and λ - the Lebesgue measure on the real line. Let $\mathcal{B}(\mathbb{R})$ be the σ-algebra of Borel sets on the real line. We shall use also the following notation: $nA = \{nx : x \in A\}$, $A - a = \{x - a : x \in A\}$ for $A \subset \mathbb{R}$, $n \in \mathbb{N}$, $a \in \mathbb{R}$ and $A' = \mathbb{R} \backslash A$. By χ_A we denote the characteristic function of a set A. Recall that

Definition 1. The point $x \in \mathbb{R}$ is a density point of a set $A \in \mathcal{L}$ if and only if

$$\lim_{h \to 0^+} \frac{\lambda(A \cap [x-h, x+h])}{2h} = 1. \tag{1}$$

Observe that the condition (1) is equivalent to the following statement

$\left\{\chi_{n(A-x)\cap[-1,1]}\right\}_{n\in\mathbb{N}}$ converges in measure to $\chi_{[-1,1]}$ (see [PWW]).

Put

$$\Phi(A) = \{x \in \mathbb{R} : x \text{ is a point of density of } A\}$$

for all $A \in \mathcal{L}$. It is well known that the family

$$\mathcal{T}_d = \{A \in \mathcal{L} : A \subset \Phi(A)\}$$

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is a topology on the real line (called the density topology) stronger than natural topology $\mathcal{T}_{nat}$ on the real line (see [O], Chapter 22 or [GNN]). Since the point of density is characterized by the convergence in measure, we shall say that a density topology is generated by the convergence in measure. Using various kinds of convergence of a sequence of measurable functions one can obtain different density-type topologies on the real line (see [W]). Thus, for example, the density-type topology generated by the convergence almost everywhere is called the simple density topology and denoted by $\mathcal{T}_s$ (see [WA]), the density-type topology generated by the complete convergence is called the complete density topology and denoted by $\mathcal{T}_c$ (see [WW]). Observe also that the natural topology $\mathcal{T}_{nat}$ can be considered as a density-type topology generated by the uniform convergence. However, in all above mentioned cases the analogue of the Lebesgue Density Theorem does not hold. Observe also that if a set $A \subset \mathbb{R}$ is measurable, then the set $\Phi_s(A)$, $\Phi_c(A)$ and $\Phi_u(A) = IntA$ (the sets of all points of simple, complete and uniform, respectively, density points of A) are measurable and that $\mathcal{T}_{nat} \subsetneqq \mathcal{T}_c \subsetneqq \mathcal{T}_s \subsetneqq \mathcal{T}_d$. In this paper we introduce the finite density topology $\mathcal{T}_f$ which is generated by the convergence everywhere except for a finite set. This topology is significantly different from the mentioned above, because one can construct a measurable set A for which the set of all finite density points is a non-measurable set (Theorem 10). We also show that $\mathcal{T}_{nat} \subsetneqq \mathcal{T}_f$ (Theorem 19).

2. Finite density point

Firstly we introduce the concept of fin-density point in the family of Lebesgue measurable sets on the real line. We study the properties of the operator assigning to a set $A \in \mathcal{L}$ the set of its fin-density points and introduce a finite density topology.

Definition 2. Let $A \in \mathcal{L}$. We shall say that:

a) 0 is a fin-density point of the set A if and only if

$$\{\chi_{nA\cap[-1,1]}\}_{n\in\mathbb{N}} \text{ converges to } \chi_{[-1,1]} \text{ everywhere except for a finite set.}$$

b) $x \in \mathbb{R}$ is a fin-density point of the set A if and only if 0 is a fin-density point of the set $A - x$.

c) $x \in \mathbb{R}$ is a fin-dispersion point of the set A if and only if x is a fin-density point of the set A'.

d) 0 is a right-hand side fin-density point of the set A if and only if $\{\chi_{nA\cap[0,1]}\}_{n\in\mathbb{N}}$ converges to $\chi_{[0,1]}$ except for a finite set.

Directly from the above definition we have the following characterization of fin-density point

Proposition 3. *Let $A \in \mathcal{L}$. Then*

a) 0 *is a fin-density point of a set A if and only if there exists a finite set $F \subset [-1,1]$ such that $[-1,1]\backslash F \subset \liminf_{n\to\infty} nA$.*
b) 0 *is a fin-dispersion point of a set A if and only if there exists a finite set $F \subset [-1,1]$ such that $[-1,1]\backslash F \subset \liminf_{n\to\infty} nA'$.*
c) 0 *is a right-hand side fin-density point of a set A if and only if there exists a finite set $F \subset [0,1]$ such that $[0,1]\backslash F \subset \liminf_{n\to\infty} nA$.*

Put $\Phi_{fin}(A) = \{x \in \mathbb{R} : x \text{ is a } fin\text{-density point of the set } A\}$ for $A \in \mathcal{L}$. Obviously we have

Proposition 4. *Let $A \in \mathcal{L}$. Then $\Phi_{fin}(A) \subset \Phi_s(A) \subset \Phi(A)$.*

Proposition 5. *For each set $A \in \mathcal{L}$, $\lambda(\Phi_{fin}(A)\backslash A) = 0$.*

Now we shall show that the Lebesgue Density Theorem does not hold for fin-density points in the place of density points.

Proposition 6. *There exists a measurable set $C \subset [0,1]$ of positive measure such that $\Phi_{fin}(C) = \emptyset$.*

Proof. In [WA] it was shown that there exists a set $C \subset [0,1]$ such that $\lambda(C) > 0$ and $\Phi_s(C) = \emptyset$. Therefore $\Phi_{fin}(C) = \emptyset$ by Proposition 4. ∎

Corollary 7. *There exists a set $C \in \mathcal{L}$ such that $\lambda(\Phi_{fin}(C) \triangle C) > 0$.*

The operator $\Phi_{fin}(A)$ has the following properties

Theorem 8. *For each sets $A, B \in \mathcal{L}$:*

1. *if $A \subset B$, then $\Phi_{fin}(A) \subset \Phi_{fin}(B)$,*
2. *$\Phi_{fin}(\emptyset) = \emptyset$ and $\Phi_{fin}(\mathbb{R}) = \mathbb{R}$,*
3. *$\Phi_{fin}(A \cap B) = \Phi_{fin}(A) \cap \Phi_{fin}(B)$.*

Proof. 1. Let A,$B \in \mathcal{L}$ and $A \subset B$. Observe that for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$ we have $n(A - x) \subset n(B - x)$. If $x \in \Phi_{fin}(A)$, then there exists a finite set $F \subset [-1,1]$ such that

$$[-1,1]\backslash F \subset \liminf_{n\to\infty} n(A - x).$$

Thus

$$[-1,1]\backslash F \subset \liminf_{n\to\infty} n(A - x) \subset \liminf_{n\to\infty} n(B - x),$$

which means that $x \in \Phi_{fin}(B)$.

2. The proof is straightforward.

3. The proof of the inclusion $\Phi_{fin}(A \cap B) \subset \Phi_{fin}(A) \cap \Phi_{fin}(B)$ follows from monotonicity of the operator Φ_{fin}. Now we prove the converse inclusion. If $x \in \Phi_{fin}(A) \cap \Phi_{fin}(B)$, then there exists a finite set $F_1 \subset [-1,1]$

such that

$$[-1,1]\backslash F_1 \subset \liminf_{n\to\infty} n(A-x)$$

and there exists a finite set $F_2 \subset [-1,1]$ such that

$$[-1,1]\backslash F_2 \subset \liminf_{n\to\infty} n(B-x).$$

Let $F = F_1 \cup F_2$. Then

$$[-1,1]\backslash F \subset \liminf_{n\to\infty} n(A-x) \cap \liminf_{n\to\infty} n(B-x) = \liminf_{n\to\infty} n((A\cap B)-x).$$

Consequently we have that $x \in \Phi_{fin}(A\cap B)$. ∎

Remark 9. For each set $A \in \mathcal{L}$ and $y \in \mathbb{R}$ we have $\Phi_{fin}(A) + y = \Phi_{fin}(A+y)$.

3. The construction of measurable set for which the set of its *fin*-density point is non-measurable

Theorem 10. *Assume Martin's axiom. There exists a measurable set $A \subset \mathbb{R}$ such that $\Phi_{fin}(\mathbb{R}\backslash A)$ is non-measurable.*

Before we prove Theorem 10 we recall the following definitions and theorems which shall be useful in the proof.

Definition 11. (see [K], p. 60) A set $H \subset \mathbb{R}$ has the property $(*)$ if the following condition is satisfied: if $B \subset H$ or $B \subset \mathbb{R}\backslash H$, and B has the Baire property, then B is of the first category.

Definition 12. (see [K], p. 58) A set $A \subset \mathbb{R}$ is called saturated non-measurable if $\lambda_*(A) = \lambda_*(\mathbb{R}\backslash A) = 0$, where λ_* is the Lebesgue inner measure.

Definition 13. (see [K], p. 258) A Hamel basis $H \subset \mathbb{R}$ is called a Burstin basis in $\mathbb{R}$ if for each set $B \in \mathcal{B}(\mathbb{R})$ such that $card(B) > \varkappa_0$ we have that $H \cap B \neq \emptyset$.

Theorem 14. (see [K], p. 258) *Every Burstin basis in $\mathbb{R}$ is saturated non-measurable.*

Theorem 15. (see [K], p. 258) *Every Burstin basis in $\mathbb{R}$ fulfills condition $(*)$.*

Theorem 16. (see [K], pp. 259, 260) *There exists a Burstin basis in $\mathbb{R}$.*

Proof of Theorem 10. Let $H \subset \mathbb{R}$ be a Burstin basis. From Theorem 15 it follows that H and $\mathbb{R}\backslash H$ have the property $(*)$. Applying Theorem 14 we conclude that the set H is saturated non-measurable. We now use the fact that we can decompose the real line into two disjoint sets E_1, E_2 such that

E_1 is a null set and E_2 is a set of the first category (see for instance [O], p. 4). Then

$$H = (H \cap E_1) \cup (H \cap E_2).$$

Obviously $H \cap E_1$ is a null set and $H \cap E_2$ is of the first category. Observe that if $A \subset \mathbb{R}$ is saturated non-measurable and $\lambda(A \triangle B) = 0$, then B is also saturated non-measurable. Similarly, if $A \subset \mathbb{R}$ fulfills property $(*)$ and $A \triangle B$ is of the first category, then B also fulfills property $(*)$. Hence $H \cap E_2 = H \setminus (H \cap E_1)$ is saturated non-measurable and $H \cap E_1 = H \setminus (H \cap E_2)$ fulfills property $(*)$.

The construction of the set A. The identity $card(H \cap E_1) = card(H \cap E_2) = c$, follows from Martin's axiom. Let $\{z_0, z_1, \dots, z_\alpha, \dots\}_{\alpha < \omega_c}$ be a well-ordering of $H \cap E_2$ and let $F_2, F_3, F_4, \dots$ be arbitrary residual and null sets. Take $z_0 \in H \cap E_2$.

Now we show that for all $i \in \mathbb{N}$ there exists a straight line $l_{z_0,i}$ such that $l_{z_0,i}$ contains $(z_0, 0), (h_{0,i}, 1)$ for some $h_{0,i} \in H \cap E_1$, in addition $h_{0,i} \neq h_{0,j}$ for $i \neq j$ and $l_{z_0,i}$ also has non-empty intersection with every set $F_n \times \{\frac{1}{n}\}$, $n \geq 2$.

Consider the set L of all straight lines l_{z_0} such that l_{z_0} contains point $(z_0, 0)$ and l_{z_0} is neither vertical nor horizontal. The set of all direction coefficients of the lines from the set L is equal to $\mathbb{R} \setminus \{0\}$ and is obviously residual and of full measure. Now we define a sequence of homeomorphisms between the set $\mathbb{R} \setminus \{z_0\}$ and the set of all direction coefficients of the lines from L in the following way

$$f_{n,z_0}(x) = \frac{1}{n(x - z_0)} \quad \text{for } x \in \mathbb{R} \setminus \{z_0\} \quad \text{and } n \in \mathbb{N}. \tag{2}$$

Homeomorphism f_{n,z_0} maps a first coordinate of the point $(x, \frac{1}{n}) \in (\mathbb{R} \setminus \{z_0\}) \times \{\frac{1}{n}\}$ into the direction coefficient $a \in \mathbb{R} \setminus \{0\}$ of the line $l_{z_0} : y = a(x - z_0)$ which goes through $\left(x, \frac{1}{n}\right)$.

Notice that $W_{n,z_0} = f_{n,z_0}(\mathbb{R} \setminus (\{z_0\} \cup F_n))$ is of the first category for each $n \geq 2$ and $D_0 = \mathbb{R} \setminus (\{0\} \cup \bigcup_{n=2}^{\infty} W_{n,z_0})$ is the set of direction coefficients of those lines l_{z_0} which have non-empty intersection with every set $F_n \times \{\frac{1}{n}\}$, $n \geq 2$. It is easily seen that the set D_0 is residual.

Now we prove that for all $i \in \mathbb{N}$ there exists direction coefficients $a_{0,i} \in D_0$ such that the line $l_{z_0,i}$: $y = a_{0,i}(x - z_0)$ contains a point $(h_{0,i}, 1)$, where $h_{0,i} \in H \cap E_1$ and $h_{0,i} \neq h_{0,j}$ for $i \neq j$ (i.e. $f_{1,z_0}^{-1}(a_{0,i}) \in H \cap E_1$ for all $i \in \mathbb{N}$).

The proof of this fact is by mathematical induction.

Let $i = 1$ and put $D_{0,1} = D_0$. On the contrary, suppose that $f_{1,z_0}^{-1}(a_{0,1}) \notin H \cap E_1$ for all $a_{0,1} \in D_0$. It means that

$$f_{1,z_0}^{-1}(D_0) \subset \mathbb{R} \setminus (H \cap E_1).$$

Since the set $\mathbb{R}\setminus(H \cap E_1)$ has the property $(*)$, then it does not contain a residual set $f_{1,z_0}^{-1}(D_{0,1})$. Thus we get the contradiction. Let us denote by $(h_{0,1}, 1)$ the point of intersection of straight line $l_{z_0,1} : y = a_{0,1}(x - z_0)$ with the set $(H \cap E_1) \times \{1\}$. Let us assume that for some $k \geq 1$ we have chosen pairwise different direction coefficients $a_{0,1}, a_{0,2}, \ldots, a_{0,k}$, pairwise different elements of $(H \cap E_1)$: $h_{0,1}, h_{0,2}, \ldots, h_{0,k}$ and put $D_{0,k} = D_0 \setminus \{a_{0,1}, a_{0,2}, \ldots, a_{0,k}\}$. Obviously the set $D_{0,k}$ is residual. The proof of existence of direction coefficient $a_{0,k+1}$ is analogus to proof of the existence $a_{0,1}$. In this way for $z_0 \in H \cap E_2$ we have found sequences $\{a_{0,i}\}_{i\in\mathbb{N}}$ and $\{h_{0,i}\}_{i\in\mathbb{N}}$ consisting of different terms with values in $\mathbb{R} \setminus \{0\}$ and $(H \cap E_1)$, respectively, such that

(3) $\quad f_{1,z_0}^{-1}(a_{0,i}) \in H \cap E_1$ and $f_{n,z_0}^{-1}(a_{0,i}) \in F_n$ for all $n \geq 2$ and for all $i \in \mathbb{N}$.

Applying the definitions of homeomorphisms $f_{1,z_0}, f_{2,z_0}, f_{3,z_0}, \ldots$ we can express (3) in the following way

$$\frac{1}{n}(h_{0,i} - z_0) + z_0 \in F_n \text{ for all } n \geq 2 \text{ and for all } i \in \mathbb{N}. \tag{4}$$

Let $\alpha < \omega_c$. Suppose that for every $z_\beta \in H \cap E_2$, $\beta < \alpha$ and for every $i \in \mathbb{N}$ we have found $a_{\beta,i} \in \mathbb{R}\setminus\{0\}$ and $h_{\beta,i} \in (H \cap E_1)\setminus \bigcup_{i\in\mathbb{N}} \{h_{\gamma,i} : \gamma < \beta\}$ such that

$$f_{1,z_\beta}^{-1}(a_{\beta,i}) \in H \cap E_1 \text{ and } f_{n,z_\beta}^{-1}(a_{\beta,i}) \in F_n \text{ for all } n \geq 2 \text{ and for all } i \in \mathbb{N}.$$

Applying the definitions of homeomorphisms $f_{1,z_\beta}, f_{2,z_\beta}, f_{3,z_\beta}, \ldots$ defined by

$$f_{n,z_\beta}(x) = \frac{1}{n(x - z_\beta)} \text{ for } x \in \mathbb{R}\setminus\{z_\beta\} \tag{5}$$

we can rewrite (5) in the following way:

$$\frac{1}{n}(h_{\beta,i} - z_\beta) + z_\beta \in F_n \text{ for all } n \geq 2 \text{ and for all } i \in \mathbb{N}. \tag{6}$$

Now, for $z_\alpha \in H \cap E_2$ we shall find $h_{\alpha,i} \in (H \cap E_1)\setminus \bigcup_{i\in\mathbb{N}}\{h_{\beta,i} : \beta < \alpha\}$ and direction coefficients $a_{\alpha,i} \in \mathbb{R}\setminus\{0\}$ such that $\frac{1}{n}(h_{\alpha,i} - z_\alpha) + z_\alpha \in F_n$ for every $n, i \in \mathbb{N}$. The set $\bigcup_{i\in\mathbb{N}}\{h_{\beta,i} : \beta < \alpha\}$ is of the first category by Martin's axiom. We will denote by l_{z_α} arbitrary straight line containing point $(z_\alpha, 0)$. We can now proceed analogously to the proof of existence $h_{0,i}$ and $a_{0,i}$ for z_0 to obtain the set

$$D_\alpha = \mathbb{R}\setminus\Big(\{0\} \cup \bigcup_{n=2}^{\infty} W_{n,z_\alpha}\Big).$$

It is the set of direction ratios of those lines l_{z_α} which have non-empty intersection with every set $F_n \times \left\{\frac{1}{n}\right\}$, $n \geq 2$. Obviously the set D_α is residual.

Now we prove that for all $i \in \mathbb{N}$ there exists direction ratio $a_{\alpha,i} \in D_\alpha$ such that the line $l_{z_\alpha,i}$: $y = a_{\alpha,i}(x - z_\alpha)$ intersects the set

$$\Big((H \cap E_1) \setminus \bigcup_{i\in\mathbb{N}} \{h_{\beta,i} : \beta < \alpha\}\Big) \times \{1\}$$

(i.e. $f^{-1}_{1,z_\alpha}(a_{\alpha,i}) \in (H \cap E_1) \setminus \bigcup_{i\in\mathbb{N}} \{h_{\beta,i} : \beta < \alpha\}$ for all $i \in \mathbb{N}$).

The proof of this fact is by mathematical induction.

Let $i = 1$ and $D_{\alpha,1} = D_\alpha$. On the contrary, suppose that

$$f^{-1}_{1,z_\alpha}(a_{\alpha,1}) \notin (H \cap E_1) \setminus \bigcup_{i\in\mathbb{N}} \{h_{\beta,i} : \beta < \alpha\} \text{ for all } a_{\alpha,1} \in D_{\alpha,1}.$$

It means that

$$\begin{aligned} f^{-1}_{1,z_\alpha}(D_{\alpha,1}) &\subset \mathbb{R} \setminus \Big((H \cap E_1) \setminus \bigcup_{i\in\mathbb{N}} \{h_{\beta,i} : \beta < \alpha\}\Big) \\ &= [\mathbb{R} \setminus (H \cap E_1)] \cup \bigcup_{i\in\mathbb{N}} \{h_{\beta,i} : \beta < \alpha\}. \end{aligned}$$

Hence

$$f^{-1}_{1,z_\alpha}(D_{\alpha,1}) \setminus \bigcup_{i\in\mathbb{N}} \{h_{\beta,i} : \beta < \alpha\} \subset \mathbb{R} \setminus (H \cap E_1).$$

Since the set $\mathbb{R} \setminus (H \cap E_1)$ has the property $(*)$, then it cannot contain a residual set $f^{-1}_{1,z_\alpha}(D_{\alpha,1}) \setminus \bigcup_{i\in\mathbb{N}} \{h_{\beta,i} : \beta < \alpha\}$. Finally, we got the contradiction.

Let us denote by $(h_{\alpha,1}, 1)$ the point of intersection of straight line $l_{z_\alpha,1} : y = a_{\alpha,1}(x - z_\alpha)$ with the set $((H \cap E_1) \setminus \bigcup_{i\in\mathbb{N}} \{h_{\beta,i} : \beta < \alpha\}) \times \{1\}$. Let us assume that for some $k \geq 1$ we have chosen pairwise different direction coefficients $a_{\alpha,1}, a_{\alpha,2}, \ldots, a_{\alpha,k}$, pairwise different elements of $(H \cap E_1)$: $h_{\alpha,1}, h_{\alpha,2}, \ldots, h_{\alpha,k}$ and put $D_{\alpha,k} = D_\alpha \setminus \{a_{\alpha,1}, a_{\alpha,2}, \ldots, a_{\alpha,k}\}$. Obviously the set $D_{\alpha,k}$ is residual. The proof of the existence of the direction coefficient $a_{\alpha,k+1}$ is analogous to the proof of the existence of $a_{\alpha,1}$.

In this way for $z_\alpha \in H \cap E_2$ we have found sequences $\{a_{\alpha,i}\}_{i\in\mathbb{N}}$ and $\{h_{\alpha,i}\}_{i\in\mathbb{N}}$ consisting of different terms with values at $\mathbb{R} \setminus \{0\}$ and $(H \cap E_1) \setminus \bigcup_{i\in\mathbb{N}} \{h_{\beta,i} : \beta < \alpha\}$, respectively, such that

$$f^{-1}_{1,z_\alpha}(a_{\alpha,i}) \in H \cap E_1 \text{ and } f^{-1}_{n,z_\alpha}(a_{\alpha,i}) \in F_n \text{ for all } n \geq 2 \text{ and for all } i \in \mathbb{N}. \tag{7}$$

We can rewrite (7) in following way

$$\frac{1}{n}(h_{\alpha,i} - z_\alpha) + z_\alpha \in F_n \text{ for all } n \geq 2 \text{ and for all } i \in \mathbb{N}. \tag{8}$$

We have constructed the set

$$A = \bigcup_{n=2}^{\infty} \bigcup_{i=1}^{\infty} \left\{ z_\alpha + \frac{1}{n}(h_{\alpha,i} - z_\alpha) : \alpha < \omega_c \right\} \cup \bigcup_{i=1}^{\infty} \{h_{\alpha,i} : \alpha < \omega_c\}.$$

Observe that for all $n \geq 2$ and $i \in \mathbb{N}$ the set $\{z_\alpha + \frac{1}{n}(h_{\alpha,i} - z_\alpha) : \alpha < \omega_c\} \subset F_n$ and $\bigcup_{i\in\mathbb{N}} \{h_{\alpha,i} : \alpha < \omega_c\} \subset H \cap E_1$. So all these sets are null sets. Hence $\lambda(A) = 0$.

We shall show below that $\Phi_{fin}(\mathbb{R}\backslash A) = \mathbb{R}\backslash(H \cap E_2)$.

1. First we shall show that $\Phi_{fin}(\mathbb{R}\backslash A) \subset \mathbb{R}\backslash(H \cap E_2)$.

Notice that from the definition of A we get that

$$\begin{aligned} z_\alpha + h_{\alpha,i} - z_\alpha &\in A, \\ z_\alpha + \frac{1}{2}(h_{\alpha,i} - z_\alpha) &\in A, \\ z_\alpha + \frac{1}{3}(h_{\alpha,i} - z_\alpha) &\in A, \\ &\dots \\ z_\alpha + \frac{1}{n}(h_{\alpha,i} - z_\alpha) &\in A, \\ &\dots \end{aligned}$$

for each $\alpha < \omega_c$, i.e. for each $z \in H \cap E_2$ and for each $n, i \in \mathbb{N}$. Hence $(h_{\alpha,i} - z_\alpha) \notin \mathbb{R}\backslash(n(A - z_\alpha)) = n((\mathbb{R}\backslash A) - z_\alpha)$ for every $n, i \in \mathbb{N}$, and $\alpha < \omega_c$. From the above we conclude that $z_\alpha \notin \Phi_{fin}(\mathbb{R}\backslash A)$ for every $\alpha < \omega_c$. Hence $(H \cap E_2) \subset \mathbb{R}\backslash\Phi_{fin}(\mathbb{R}\backslash A)$.

2. To prove the converse inclusion it is sufficient to show that if $x \in \mathbb{R} \setminus (H \cap E_2)$, then for each $a \in [-1, 1] \setminus \{0\}$ there exists $m \in \mathbb{N}$ such that for each $k > m$ we have $\frac{a}{k} + x \notin A$. To prove the last statement we show that at most two of the numbers $\frac{a}{k} + x$, $k = 1, 2, \dots$ belong to A. To obtain a contradiction, suppose that there exist three different numbers $t_1, t_2, t_3 \in A$ such that

$$\begin{aligned} t_1 &= \frac{a}{k_1} + x, \\ t_2 &= \frac{a}{k_2} + x, \\ t_3 &= \frac{a}{k_3} + x \end{aligned}$$

for different k_1, k_2, k_3, $k_i \in \mathbb{N}$, $i = 1, 2, 3$. From the above it follows that

$$a = k_1(t_1 - x) = k_2(t_2 - x) = k_3(t_3 - x).$$

Hence we have $x = \frac{k_2t_2 - k_1t_1}{k_2 - k_1}$ and $x = \frac{k_3t_3 - k_1t_1}{k_3 - k_1}$. Consequently

$$(k_2t_2 - k_1t_1)(k_3 - k_1) = (k_3t_3 - k_1t_1)(k_2 - k_1).$$

Since $t_i = z_{\alpha_i} + \frac{1}{n_i}(h_{\alpha_i,i} - z_{\alpha_i})$ for $i = 1, 2, 3$, where z_{α_i} and $h_{\alpha_i,i}$ are different for $i = 1, 2, 3$, we obtain

$$z_{\alpha_2}\left(k_2k_3 - \frac{1}{n_2}k_2k_3 - k_1k_2 + \frac{1}{n_2}k_1k_2\right) +$$
$$z_{\alpha_1}\left(-k_1k_3 + \frac{1}{n_1}k_1k_3 + k_1k_2 - \frac{1}{n_1}k_1k_2\right) +$$
$$z_{\alpha_3}\left(-k_2k_3 + \frac{1}{n_3}k_2k_3 + k_1k_3 - \frac{1}{n_3}k_1k_3\right) +$$
$$h_{\alpha_1,1}\left(-\frac{1}{n_1}k_1k_3 + \frac{1}{n_1}k_1k_2\right) + h_{\alpha_2,2}\left(\frac{1}{n_2}k_2k_3 - \frac{1}{n_2}k_1k_2\right) +$$
$$h_{\alpha_1,3}\left(-\frac{1}{n_3}k_2k_3 + \frac{1}{n_3}k_1k_3\right) = 0.$$

Since $z_{\alpha_i} \in H \cap E_2$, $h_{\alpha_i,i} \in H \cap E_1$ are linearly independent we get the system of equalities

$$\begin{cases} k_2k_3 - \frac{1}{n_2}k_2k_3 - k_1k_2 + \frac{1}{n_2}k_1k_2 = 0 \\ -k_1k_3 + \frac{1}{n_1}k_1k_3 + k_1k_2 - \frac{1}{n_1}k_1k_2 = 0 \\ -k_2k_3 + \frac{1}{n_3}k_2k_3 + k_1k_3 - \frac{1}{n_3}k_1k_3 = 0 \\ -\frac{1}{n_1}k_1k_3 + \frac{1}{n_1}k_1k_2 = 0 \\ \frac{1}{n_2}k_2k_3 - \frac{1}{n_2}k_1k_2 = 0 \\ -\frac{1}{n_3}k_2k_3 + \frac{1}{n_3}k_1k_3 = 0. \end{cases}$$

From the last three equation we obtain immediately $k_1 = k_2 = k_3$, contrary to the fact that k_i are different for $i = 1, 2, 3$.

Hence

$$\Phi_{fin}(\mathbb{R}\backslash A) = \mathbb{R}\backslash(H \cap E_2).$$

The set $H \cap E_2$ is saturated non-measurable, so $\Phi_{fin}(\mathbb{R}\backslash A)$ is non-measurable. Consequently, we have that $\mathbb{R}\backslash A$ is measurable, but $\Phi_{fin}(\mathbb{R}\backslash A)$ is nonmeasurable. ∎

Similarly, we can prove

Theorem 17. *Assume Martin's axiom. There exists a set $A \subset \mathbb{R}$ having the Baire property such that the set $\Phi_{fin}(\mathbb{R}\backslash A)$ has not the Baire property.*

4. Finite density topology

We define

$$\mathcal{T}_f = \{A \in \mathcal{L} : A \subset \Phi_{fin}(A)\}. \tag{9}$$

Theorem 18. *The family $\mathcal{T}_f$ is a topology on the real line.*

Proof. 1. From the properties of Φ_{fin} it follows immediately that $\emptyset = \Phi_{fin}(\emptyset)$ and $\mathbb{R} = \Phi_{fin}(\mathbb{R})$. Hence $\emptyset, \mathbb{R} \in \mathcal{T}_f$.

2. Let A, $B \in \mathcal{T}_f$ then $A \subset \Phi_{fin}(A)$, $B \subset \Phi_{fin}(B)$ and $A, B \in \mathcal{L}$. Hence $A \cap B \in \mathcal{L}$ and $A \cap B \subset \Phi_{fin}(A) \cap \Phi_{fin}(B) = \Phi_{fin}(A \cap B)$. We thus obtain that $A \cap B \in \mathcal{T}_f$.

3. To prove that $\mathcal{T}_f$ is closed under arbitrary unions observe that from Proposition 4 it follows that $\mathcal{T}_f \subset \mathcal{T}_d$. Let $A_t \in \mathcal{T}_f$ for all $t \in T$. Obviously $A_t \in \mathcal{T}_d$ for all $t \in T$. It is well known that $\mathcal{T}_d \subset \mathcal{L}$ and $\mathcal{T}_d$ is a topology. From this we have $\bigcup\limits_{t \in T} A_t \in \mathcal{L}$. From monotonicity of the operator Φ_{fin} we have that

$$A_t \subset \Phi_{fin}(A_t) \subset \Phi_{fin}\Big(\bigcup_{t \in T} A_t\Big) \quad \text{for each} \ \ t \in T.$$

Finally, we obtain $\bigcup\limits_{t \in T} A_t \subset \Phi_{fin}(\bigcup_{t \in T} A_t)$. ∎

We call the topology $\mathcal{T}_f$ the finite density topology.

Theorem 19. *Topology $\mathcal{T}_f$ is stronger than the natural topology.*

Proof. We first prove that any open set in natural topology is open in fin-density topology. Consider any nonempty set $I \in \mathcal{T}_{nat}$. Obviously $I \in \mathcal{L}$. Let $x \in I$. There exists $\delta > 0$ such that $(-\delta, \delta) \subset I - x$. We have

$$\mathbb{R} = \liminf_{n \to \infty} n\,(-\delta, \delta) \subset \liminf_{n \to \infty} n(I - x).$$

We obtain that $[-1, 1] \subset \liminf\limits_{n \to \infty} n\,(I - x)$. Hence $I \in \mathcal{T}_f$.

We now give an example of a set $A \in \mathcal{L}$ such that $A \in \mathcal{T}_f \backslash \mathcal{T}_{nat}$. Choose $x_1 \in (0, 1]$, $x_2 \in \left(0, \frac{1}{2}\right]$, $x_1 > x_2$ such that $\frac{x_2}{x_1} \notin \mathbb{Q}$, next we choose $x_3 \in \left(0, \frac{1}{3}\right]$, $x_3 < x_2$ such that $\frac{x_3}{x_1}, \frac{x_3}{x_2} \notin \mathbb{Q}$ and so on. In this way we obtain the sequence $(x_n)_{n \in \mathbb{N}}$ which includes at most one rational number and is convergent to 0. Put $A = \mathbb{R} \backslash \{x_1, x_2, \dots\}$. Obviously $A \notin \mathcal{T}_{nat}$, $A \backslash \{0\} \in \mathcal{T}_{nat}$ and by the first part of the proof we conclude that $A \backslash \{0\} \subset \Phi_{fin}(A \setminus \{0\})$. To prove that $0 \in \Phi_{fin}(A)$ we show that $[-1, 1] \subset \liminf\limits_{n \to \infty} nA$. The last fact is equivalent to the statement: for each $x \in [-1, 1]$ there exists a natural number n_0 such that for $n \geq n_0$, $\frac{x}{n} \in A$. Let $x \in [-1, 0]$. Then $x \in nA$ for every $n \in \mathbb{N}$. Let us take $x \in (0, 1]$ and the sequence $\left(\frac{x}{k}\right)_{k \in \mathbb{N}}$. On the contrary, suppose that there exist two different numbers $\frac{x}{k_1}$, $\frac{x}{k_2} \notin A$, where $k_1, k_2 \in \mathbb{N}$. Then $\frac{x}{k_1} = x_{n_1}$ and $\frac{x}{k_2} = x_{n_2}$ and $n_1 \neq n_2$, where $x_{n_1}, x_{n_2} \notin A$. We obtain that

$$\frac{x_{n_2}}{x_{n_1}} = \frac{k_1}{k_2}.$$

The number $\frac{k_1}{k_2}$ is rational, which is in contradiction with the definition of the sequence $(x_n)_{n \in \mathbb{N}}$. We have proved that at most one of the number $\left(\frac{x}{k}\right)_{k \in \mathbb{N}}$ is not a member of a set A. Consequently $A \in \mathcal{T}_f \backslash \mathcal{T}_{nat}$. ∎

Proposition 20. *The set of irrational numbers is open in $\mathcal{T}_f$.*

Proof. Let $y \in \mathbb{R}\backslash\mathbb{Q}$. We shall now prove more that

$$[-1,1] \subset \liminf_{n\to\infty} n((\mathbb{R}\backslash\mathbb{Q}) - y).$$

This condition is equivalent to the following one: for each $x \in [-1,1]$ there exists $k \in \mathbb{N}$ such that for $n > k$ we have $\frac{x}{n} + y \in \mathbb{R}\backslash\mathbb{Q}$.

To prove the last statement we show that there exists at most one positive integer n such that $x \notin n\left((\mathbb{R}\backslash\mathbb{Q}) - y\right)$. Converslely, suppose that there are n_1, $n_2 \in \mathbb{N}$, $n_1 \neq n_2$, and $x \notin n_1((\mathbb{R}\backslash\mathbb{Q}) - y)$ and $x \notin n_2((\mathbb{R}\backslash\mathbb{Q}) - y)$.

Hence

$$\frac{x}{n_1} + y \notin \mathbb{R}\backslash\mathbb{Q} \;\text{ and }\; \frac{x}{n_2} + y \notin \mathbb{R}\backslash\mathbb{Q}\ ,$$

$$\frac{x}{n_1} + y = \frac{p_1}{q_1} \;\text{ and }\; \frac{x}{n_2} + y = \frac{p_2}{q_2}\ , \text{ where } p_1, p_2 \in \mathbb{Z}\ q_1, q_2 \in \mathbb{N}.$$

From the above equalities we obtain that

$$y = \frac{n_2\frac{p_2}{q_2} - n_1\frac{p_1}{q_1}}{n_2 - n_1} \in \mathbb{Q},$$

contrary to $y \in \mathbb{R}\backslash\mathbb{Q}$. Therefore $\mathbb{R}\backslash\mathbb{Q} \in \mathcal{T}_f$. ∎

Corollary 21. *The set of rational numbers is closed in $\mathcal{T}_f$.*

The idea of the next proposition and its proof came from Lemma 5.1 in [H].

Proposition 22. *There exists a nonempty perfect set $F \subset \mathbb{R}$ such that $(\mathbb{R}\backslash F) \cup \{x\} \in \mathcal{T}_f$ for each $x \in F$.*

Proof. Let H be any Hamel basis of the space of reals over the field of rational numbers, containing a nonempty perfect set F (see [K], p. 270). Since $\mathbb{R}\backslash F \in \mathcal{T}_{nat}$, Theorem 19 gives that $\mathbb{R}\backslash F \in \mathcal{T}_f$, which means that $\mathbb{R}\backslash F \subset \Phi_{fin}(\mathbb{R}\backslash F)$. We have to prove that each point $x \in F$ is a fin-density point of $(\mathbb{R}\backslash F) \cup \{x\}$. We shall show that

$$[-1,1] \subset \liminf_{n\to\infty} n\left[((\mathbb{R}\backslash F) \cup \{x\}) - x\right].$$

Let $a \in [-1,1]$. Clearly, we may assume that $a \neq 0$. There exists at most one positive integer n, such that $a \notin n((\mathbb{R}\backslash F) - x)$. To obtain a contradiction, suppose that we have n_1, $n_2 \in \mathbb{N}$, $n_1 \neq n_2$ such that $a \notin n_1((\mathbb{R}\backslash F) - x)$ and $a \notin n_2((\mathbb{R}\backslash F) - x)$. Consequently, $\frac{a}{n_1} + x = z_2$ and $\frac{a}{n_1} + x = z_1$, where $z_1, z_2 \in F$ and z_1, z_2, x are different. Hence we have:

$$x(n_1 - n_2) - z_2 n_2 + z_1 n_1 = 0.$$

Since z_1, z_2, x are different and z_1, z_2 and x are elements of a Hamel basis, therefore they are linearly independent. We obtain $n_1 = n_2 = 0$, contrary to the fact that $n_1 \neq n_2$. Finally,

$$(\mathbb{R}\backslash F) \cup \{x\} \subset \Phi_{fin}((\mathbb{R}\backslash F) \cup \{x\}) \text{ for each } x \in F.$$

It means that $(\mathbb{R}\backslash F) \cup \{x\} \in \mathcal{T}_f$. ∎

From the previous theorem it follows that $(\mathbb{R}\backslash F) \cup \{x\} \notin \mathcal{T}_{nat}$ for each $x \in F$.

Proposition 23. *The family $\mathcal{T}_{f\mathcal{B}} = \{A \in \mathcal{B}(\mathbb{R}) : A \subset \Phi_{fin}(A)\}$ is not a topology on the real line.*

Proof. By Proposition 22 there exists a nonempty perfect set $F \subset \mathbb{R}$ such that $(\mathbb{R}\backslash F) \cup \{x\} \subset \Phi_{fin}((\mathbb{R}\backslash F) \cap \{x\})$ for every $x \in F$. Let $C \subset F$, where C is not a Borel set. Consider $\{(\mathbb{R}\backslash F) \cup \{x\} : x \in C\}$. The set $(\mathbb{R}\backslash F) \cup \{x\}$ is a Borel set for each $x \in C$ and $(\mathbb{R}\backslash F) \cup \{x\} \in \mathcal{T}_{f\mathcal{B}}$. Simultaneously

$$\bigcup_{x \in C} ((\mathbb{R}\backslash F) \cup \{x\}) = (\mathbb{R}\backslash F) \cup C.$$

Obviously the set $\bigcup_{x \in C} ((\mathbb{R}\backslash F) \cup \{x\})$ is not a Borel set, therefore $\mathcal{T}_{f\mathcal{B}}$ is not closed under arbitrary unions. Thus finally, $\mathcal{T}_{f\mathcal{B}}$ is not a topology. ∎

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