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THE BOUNDARY VALUE PROBLEM OF HIGHER ORDER DIFFERENTIAL EQUATIONS WITH DELAY*

Abstract. In the paper, Guo–Krasnoselskii's fixed point theorem is adapted to study the existence of positive solutions to a class of boundary value problems for higher order differential equations with delay. The sufficient conditions, which assure that the equation has one positive solution or two positive solutions, are derived. These conclusions generalize some existing ones.

1. Introduction

The boundary value problems (bvps for short) for delay differential equations arise in a variety of areas of applied mathematics, physics and variational problems of control theory. In recent years, many researchers have done a great deal of research works upon bvps of lower order differential equations with delay, and some good results were produced, see, for example [1–9]. But higher order cases have not been focused. In fact, the bvps of higher order delay differential equations also have extensive applications in many fields [13]. Recently, Graef and Yang [10] considered the following multi-point higher order bvp of ordinary differential equation

$$u^{(n)}(t) + \lambda g(t) f(u(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

with boundary condition

$$u(0) = u'(0) = \cdots = u^{(n-3)}(0) = u^{(n-2)}(0) = \sum_{i=1}^{m} a_i u^{(n-2)}(1) = 0,$$

by using Krasnose'skii fixed point theorem, the authors obtained some sufficient conditions for the existence of positive solutions. While in [11], Shen

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and Dong studied the following bvp of higher order delay differential equation

$$u^{(n)}(t) + \lambda g(t) f(u(t-\tau)) = 0, \quad 0 < t < 1, \tag{1.2}$$

with boundary condition

$$u(t) = u'(t) = \cdots = u^{(n-3)}(t) = u^{(n-2)}(t) = 0, \quad -\tau \le t \le 0,$$
$$u^{(n-2)}(1) = 0.$$

Motivated by the above works, we will consider the existence of positive solutions for the bvp of higher order differential equation with delay as

$$\begin{cases} -u^{(n)}(t) = \lambda p(t) f[t, u(t-\tau)], \quad 0 < t < 1, \lambda > 0, \\ u(t) = u^{'}(t) = \cdots = u^{(n-3)}(t) = u^{(n-2)}(t) = 0, \quad -\tau \le t \le 0, \\ u^{(n-2)}(1) = (n-1)! a u(\eta). \end{cases} \tag{1.3}$$

In equation (1.3), we assume that the following conditions $(H_1)-(H_4)$ hold

(H_1) $f \in C(J \times R, R)$, $J = [0,1]$, $0 < a \le 1$, $0 < \eta < 1, 0 < \tau < 1$.
(H_2) $p(s) \in C(J_1, R^+)$, $J_1 = (0,1)$.
(H_3) $\int_0^1 s(1-s)p(s)ds < \infty, \int_{\theta+\tau}^{1-\theta+\tau} G_2(s,s)p(s)ds > 0$, $0 < \theta \le 1-\theta \le 1-\tau$, where G_2 is second order Green function as defined later.
(H_4) $u \in C[-\tau, 1] \cap C^n(0,1)$; $u(t) \ge 0$, $t \in [-\tau, 1]$.

Here we allow that $p(t)$ has some suitable singularity at the ends of $(0,1)$.

Many differential equations, which have been researched, become special cases of (1.3). For example, the both differential equations (1.1), (1.2) mentioned above are special cases of (1.3). Particularly, the existence of positive solutions for bvp (1.3) when $n = 2$ has been widely studied by many authors, we refer the reader to [5–7, 9] and references therein.

For the existence of positive solutions to bvps of second/ higher order differential equations, we mainly adopt the scheme which transform it into integral equations. During the process of transformation, several kinds of Green functions play important role. Based on the transformation, a cone and a completely continuous operator are defined over a Banach space. Then we apply all kinds of fixed-point theorems to solve the problems. In this paper, the Green function is the same as that defined in paper [11]. That is, the second order Green function is defined by

$$G_2(t,s) = \begin{cases} (1-t)s, & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1, \end{cases}$$

and for $n \geq 3$, the n-order Green function is defined by

$$G_n(t,s) = \int_0^t G_{n-1}(v,s)dv.$$

Lemma 1.1. *$G_n(t,s)$ satisfy:*

(i) $G_n(t,s) \leq G_2(s,s), \quad (t,s) \in [0,1] \times [0,1], \quad n \geq 2, n \in N.$

(ii) *Let* $0 < \theta \leq 1-\theta \leq 1-\tau, \quad J_\theta = [\theta, 1-\theta]$, *for* $t \in J_\theta, s \in [0,1]$, *one has*

$$G_2(t,s) \geq \min\{t, 1-t\}G_2(s,s) \geq \theta G_2(s,s), \tag{1.4}$$

$$G_n(t,s) \geq \theta^{n-1}G_2(s,s), t \in J_\theta, \quad n \geq 2, n \in N. \tag{1.5}$$

Proof. At first, we prove the conclusion (i) of Lemma 1.1 by induction. Clearly, $G_2(t,s) \leq G_2(s,s)$. Assuming that when $n = k$, $G_k(t,s) \leq G_2(s,s)$. Then for $n = k+1$, when $(t,s) \in [0.1] \times [0,1]$

$$G_{k+1}(t,s) = \int_0^t G_k(v,s)dv \leq \int_0^1 G_2(s,s)dv = G_2(s,s).$$

Therefore, the conclusion (i) of Lemma 1.1 hold.

For the conclusion (ii) of Lemma 1.1, formula (1.4) is clear. We prove the relation formula (1.5) in the following. For $t \in J_\theta$, by (1.4) we have $G_2(t,s) \geq \theta G_2(s,s)$. Assuming when $n = k$, $G_k(t,s) \geq \theta^{k-1}G_2(s,s)$. Then when $n = k+1$, $t \in J_\theta$

$$G_{k+1}(t,s) = \int_0^t G_k(v,s)dv \geq \int_0^t \theta^{k-1}G_2(s,s)dv = t\theta^{k-1}G_2(s,s) \geq \theta^k G_2(s,s).$$

Therefore, (1.5) holds.

Let

$$\begin{aligned} E = \{u \in C[-\tau,1] : u(t) \geq 0, \text{ for } t \in J; u(t) = u^{'}(t) = \cdots = u^{(n-3)}(t) \\ = u^{(n-2)}(t) = 0, \text{ for } t \in [-\tau,0];\ u^{(n-2)}(1) = (n-1)!u(\eta)\}. \end{aligned}$$

With the norm $\|\cdot\|$ given by $\|u\| = \sup\{|u(t)| : -\tau \leq t \leq 1\}$, $(E, \|\cdot\|)$ is a Banach space. It is obvious that$\|\cdot\| = \|\cdot\|_{[0,1]}$ for $u \in E$. ∎

Define a cone $K \in E$ by

$$K = \{u \in E : u(t) \geq 0, \text{ for } t \in [0,1];\ \min_{t \in J_\theta} u(t) \geq \gamma\|u\|\},$$

where $\gamma = \frac{\theta^{n-1}(1-a\eta^{n-1})}{1+a-a\eta^{n-1}}$.

Definition 1.1. $u(t)$ is the positive solution of BVP(1.1) if and only if it satisfies the following conditions:

1. $u(t) \subset C[-\tau,1] \cap C^n(0,1); u(t) \geq 0, \quad t \in J$;

2. $u(t) = u'(t) = \cdots = u^{(n-3)}(t) = u^{(n-2)}(t) = 0, \quad for \quad t \in [-\tau, 0]$ and $u(1) = (n-1)!au(\eta) \quad (0 < \eta < 1)$;
3. $u^{(n)}(t) = -\lambda p(t) f(t, u(t-\tau)), \quad \forall t \in J_1$.

If $u(t)$ is the solution of bvp (1.3), then $u(t)$ can be represented as

$$u(t) = \begin{cases} 0, & -\tau \le t \le 0, \\ \lambda \int_0^1 G_n(t,s)p(s)f(s,u(s-\tau))ds & \\ \quad + \frac{a\lambda t^{n-1}}{1-a\eta^{n-1}} \int_0^1 G_n(\eta,s)p(s)f(s,u(s-\tau))ds, & 0 < t < 1. \end{cases}$$

Define an operator $\Phi : K \to K$ by the formula

$$\Phi u(t) = \begin{cases} 0, & -\tau \le t \le 0 \\ \lambda \int_0^1 G_n(t,s)p(s)f(s,u(s-\tau))ds & \\ \quad + \frac{a\lambda t^{n-1}}{1-a\eta^{n-1}} \int_0^1 G_n(\eta,s)p(s)f(s,u(s-\tau))ds, & 0 < t < 1. \end{cases}$$

The operator Φ has the following properties.

Lemma 1.2. *The fixed point of the Φ is the solution of equation* (1.3).

The proof of the Lemma is easy, and we omit it here. By the Lemma, we know that positive solutions of the differential equation (1.3) are equivalent to a fixed point of Φ in K.

Lemma 1.3. *$\Phi : K \to K$ is a completely continuous operator.*

Proof. Clearly $\|\Phi u\| = \|\Phi u\|_{[0,1]}, \forall u(t) \in K$,

$$\begin{aligned} \|\Phi u\| = \|\Phi u\|_{[0,1]} &\le \lambda \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds \\ &\quad + \frac{a\lambda}{1-a\eta^{n-1}} \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds, \\ &\le \frac{\lambda(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_2(s,s)p(s)f(s,u(s-\tau))ds. \end{aligned}$$

We have

$$\begin{aligned} \Phi u(t) &\ge \lambda \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds \\ &\ge \lambda\theta^{n-1} \int_0^1 G_2(s,s)p(s)f(s,u(s-\tau))ds \\ &\ge \frac{\theta^{n-1}(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \|\Phi u\| = \gamma \|\Phi u\|. \end{aligned}$$

Then $\Phi : K \to K$. Because Φ is a sequential compact set, we can conclude that Φ is a completely continuous operator by Arzela–Ascoli Theorem. ∎

One of the main tool of this paper is the following Guo–Kranoselskii fixed point theorem in cones [12].

Lemma 1.4. [12] *Let E be a Banach space and K a conic in E. Ω_1, Ω_2 are two open subsets in E, and $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. If $\Phi : K \cap (\Omega_2 \backslash \Omega_1) \to K$ is a completely continuous operator and satisfies*

(i) $\|\Phi u\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|\Phi u\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$, *or*
(ii) $\|\Phi u\| \leq \|u\|, u \in K \cap \partial\Omega_2$ *and* $\|\Phi u\| \geq \|u\|, u \in K \cap \partial\Omega_1$,

then E has a fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.

2. Main results

Let

$$M_0 = \lim_{u\to 0} \inf \min_{t\in(0,,1)} \frac{f(t,u)}{u}, \quad M_\infty = \lim_{u\to\infty} \inf \min_{t\in(0,,1)} \frac{f(t,u)}{u}.$$

$$M^0 = \lim_{u\to 0} \sup \max_{t\in(0,,1)} \frac{f(t,u)}{u}, \quad M^\infty = \lim_{u\to\infty} \sup \max_{t\in(0,,1)} \frac{f(t,u)}{u}.$$

$$M_1 = \gamma\theta^{n-1} \int_{\theta+\tau}^{1-\theta+\tau} G_2(s,s)p(s)ds, \quad M_2 = \frac{(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_2(s,s)p(s)ds.$$

In the following, we discuss the existence of the positive solutions for arbitrary values and some compositions of M_0, M_∞, M^0 and M^∞. In Theorem 2.1 and Theorem 2.2, we take a suitable positive $\varepsilon > 0$ such that $M_0 - \varepsilon > 0, M_\infty - \varepsilon > 0$.

Theorem 2.1. *If the conditions (H_1)–(H_4) and the following conditions hold*

$$0 < M_\infty < +\infty, \tag{2.1}$$

$$0 < M^0 < +\infty, \tag{2.2}$$

$$\frac{1}{M_1(M_\infty - \varepsilon)} \leq \lambda \leq \frac{1}{M_2(M^0 + \varepsilon)}, \tag{2.3}$$

then the equation (1.3) has at least one solution.

Proof. By (2.2), (2.3), for a given $\varepsilon > 0$, $\exists r_1 > 0$, when $0 < u \leq r_1$, $f(t,u) \leq (M^0 + \varepsilon)u$.

Let $\Omega_1 = \{t \in [-\tau, 1] : \|u\| < r_1\}$, for $u \in K \cap \partial\Omega_1$, we have

$$\begin{aligned}
\|\Phi u\| &\le \lambda \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds \\
&\quad + \frac{\lambda a}{1-a\eta^{n-1}} \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds \\
&= \frac{\lambda(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds\} \\
&\le \frac{\lambda(M^0+\varepsilon)(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_2(s,s)p(s)u(s-\tau)ds \\
&= \frac{\lambda(M^0+\varepsilon)(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^{1-\tau} G_2(s+\tau,s+\tau)p(s+\tau)u(s)ds \\
&\le \frac{\lambda(M^0+\varepsilon)(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_2(s,s)p(s)ds\|u\| \\
&= \lambda M_2(M^0+\varepsilon)\|u\| \le \|\mathrm{u}\|.
\end{aligned}$$

For the same $\varepsilon > 0$ as given above, from (2.1) and (2.3), $\exists R_1 > r_1$, that we can derive that when $u(t) \ge R_1$, $f(t,u) > (M_\infty - \varepsilon)u$.

Let $\Omega_2 = \{t \in [-\tau, 1] : \|u\| < R_1\}$, for $u \in K \cap \partial\Omega_2$. We get

$$\begin{aligned}
\|\Phi u\| &\ge \lambda \sup_{t\in J} \int_0^1 G_n(t,s)p(s)f(s,u(s-\tau))ds \\
&\ge \lambda(M_\infty-\varepsilon) \sup_{t\in J_1} \int_0^1 G_n(t,s)p(s)u(s-\tau)ds \\
&= \lambda(M_\infty-\varepsilon) \sup_{t\in J_1} \int_0^{1-\tau} G_2(t,s+\tau)p(s+\tau)u(s)ds \\
&\ge \lambda(M_\infty-\varepsilon)\gamma \sup_{t\in J_1} \int_\theta^{1-\theta} G_n(t,s+\tau)p(s+\tau)ds\|u\| \\
&= \lambda(M_\infty-\varepsilon)\gamma \sup_{t\in J_1} \int_{\theta+\tau}^{1-\theta+\tau} G_n(t,s)p(s)ds\|u\| \\
&\ge \lambda(M_\infty-\varepsilon)\gamma\theta^{n-1} \int_{\theta+\tau}^{1-\theta+\tau} G_2(s,s)p(s)ds\|u\| \\
&= \lambda M_1(M_\infty-\varepsilon)\|u\| \ge \|u\|.
\end{aligned}$$

Therefore, by Lemma 1.4, Φ has a fixed point $u(t) \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$, and $u(t)$ is a positive solution of equation (1.3). Thus Theorem 2.1 is proved. ∎

Theorem 2.2. *If the conditions* (H_1)–(H_4) *and the following conditions*

$$0 < M_0 < +\infty, \tag{2.4}$$

$$0 < M^\infty < +\infty, \tag{2.5}$$

$$\frac{1}{M_1(M_0 - \varepsilon)} \le \lambda \le \frac{1}{M_2(M^\infty + \varepsilon)}, \tag{2.6}$$

hold, then the equation (1.3) *has at least one solution.*

Proof. By (2.4), (2.6), for a given $\varepsilon > 0$, $\exists r_2 > 0$, when $0 < u \le r_2$, $f(t,u) \ge (M_0 - \varepsilon)u$.

Let $\Omega_1 = \{t \in [-\tau, 1] : \|u\| < r_2\}$, for $u \in K \cap \partial\Omega_1$. We have

$$\begin{aligned}
\|\Phi u\| &\ge \lambda \sup_{t\in J_1} \int_0^1 G_n(t,s)p(s)f(s,u(s-\tau))ds \\
&\ge \lambda(M_0 - \varepsilon) \sup_{t\in J_1} \int_0^1 G_n(t,s)p(s)u(s-\tau)ds \\
&= \lambda(M_0 - \varepsilon) \sup_{t\in J_1} \int_0^{1-\tau} G_2(t,s+\tau)p(s+\tau)u(s)ds \\
&\ge \lambda(M_0 - \varepsilon)\gamma \sup_{t\in J_1} \int_\theta^{1-\theta} G_n(t,s+\tau)p(s+\tau)ds\|u\| \\
&= \lambda(M_0 - \varepsilon)\gamma \sup_{t\in J_1} \int_{\theta+\tau}^{1-\theta+\tau} G_n(t,s)p(s)ds\|u\| \\
&\ge \lambda(M_0 - \varepsilon)\gamma\theta^{n-1} \int_{\theta+\tau}^{1-\theta+\tau} G_2(s,s)p(s)ds\|u\| \\
&= \lambda M_1(M_0 - \varepsilon)\|u\| \ge \|u\|.
\end{aligned}$$

For the same $\varepsilon > 0$ as mentioned above, from (2.5) and (2.6), $\exists R_2 > r_2$, we can derive that when $u(t) \ge R_2$, $f(t,u) > (M^\infty + \varepsilon)u$.

Let $\Omega_2 = \{t \in [-\tau, 1] : \|u\| < R_2\}$, for $u \in K \cap \partial\Omega_2$. We get

$$\begin{aligned}
\|\Phi u\| \le \lambda \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds \\
+ \frac{\lambda a}{1 - a\eta^{n-1}} \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds
\end{aligned}$$

$$= \frac{\lambda(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds\}$$

$$\leq \frac{\lambda(M^\infty+\varepsilon)(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_2(s,s)p(s)u(s-\tau)ds$$

$$= \frac{\lambda(M^\infty+\varepsilon)(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^{1-\tau} G_2(s+\tau,s+\tau)p(s+\tau)u(s)ds$$

$$\leq \frac{\lambda(M^\infty+\varepsilon)(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_2(s,s)p(s)ds\|u\|$$

$$= \lambda M_2(M^\infty+\varepsilon)\|u\| \leq \|\mathrm{u}\|.$$

Therefore, by Lemma 1.4, Φ has a fixed point $u(t) \in K \cap (\overline{\Omega_2}\backslash\Omega_1)$, and $u(t)$ is a positive solution of equation (1.3), completing the proof of Theorem 2.2. ∎

Theorem 2.3. *If the conditions $(H_1)-(H_4)$ hold and $M_\infty = \infty, M^0 = 0$. Then there exists two positive numbers λ_1, λ_2, when $\lambda_1 \leq \lambda \leq \lambda_2$, bvp (1.3) has at least a positive solution.*

Proof. Since $M_\infty = \infty$, we can choose a positive constant $M > 0$ such that $f(t,u) \geq M = \alpha R_3$ $(\alpha > 0)$, for any $u \geq R_3, t \in J$.

Let

$$\lambda_1 = [\alpha\theta^{n-1} \int_\theta^{1-\theta} G_2(s,s)p(s)ds]^{-1}, \Omega_2 = \{t \in [-\tau,1] : \|u\| < R_3\}.$$

For $u \in K \cap \partial\Omega_2, \lambda \geq \lambda_1$, we have

$$\|\Phi u\| \geq \lambda \sup_{t\in J_1} \int_0^1 G_n(t,s)p(s)f(s,u(s-\tau))ds \geq \lambda M \sup_{t\in J_1} \int_0^1 G_n(t,s)p(s)ds$$

$$\geq \lambda M \sup_{t\in J_1} \int_\theta^{1-\theta} G_n(t,s)p(s)ds \geq \lambda M\theta^{n-1} \int_\theta^{1-\theta} G_2(s,s)p(s)ds$$

$$\geq \lambda\alpha R_3\theta^{n-1} \int_\theta^{1-\theta} G_2(s,s)p(s)ds = \frac{\lambda}{\lambda_1}R_3 \geq R_3 = \|u\|.$$

Because $M^0 = 0$, we choose a value small enough for $\varepsilon > 0$, so that

$$\lambda_2 = [\frac{\varepsilon(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_2(s,s)p(s)ds]^{-1} > \lambda_1,$$

and $\exists 0 < r_3 < R_3$, such that $f(t,u) \leq \varepsilon u$ for any $u \leq r_3$.

Let $\Omega_1 = \{t \in [-\tau, 1] : \|u\| < r_3\}$, for $u \in K \cap \partial\Omega_1$. We have

$$\begin{aligned}
\|\Phi u\| &\leq \lambda \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds \\
&\qquad + \frac{\lambda a}{1-a\eta^{n-1}} \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds \\
&= \frac{\lambda(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds \\
&\leq \frac{\lambda\varepsilon(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_2(s,s)p(s)u(s-\tau)ds \\
&= \frac{\lambda\varepsilon(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^{1-\tau} G_2(s+\tau,s+\tau)p(s+\tau)u(s)ds \\
&\leq \frac{\lambda\varepsilon(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_2(s,s)p(s)ds\|u\| = \frac{\lambda}{\lambda_2}\|u\| \leq \|u\|.
\end{aligned}$$

Therefore, by Lemma 1.4, Φ has at least one fixed point $u(t) \in K \cap (\overline{\Omega_2}\backslash\Omega_1)$, and equation (1.3) has at least one positive solution. Theorem 2.3 is proved. ■

Remark 2.1. If $M_0 = \infty, M^\infty = 0$, similarly we can verify that bvp (1.3) has at least one positive solution.

Theorem 2.4. *If the conditions (H_1)–(H_4) and the following conditions hold:*

(1) $M^0 = 0$ *and* $M^\infty = 0$.
(2) *There exist positive numbers $\rho > 0, \delta > 0$, such that $f(t,u) \geq \delta$ for any $u \geq \rho, t \in J$.*

Then there exist two positive numbers λ_1, λ_2, when $\lambda_1 \leq \lambda \leq \lambda_2$, the equation (1.3) has at least two solutions.

Proof. Let

$$\lambda_1 = \left[\delta\theta^{n-1} \int_\theta^{1-\theta} G_2(s,s)p(s)ds\right]^{-1},$$

$$\lambda_2 = \left[\frac{\varepsilon(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_2(s,s)p(s)ds\right]^{-1}.$$

Here we take $\varepsilon > 0$ satisfying $\lambda_1 \leq \lambda_2$.

Because $M^0 = 0$, for a given $\varepsilon > 0$, $\exists 0 < r_4 < \rho$, such that $f(t,u) \leq \varepsilon u$ for all $0 < u \leq r_4, t \in J$.

Let $\Omega_1 = \{u \in E : \|u\| < r_4\}$, for $u \in K \cap \partial\Omega_1$. We have

$$\begin{aligned}
\|\Phi u\| &\leq \lambda \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds \\
&\quad + \frac{\lambda a}{1-a\eta^{n-1}} \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds \\
&= \frac{\lambda(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds\} \\
&\leq \frac{\lambda\varepsilon(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_2(s,s)p(s)u(s-\tau)ds \\
&= \frac{\lambda\varepsilon(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^{1-\tau} G_2(s+\tau,s+\tau)p(s+\tau)u(s)ds \\
&\leq \frac{\lambda\varepsilon(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_2(s,s)p(s)ds\|u\| = \frac{\lambda}{\lambda_2}\|u\| \leq \|\mathrm{u}\|.
\end{aligned}$$

Let $\Omega_2 = \{u \in E : \|u\| < \rho\}$, for $u \in K \cap \partial\Omega_2$. We have

$$\begin{aligned}
\|\Phi u\| &\geq \lambda \sup_{t\in J_1} \int_0^1 G_n(t,s)p(s)f(s,u(s-\tau))p(s)ds \geq \lambda\delta \sup_{t\in J_1} \int_\theta^{1-\theta} G_n(t,s)p(s)ds \\
&\geq \lambda\delta\theta^{n-1} \int_\theta^{1-\theta} G_2(s,s)p(s)ds = \frac{\rho\lambda}{\lambda_1} \geq \rho = \|u\|.
\end{aligned}$$

Therefore, by Lemma 1.4, Φ has at least one fixed point $u_1 \in K\cap(\overline{\Omega_2}\backslash\Omega_1)$, and equation (1.3) has at least one positive solution u_1 satisfying $\|u_1\| < \rho$.

By $M^\infty = 0$, for the same value of ε as mentioned above, $\exists R_4 > \rho$, when $u \geq R_4, f(t,u) \leq \varepsilon u$.

Let $\Omega_3 = \{u \in E : \|u\| < R_4\}$. We can prove that Φ has at least one fixed point $u_2 \in K \cap (\overline{\Omega_3}\backslash\Omega_2)$ when $\lambda_1 \leq \lambda \leq \lambda_2$, then equation (1.3) has another positive solution u_2 which satisfies $\|u_2\| > \rho$. So the equation (1.3) at least has two positive solutions. ■

Theorem 2.5. *If the conditions (H_1)–(H_4) and the following conditions hold:*

(1) $M_0 = \infty$ *and* $M_\infty = \infty$.
(2) *There exist two positive constants* α, β $(\beta \geq \lambda)$ *such that*

$$
\begin{aligned}
(2.7)\qquad & 0 < f(t,u) \\
& < \left[\frac{\alpha(1-a\eta^{n-1})}{\beta(1+a-a\eta^{n-1})} - \int_0^\tau G_2(s,s)p(s)f(s,0)ds\right]\frac{1}{\int_\tau^1 G_2(s,s)p(s)ds}, \\
& \qquad t \in J_1, \theta\alpha < u < \alpha.
\end{aligned}
$$

then (1.3) *has at least two solutions.*

Proof. Since f is continuous in u, there exists a positive constant $\alpha_1 : \alpha_1 < \alpha$ such that

$$
\begin{aligned}
(2.8)\qquad & 0 < f(t,u) \\
& < \left[\frac{\alpha_1(1-a\eta^{n-1})}{\beta(1+a-a\eta^{n-1})} - \int_0^\tau G_2(s,s)p(s)f(s,0)ds\right]\frac{1}{\int_\tau^1 G_2(s,s)p(s)ds}, \\
& \qquad s \in J, \theta\alpha_1 < u < \alpha_1.
\end{aligned}
$$

Because $M_0 = \infty$, there exists $L > 0, 0 < r_5 < \alpha_1$ such that

$$
(2.9)\qquad f(t,u) \geq Lu, 0 < u < r_5; \lambda L\theta^{n-1}\gamma \int_{\theta+\tau}^{1-\theta+\tau} G_2(s,s)p(s)ds \geq 1.
$$

Let $\Omega_1 = \{u \in E : \|u\| < r_5\}$, for $u \in K \cap \partial\Omega_1$. We have by (2.9) that

$$
\begin{aligned}
\|\Phi u\| &\geq \lambda \sup_{t\in J_1} \int_0^1 G_n(t,s)p(s)f(s,u(s-\tau))ds \\
&\geq \lambda L \sup_{t\in J_1} \int_0^1 G_n(t,s)p(s)u(s-\tau)ds \\
&\geq \lambda L \sup_{t\in J_1} \int_0^{1-\tau} G_n(t,s+\tau)p(s+\tau)u(s)ds \\
&\geq \lambda L\gamma \sup_{t\in J_1} \int_\theta^{1-\theta} G_n(t,s+\tau)p(s+\tau)ds\|u\| \\
&\geq \lambda L\gamma \sup_{t\in J_1} \int_{\theta+\tau}^{1-\theta+\tau} G_n(t,s)p(s)ds\|u\| \\
&\geq \lambda L\gamma\theta^{n-1} \int_{\theta+\tau}^{1-\theta+\tau} G_2(s,s)p(s)ds\|u\| \geq \|u\|.
\end{aligned}
$$

On the other hand, let $\Omega_2 = \{u \in E : \|u\| < \alpha_1\}$, for $u \in K \cap \partial\Omega_2$ and

$\lambda \le \beta$. We have by (2.8) that

$$\begin{aligned}
\|\Phi u\| &\le \lambda \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds \\
&\quad + \frac{\lambda a}{1-a\eta^{n-1}} \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds \\
&= \frac{\lambda(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_n(s,s)p(s)f(s,u(s-\tau))ds\} \\
&\le \frac{\lambda(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \int_0^1 G_2(s,s)p(s)f(s,u(s-\tau))ds \\
&= \frac{\lambda(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \\
&\quad \cdot \left[\int_0^\tau G_2(s,s)p(s)f(s,0)ds + \int_\tau^1 G_2(s,s)p(s)f(s,u(s-\tau))ds\right] \\
&\le \frac{\lambda(1+a-a\eta^{n-1})}{1-a\eta^{n-1}} \cdot \frac{\alpha_1(1-a\eta^{n-1})}{\beta(1+a-a\eta^{n-1})} \le \alpha_1 = \|u\|.
\end{aligned}$$

Therefore, by Lemma 1.4, Φ has at least one fixed point $u_1 \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$, and u_1 is the positive solution of equation (1.3) such that $\|u_1\| < \alpha_1 < \alpha$. By $M_\infty = \infty$ and (2.7), analogously, we can prove that equation (1.3) has at least another positive solution u_2 which satisfies $\|u_2\| > \alpha$. So the equation (1.3) has at least two positive solutions. ∎

3. Examples

Example 1. Consider the problem

$$(3.1) \qquad \begin{cases} -u^{(4)}(t) = \frac{1}{\sqrt{t}}\sqrt{t^2+1}\frac{u^2(t-\frac12)[1+u(t-\frac12)]}{2+u(t-\frac12)}, & 0<t<1, \\ u(t)=u'(t)=u'(t)=u''(t)=0, & -\tau \le t \le 0, \\ u''(1) = 3!\frac12 u(\frac12), \end{cases}$$

where $f(t,u) = \sqrt{t^2+1}\frac{u^2(t-\frac12)(1+u(t-\frac12)}{2+u(t-\frac12)}$; $p(t) = \frac{1}{\sqrt{t}}$, $t=0$ is its singularity.

Here we have $M_\infty = \infty, M^0 = 0$. If we choose $M = 100, \theta = \frac14, \varepsilon_1 = \frac{100}{11}$, then when $u \ge 11$, $f(t,u) \ge M$. We can calculate that $\lambda_1 = 168.3$.

Choosing $\varepsilon_2 = 0.01, r_3 = \frac{\varepsilon_2}{2}$, we have $f(t,u) \le 2u^2 \le \varepsilon_2 u$ for any $u \le r_3$ and we can calculate that $\lambda_2 = 244.565$.

By Theorem 2.3, we have that if $168.3 = \lambda_1 \le \lambda \le \lambda_2 = 244.565$, the problem (3.1) has at least one positive solution.

Example 2. Consider the boundary value problem

$$
\begin{cases}
-u^{(4)}(t) = \frac{1}{\sqrt{1-t}}(t^2+1)\sqrt{u(t-\frac{1}{2})}\tanh u(t-\frac{1}{2}), \quad 0 < t < 1, \\
\quad u(t) = u'(t) = u'(t) = u''(t) = 0, \quad -\tau \leq t \leq 0, \\
u''(1) = 3!\frac{1}{2}u(\frac{1}{2}),
\end{cases} \tag{3.2}
$$

where $f(t,u) = (t^2+1)\sqrt{u(t-\frac{1}{2})}\tanh u(t-\frac{1}{2})$; $p(t) = \frac{1}{\sqrt{1-t}}, t = 1$ is its singularity.

If we choose $\varepsilon = 0,01, \rho = 1, \delta = \frac{e^2-1}{e^2+1}$, then calculates easily that

$$\lambda_1 = 46.545, \lambda_2 = 260.87$$

and

$$M^0 = 0, M^\infty = 0, f(t,u) \geq \delta.$$

From Theorem 2.4, we have that if $46.545 = \lambda_1 \leq \lambda \leq \lambda_2 = 260.87$, the problem (3.2) has at least two positive solutions.

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