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NON LINEAR DIFFERENTIAL POLYNOMIALS SHARING FIXED POINTS WITH FINITE WEIGHTS

Abstract. We employ the notion of weighted sharing to investigate the uniqueness of meromorphic functions when two nonlinear differential polynomials share fixed points. The results of the paper improve and generalize the recent results due to Xu–Lu–Yi [10].

1. Introduction, definitions and results

In this paper, by meromorphic functions, we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [5], [13] and [14]. It will be convenient, to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r,h)$ any quantity satisfying

$$S(r,h) = o(T(r,h)) \qquad (r \longrightarrow \infty, r \notin E).$$

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM. A finite value z_0 is said to be a fixed point of $f(z)$ if $f(z_0) = z_0$.

Throughout this paper, we need the following definition.

$$\Theta(a,f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r,a;f)}{T(r,f)},$$

where a is a value in the extended complex plane.

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In 1959, Hayman (see [4], Corollary of Theorem 9) proved the following theorem.

Theorem A. *Let f be a transcendental meromorphic function and $n(\geq 3)$ is an integer. Then $f^n f' = 1$ has infinitely many solutions.*

Corresponding to which, Yang and Hua [11] obtained the following result.

Theorem B. *Let f and g be two non-constant meromorphic functions, $n \geq 11$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

In 2000, Fang [2] proved the following result.

Theorem C. *Let f be a transcendental meromorphic function, and let n be a positive integer. Then $f^n f' - z = 0$ has infinitely many solutions.*

Corresponding to Theorem C, Fang and Qiu [3] proved the following result.

Theorem D. *Let f and g be two non-constant meromorphic functions, and let $n \geq 11$ be a positive integer. If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a complex number t such that $t^{n+1} = 1$.*

Using the idea of sharing fixed points, recently Xu–Lu–Yi [10] proved the following uniqueness theorems for meromorphic functions where an additional condition namely the sharing of poles are taken under consideration.

Theorem E. *Let f and g be two non-constant meromorphic functions, and let n, k be two positive integers with $n > 3k+10$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2(c_1 c_2)^n c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem F. *Let f and g be two non-constant meromorphic functions satisfying $\Theta(\infty, f) > \frac{2}{n}$, and let n, k be two positive integers with $n \geq 3k + 12$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share z CM, f and g share ∞ IM, then $f \equiv g$.*

Now one may ask the following question which is the motivation of the paper.

Question 1. Is it possible simultaneously to relax the nature of sharing the fixed point and reduce the lower bound of n in Theorem E and Theorem F ?

In the paper, we will not only affirmatively solve the above question but also obtain a more generalised result. Relaxation of the sharing can be done by the following definition known as weighted sharing of values introduced by I. Lahiri [7, 8] which measure how close a shared value is to being shared CM or to being shared IM.

Definition 1. Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \le k$ and k+1 times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\le k)$ if and only if it is an a-point of g with multiplicity $m(\le k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly, if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

We now state the main results of the paper.

Theorem 1. *Let f and g be two transcendental meromorphic functions, and let $n(\ge 1)$, $k(\ge 1)$ and $m(\ge 0)$ be three integers such that $n > 3k+m+6$. Let $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share $(z, 2)$, f and g share $(\infty, 0)$. Then each of the following holds:*

(i) *when $m = 0$, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2(c_1c_2)^n c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^n = 1$;*

(ii) *when $m = 1$ and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$, then $f \equiv g$;*

(iii) *when $m \ge 2$, then either $f \equiv g$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where*

$$R(x, y) = x^n(x-1)^m - y^n(y-1)^m.$$

Theorem 2. *Let f and g be two rational meromorphic functions, and let $n(\ge 1)$, $k(\ge 1)$ and $m(\ge 0)$ be three integers such that $n > 3k + m + 9$. Let $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share $(z, 2)$, f and g share $(\infty, 0)$. Then* (i)–(iii) *of Theorem* 1 *hold.*

Remark 1. Obviously Theorems 1 and 2 both are two-fold improvements of Theorem E and Theorem F in a compact form.

We now explain some definitions and notations which are used in the paper.

Definition 2. [6] Let $a \in \mathbb{C} \cup \{\infty\}$. For a positive integer p we denote by $N(r, a; f \mid\leq p)$ the counting function of those a-points of f (counted with multiplicities), whose multiplicities are not greater than p. By $\overline{N}(r, a; f \mid\leq p)$ we denote the corresponding reduced counting function.

In an analogous manner we define $N(r, a; f \mid\geq p)$ and $\overline{N}(r, a; f \mid\geq p)$.

Definition 3. [8] Let k be a positive integer or infinity. We denote by $N_k(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \cdots + \overline{N}(r, a; f \mid\geq k).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 4. [1] Let f and g be two non-constant meromorphic functions such that f and g share the $(a, 2)$ for $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p and also an a-point of g with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ ($\overline{N}_L(r, a; g)$) the reduced counting function of those a-points of f and g, where $p > q \geq 3$ ($q > p \geq 3$). Also we denote by $\overline{N}_E^{(3}(r, a; f)$ the reduced counting function of those a-points of f and g, where $p = q \geq 3$.

Definition 5. [7, 8] Let f and g be two non-constant meromorphic functions such that f and g share the value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We shall denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 1. [12] *Let f be a non-constant meromorphic function,*

$$R(z, f) = \frac{P(z, f)}{Q(z, f)} = \frac{\sum\limits_{i=0}^{p} a_i(z) f^i}{\sum\limits_{j=0}^{q} b_j(z) f^j}$$

be a non-constant irreducible rational in f with meromorphic coefficients $a_i(z)$, $b_j(z)$ such that $T(r, a_i) = S(r, f)$, $i = 0, 1, \ldots, p$, $T(r, b_j) = S(r, f)$, $j = 0, 1, \ldots, q$. Then the characteristic function of $R(z, f)$ satisfies

$$T(r, R(z, f)) = dT(r, f) + S(r, f),$$

where $d=\max\{p,q\}$.

Lemma 2. [16] *Let* f *be a non-constant meromorphic function, and* p, k *be positive integers. Then*

$$N_p\big(r,0;f^{(k)}\big)\le T\big(r,f^{(k)}\big)-T(r,f)+N_{p+k}(r,0;f)+S(r,f), \tag{2.1}$$

$$N_p\big(r,0;f^{(k)}\big)\le k\overline{N}(r,\infty;f)+N_{p+k}(r,0;f)+S(r,f). \tag{2.2}$$

Lemma 3. [1] *Let* F, G *be two non-constant meromorphic functions sharing* $(1,2)$, (∞,k) *where* $0\le k<\infty$ *and* $H\not\equiv 0$. *Then*

(i) $T(r,F)\le N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)$
$+\overline{N}_*(r,\infty;F,G)-m(r,1;G)-\overline{N}_E^{(3}(r,1;F)-\overline{N}_L(r,1;G)+S(r,F)$
$+S(r,G)$;

(ii) $T(r,G)\le N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)$
$+\overline{N}_*(r,\infty;F,G)-m(r,1;F)-\overline{N}_E^{(3}(r,1;G)-\overline{N}_L(r,1;F)+S(r,F)$
$+S(r,G)$.

Lemma 4. [5, 13] *Let* f *be a transcendental meromorphic function, and let* $a_1(z)$, $a_2(z)$ *be two distinct meromorphic functions such that* $T(r,a_i(z))=S(r,f)$, $i=1,2$. *Then*

$$T(r,f)\le\overline{N}(r,\infty;f)+\overline{N}(r,a_1;f)+\overline{N}(r,a_2;f)+S(r,f).$$

Lemma 5. *Let* f *and* g *be two rational (transcendental) meromorphic functions and let* $n(\ge 1)$, $k(\ge 1)$, $m(\ge 0)$ *be three integers. Suppose that* $F_1=\frac{(f^n(f-1)^m)^{(k)}}{a_0+a_1z}$ *and* $G_1=\frac{(g^n(g-1)^m)^{(k)}}{a_0+a_1z}$. *If there exist two nonzero constants* c_1 *and* c_2 *such that* $\overline{N}(r,c_1;F_1)=\overline{N}(r,0;G_1)$ *and* $\overline{N}(r,c_2;G_1)=\overline{N}(r,0;F_1)$, *then* $n\le 3k+m+5$ *(*$n\le 3k+m+3$*), where* a_0, a_1 *are complex constants which are not simultaneously zero.*

Proof. By the second fundamental theorem of Nevanlinna we have

$$\begin{aligned} T(r,F_1)&\le\overline{N}(r,0;F_1)+\overline{N}(r,\infty;F_1)+\overline{N}(r,c_1;F_1)+S(r,F_1)\\ &\le\overline{N}(r,0;F_1)+\overline{N}(r,0;G_1)+\overline{N}(r,\infty;F_1)+S(r,F_1). \end{aligned} \tag{2.3}$$

By (2.1), (2.2), (2.3) and Lemma 1 we obtain

$$\begin{aligned} &(n+m)T(r,f)\\ &\le T(r,F_1)-\overline{N}(r,0;F_1)+N_{k+1}(r,0;f^n(f-1)^m)+\log r+S(r,f)\\ &\le\overline{N}(r,0;G_1)+N_{k+1}(r,0;f^n(f-1)^m)+\overline{N}(r,\infty;f)+2\log r+S(r,f)\\ &\le N_{k+1}(r,0;f^n(f-1)^m)+N_{k+1}(r,0;g^n(g-1)^m)+\overline{N}(r,\infty;f)\\ &\quad+k\overline{N}(r,\infty;g)+2\log r+S(r,f)+S(r,g)\\ &\le(k+m+2)T(r,f)+(2k+m+1)T(r,g)+2\log r+S(r,f)+S(r,g). \end{aligned} \tag{2.4}$$

Similarly, we obtain

$$(n+m)T(r,g) \le (k+m+2)T(r,g) + (2k+m+1)T(r,f) + 2\log r + S(r,f) + S(r,g). \tag{2.5}$$

Combining (2.4), (2.5) and noting that $T(r,f) \ge \log r$ and $T(r,g) \ge \log r$ we get

$$(n-3k-m-5)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g),$$

which gives $n \le 3k+m+5$. When f and g are transcendental meromorphic functions then noting that $\log r = o(1)T(r,f) = o(1)T(r,g)$ we can prove the lemma similarly. This completes the proof of the lemma. ∎

Lemma 6. [15] *Suppose that f and g be two nonconstant meromorphic functions. Let*

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right). \tag{2.6}$$

If F, G share $(\infty,0)$ and $V \equiv 0$, then $F \equiv G$.

Lemma 7. *Suppose that f and g be two nonconstant meromorphic functions. Let V be given by (2.6), where $F = \frac{(f^n(f-1)^m)^{(k)}}{z}$, $G = \frac{(g^n(g-1)^m)^{(k)}}{z}$, and $n(\ge 1)$, $k(\ge 1)$, $m(\ge 0)$ are positive integers and suppose that $V \not\equiv 0$. If f, g share $(\infty,0)$ and F, G share $(1,2)$, then the poles of F and G are zeros of V and*

$$\begin{aligned}\big(n+m+k-1\big)\overline{N}(r,\infty;f\mid\ge 1) &= \big(n+m+k-1\big)\overline{N}(r,\infty;g\mid\ge 1)\\ &\le \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

Proof. Since f, g share $(\infty,0)$, we note that the order of the possible poles of F and G are at least $n+m+k$. So F, G share $(\infty, n+m+k-1)$. Now using the Milloux theorem [5] , p. 55, and Lemma 1, we obtain from the definition of V that

$$m(r,V) = S(r,f)+S(r,g). \tag{2.7}$$

Thus

$$\begin{aligned}\big(n+m+k-1\big)\overline{N}(r,\infty;f\mid\ge 1) &= \big(n+m+k-1\big)\overline{N}(r,\infty;g\mid\ge 1)\\ &= \big(n+m+k-1\big)\overline{N}\big(r,\infty;F\mid\ge n+m+k\big)\\ &\le N(r,0;V) \le T(r,V)+O(1)\\ &\le N(r,\infty;V)+m(r,V)+O(1)\\ &\le N(r,\infty;V)+S(r,f)+S(r,g)\\ &\le \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)\\ &\quad +S(r,f)+S(r,g).\end{aligned}$$

This completes the proof of the lemma. ∎

Lemma 8. *Let f and g be two nonconstant meromorphic functions such that*

$$\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n},$$

where $n(\geq 3)$ is an integer. Then

$$f^n(af+b) \equiv g^n(ag+b)$$

implies $f \equiv g$, where a, b are two nonzero constants.

Proof. We omit the proof since it can be carried out in the line of Lemma 6 [9]. ■

3. Proof of the Theorem

Proof of Theorem 2. Let $F(z)$ and $G(z)$ be given as in Lemma 7. Then $F(z)$, $G(z)$ are rational meromorphic functions that share $(1, 2)$ and $(\infty, 0)$. So

$$\overline{N}_*(r, \infty; F, G) \leq \overline{N}(r, \infty; F \mid \geq n+m+k) = \overline{N}(r, \infty; f \mid \geq 1).$$

If possible, we suppose that $H \not\equiv 0$. Then $F \not\equiv G$. So, from Lemma 6 we have $V \not\equiv 0$. From Lemma 1 and (2.1) we obtain

$$\begin{aligned}(3.1)\qquad N_2(r, 0; F) &\leq N_2\big(r, 0; (f^n(f-1)^m)^{(k)}\big) + S(r, f)\\ &\leq T\big(r, (f^n(f-1)^m)^{(k)}\big) - (n+m)T(r, f) + N_{k+2}(r, 0; f^n(f-1)^m) + S(r, f)\\ &\leq T(r, F) - (n+m)T(r, f) + N_{k+2}(r, 0; f^n(f-1)^m) + \log r + S(r, f).\end{aligned}$$

In a similar way we obtain

$$\begin{aligned}(3.2)\qquad N_2(r, 0; G) \leq{}& T(r, G) - (n+m)T(r, g) + N_{k+2}(r, 0; g^n(g-1)^m)\\ &+ \log r + S(r, g).\end{aligned}$$

Again by (2.2) we have

$$(3.3)\qquad N_2(r, 0; F) \leq k\overline{N}(r, \infty; f) + N_{k+2}(r, 0; f^n(f-1)^m) + S(r, f).$$

$$(3.4)\qquad N_2(r, 0; G) \leq k\overline{N}(r, \infty; g) + N_{k+2}(r, 0; g^n(g-1)^m) + S(r, g).$$

From (3.1) and (3.2) we get

$$\begin{aligned}(3.5)\qquad (n+m)\{T(r, f) + T(r, g)\}&\\ \leq{}& T(r, F) + T(r, G) + N_{k+2}(r, 0; f^n(f-1)^m)\\ &+ N_{k+2}(r, 0; g^n(g-1)^m) - N_2(r, 0; F) - N_2(r, 0; G)\\ &+ 2\log r + S(r, f) + S(r, g).\end{aligned}$$

Then using Lemma 1, Lemma 3, (3.3) and (3.4) we obtain from (3.5)

$$
\begin{aligned}
&(n+m)\{T(r,f)+T(r,g)\} \\
&\qquad \le N_2(r,0;F)+N_2(r,0;G)+2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G) \\
&\qquad\quad +2\overline{N}_*(r,\infty;F,G)+N_{k+2}(r,0;f^n(f-1)^m) \\
&\qquad\quad +N_{k+2}(r,0;g^n(g-1)^m)-\overline{N}_L(r,1;F)-\overline{N}_L(r,1;G) \\
&\qquad\quad +2\log r+S(r,f)+S(r,g) \\
&\qquad \le 2N_{k+2}(r,0;f^n(f-1)^m)+2N_{k+2}(r,0;g^n(g-1)^m) \\
&\qquad\quad +(k+2)\overline{N}(r,\infty;f)+(k+2)\overline{N}(r,\infty;g)+2\overline{N}_*(r,\infty;F,G) \\
&\qquad\quad -\overline{N}_*(r,1;F,G)+6\log r+S(r,f)+S(r,g) \\
&\qquad \le 2(k+m+2)\{T(r,f)+T(r,g)\}+(k+2)(\overline{N}(r,\infty;f) \\
&\qquad\quad +\overline{N}(r,\infty;g))+2\overline{N}_*(r,\infty;F,G)-\overline{N}_*(r,1;F,G) \\
&\qquad\quad +6\log r+S(r,f)+S(r,g).
\end{aligned}
$$

Using Lemma 2, Lemma 7 and noting that f and g are rational functions we obtain from above

$$
\begin{aligned}
&(n-3k-m-9)\{T(r,f)+T(r,g)\} \\
&\qquad \le \frac{2}{n+m+k-1}\left[\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)\right] \\
&\qquad\quad -\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g) \\
&\qquad \le \frac{2}{n+m+k-1}[N_{k+1}(r,0;f^n(f-1)^m)+k\overline{N}(r,\infty;f) \\
&\qquad\quad +N_{k+1}(r,0;g^n(g-1)^m)+k\overline{N}(r,\infty;g)+\overline{N}_*(r,1;F,G)] \\
&\qquad\quad -\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).
\end{aligned}
$$

From this we obtain

$$
\begin{aligned}
&[(n-3k-m-9)(n+m+k-1)-(4k+2m+2)]\{T(r,f)+T(r,g)\} \\
&\qquad\qquad \le S(r,f)+S(r,g),
\end{aligned}
$$

which leads to a contradiction as $n>3k+m+9$.

We now assume that $H\equiv 0$. That is

$$\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right)=0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F-1}=\frac{A}{G-1}+B, \tag{3.6}$$

where $A(\neq 0)$ and B are constants.

Now we consider the following three cases.

Case 1. Let $B \neq 0$ and $A = B$. Then from (3.6) we get

$$\frac{1}{F-1} = \frac{BG}{G-1}. \tag{3.7}$$

If $B = -1$, then from (3.7) we obtain

$$FG = 1, \text{ i.e.,}$$

$$(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} = z^2. \tag{3.8}$$

We now consider the following two subcases.

Subcase (i). Let $m = 0$. Then from (3.8) we have

$$(f^n)^{(k)}(g^n)^{(k)} = z^2.$$

Then we obtain $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2(c_1c_2)^n c^2 = -1$ (see P. 15 [10]).

Subcase (ii). Next we assume that $m \geq 1$. From our assumption it is clear that $f \neq 0$ and $f \neq \infty$. Let $f(z) = e^\alpha$, where α is a nonconstant entire function. Then by induction we get

$$(f^{n+m})^{(k)} = t_m(\alpha', \alpha'', \ldots, \alpha^{(k)})e^{(n+m)\alpha}, \tag{3.9}$$

$$.$$

$$(-1)^i \, {}^mC_i(f^{n+m-i})^{(k)} = t_{m-i}(\alpha', \alpha'', \ldots, \alpha^{(k)})e^{(n+m-i)\alpha}, \tag{3.10}$$

$$.$$

$$(-1)^m(f^n)^{(k)} = t_0(\alpha', \alpha'', \ldots, \alpha^{(k)})e^{n\alpha}, \tag{3.11}$$

where $t_i(\alpha', \alpha'', \ldots, \alpha^{(k)})$ $(i = 0, 1, \ldots, m)$ are differential polynomials in α', α'', $\ldots$, $\alpha^{(k)}$. Obviously

$$t_i(\alpha', \alpha'', \ldots, \alpha^{(k)}) \not\equiv 0$$

for $i = 0, 1, 2, \ldots, m$, and

$$(f^n(f-1)^m)^{(k)} \not\equiv 0.$$

From (3.9)–(3.11) we obtain

$$\begin{aligned} \overline{N}(r, 0; t_m e^{m\alpha(z)} + \cdots + t_{m-i}e^{(m-i)\alpha(z)} + \cdots + t_0) \\ \leq N(r, 0; z^2) = S(r, f). \end{aligned} \tag{3.12}$$

Since α is an entire function, we obtain $T(r, \alpha^{(j)}) = S(r, f)$ for $j = 1, 2, \ldots, k$. Hence $T(r, t_i) = S(r, f)$ for $i = 0, 1, 2, \ldots, m$. So from (3.12), Lemmas 1 and

4 we obtain

$$\begin{aligned}
mT(r,f) &= T(r, t_m e^{m\alpha} + \cdots + t_{m-i}e^{(m-i)\alpha} + \cdots + t_1 e^{\alpha}) + S(r,f) \\
&\leq \overline{N}(r,0; t_m e^{m\alpha} + \cdots + t_{m-i}e^{(m-i)\alpha} + \cdots + t_1 e^{\alpha}) \\
&\quad + \overline{N}(r,0; t_m e^{m\alpha} + \cdots + t_{m-i}e^{(m-i)\alpha} + \cdots + t_1 e^{\alpha} + t_0) + S(r,f) \\
&\leq \overline{N}(r,0; t_m e^{(m-1)\alpha} + \cdots + t_{m-i}e^{(m-i-1)\alpha} + \cdots + t_1) + S(r,f) \\
&\leq (m-1)T(r,f) + S(r,f),
\end{aligned}$$

which is a contradiction.

If $B \neq -1$, from (3.7), we have $\frac{1}{F} = \frac{BG}{(1+B)G-1}$ and so $\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r,0;F)$. Now, from the second fundamental theorem of Nevanlinna, we get

$$\begin{aligned}
T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r, \frac{1}{1+B}; G\right) + \overline{N}(r,\infty;G) + S(r,G) \\
&\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + S(r,G).
\end{aligned}$$

Using (2.1) and (2.2) we obtain from above inequality

$$\begin{aligned}
T(r,G) \leq\; & N_{k+1}(r,0; f^n(f-1)^m) + k\overline{N}(r,\infty;f) + T(r,G) + N_{k+1}(r,0;g^n(g-1)^m) \\
& - (n+m)T(r,g) + \overline{N}(r,\infty;g) + 2\log r + S(r,g).
\end{aligned}$$

Hence

$$(n+m)T(r,g) \leq (2k+m+2)T(r,f) + (k+m+3)T(r,g) + S(r,g).$$

In a similar way we can obtain

$$(n+m)T(r,f) \leq (k+m+3)T(r,f) + (2k+m+2)T(r,g) + S(r,g).$$

Thus, combining the above two, we obtain

$$(n-3k-m-5)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g),$$

a contradiction as $n > 3k+m+9$.

Case 2. Let $B \neq 0$ and $A \neq B$. Then from (3.6) we get $F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and so $\overline{N}(r, \frac{B-A+1}{B+1}; G) = \overline{N}(r,0;F)$. Proceeding as in **Subcase (i)** we obtain a contradiction.

Case 3. Let $B = 0$ and $A \neq 0$. Then from (3.6) $F = \frac{G+A-1}{A}$ and $G = AF - (A-1)$. If $A \neq 1$, we have $\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r,0;G)$ and $\overline{N}(r, 1-A; G) = \overline{N}(r,0;F)$. So, by Lemma 5, we have $n \leq 3k+m+5$, a contradiction. Thus $A = 1$ and hence $F = G$. That is

$$[f^n(f-1)^m]^{(k)} = [g^n(g-1)^m]^{(k)}.$$

Integrating we get

$$[f^n(f-1)^m]^{(k-1)} = [g^n(g-1)^m]^{(k-1)} + c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1} \neq 0$, from Lemma 5 we obtain $n \leq 3k + m + 2$, a contradiction. Hence $c_{k-1} = 0$. Repeating k-times, we obtain

$$f^n(f-1)^m = g^n(g-1)^m. \tag{3.13}$$

Now we consider following three subcases.

Subcase (i). Let $m = 0$. Then $f^n = g^n$ and so $f \equiv tg$ for a constant t such that $t^n = 1$.

Subcase (ii). Let $m = 1$. Then from (3.13) we have

$$f^n(f-1) \equiv g^n(g-1). \tag{3.14}$$

So by Lemma 8 we obtain $f \equiv g$.

Subcase (iii). Let $m \geq 2$. Then from (3.13) we obtain

$$f^n[f^m + \cdots + (-1)^i \ {}^mC_i f^{m-i} + \cdots + (-1)^m] = g^n[g^m + \cdots + (-1)^i \ {}^mC_i g^{m-i} + \cdots + (-1)^m]. \tag{3.15}$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ in (3.15) we obtain

$$g^{n+m}(h^{n+m} - 1) + \cdots + (-1)^i \ {}^mC_i g^{n+m-i}(h^{n+m-i} - 1) + \cdots + (-1)^m g^n(h^n - 1) = 0,$$

which imply $h = 1$. Hence $f \equiv g$.

If h is not a constant, then from (3.15) we can say that f and g satisfy the algebraic equation $R(f, g) = 0$, where

$$R(x, y) = x^n(x-1)^m - y^n(y-1)^m.$$

This completes the proof of the theorem. ∎

Proof of Theorem 1. Let $F(z)$ and $G(z)$ be given as in Lemma 7. Then $F(z)$, $G(z)$ are transcendental meromorphic functions that share $(1, 2)$ and $(\infty, 0)$. Proceeding in a similar way as in the proof of Theorem 2 and noting that $\log r = o\{T(r, f)\}$, we obtain the conclusions of the theorem. ∎

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