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A CLASS OF UNIVALENT FUNCTIONS INVOLVING A DIFFERENTIO-INTEGRAL OPERATOR

Abstract. This paper focuses on a generalized linear operator I^m which is a combination of both differential and integral operators. Involving this operator, a class $TS_k(I^m;\alpha,\beta)$ $(\subseteq TS(I^m;\alpha,\beta))$ with respect to k-symmetric points is defined. Results based on coefficient inequalities and bounds for this class are obtained. Various integral representations and some consequent results for $TS(I^m;\alpha,\beta)$ class are also determined. Further, results on partial sums are discussed.

1. Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n\, z^n, \tag{1.1}$$

which are analytic and univalent in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let T denote the subclass of S consisting of functions, whose coefficients from the second one are real and non-positive, of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n|\ z^n. \tag{1.2}$$

The convolution $*$ of two power series $\sum_{n=1}^{\infty} a_n\, z^n$ and $\sum_{n=1}^{\infty} b_n\, z^n$ is defined by

$$\sum_{n=1}^{\infty} a_n\, z^n * \sum_{n=1}^{\infty} b_n\, z^n = \sum_{n=1}^{\infty} a_n b_n\, z^n.$$

Based on the work of Silverman [16] on the class T of univalent functions, further studies involve various types of linear operators. We involve in this study, for some $m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, a generalized linear operator $I^m \equiv$

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$I^m(A,B,\mu,\lambda): T \to T$ which is defined for $A \ge 0, B \ge 0, \mu \ge 0$ and $\lambda > -1$, by

$$I^m(A,B,\mu,\lambda)f(z) = \phi^m(A,B,\mu)(z) * R^\lambda f(z), \tag{1.3}$$

where

$$R^\lambda f(z) = z - \sum_{n=2}^{\infty} \frac{(\lambda+1)_{n-1}}{(1)_{n-1}} \; |a_n| \; z^n \tag{1.4}$$

is the Ruscheweyh derivative [10] of the function $f \in T$ of the form (1.2) with the Pochammer symbol $(x)_n$ and

$$\phi^m(A,B,\mu)(z) = z + \sum_{n=2}^{\infty} \frac{\left(1 + \frac{A}{1+\mu}(n-1) + \mu\right)^{m-1}}{(1+B(n-1))^m} \; z^n.$$

The operator I^m is recently considered in [4].

Let, for $A \ge 0, \mu \ge 0$, an integral operator $\mathbf{I}_{A,\mu}: T \to T$ be defined by

$$\mathbf{I}_{A,\mu} f(z) = \begin{cases} \dfrac{(1+\mu) \; z^{1-\frac{1+\mu}{A}}}{A} \displaystyle\int_0^z t^{\frac{1+\mu}{A}-2} \; f(t), & A \ne 0, \\ f(z), & A = 0 \end{cases}$$

and a differential operator $\mathbf{D}_{A,\mu}: T \to T$ be defined by

$$\mathbf{D}_{A,\mu} f(z) = \frac{A}{1+\mu} \; z^{(2-\frac{1+\mu}{A})} \left(z^{(\frac{1+\mu}{A}-1)} \; f(z) \right)'.$$

We denote $\mathbf{I}_{A,0} \equiv \mathbf{I}_A$.

It is observed that the operator I^m is a *differentio-integral* operator and is a combination of the operators $\mathbf{I}_{A,\mu}$ and $\mathbf{D}_{A,\mu}$ defined on the Ruscheweyh derivative $R^\lambda f(z)$. We may express it as follows

$$\begin{aligned} I^0 f(z) &\equiv \mathbf{I}_{A,\mu} R^\lambda f(z) = \frac{(1+\mu) \; z^{1-\frac{1+\mu}{A}}}{A} \int_0^z t^{\frac{1+\mu}{A}-2} \; R^\lambda f(t), \; A \ne 0, \\ &= R^\lambda f(z), \; A = 0, \\ I^1 f(z) &\equiv \mathbf{I}_B \; \mathbf{J}^1 f(z) = \frac{z^{1-\frac{1}{B}}}{B} \int_0^z t^{\frac{1}{B}-2} \mathbf{J}^1 f(t), \; B \ne 0, \\ &= \mathbf{J}^1 f(z), \; B = 0, \\ I^2 f(z) &\equiv \mathbf{I}_B \mathbf{I}_B \mathbf{J}^2 f(z) = \mathbf{I}_B^2 \; \mathbf{J}^2 f(z), \; B \ne 0, \\ &= \mathbf{J}^2 f(z), B = 0, \end{aligned}$$

hence, for $m \geq 2$,

$$I^m f(z) \equiv \underbrace{\mathbf{I}_B \mathbf{I}_B ... \mathbf{I}_B}_{m \text{ times}} \mathbf{J}^m f(z) = \mathbf{I}_B^m \ \mathbf{J}^m f(z), \ B \neq 0 \tag{1.5}$$
$$= \mathbf{J}^m f(z), \ B = 0$$

where for some $m \in \mathbb{N}_0$, the operator $\mathbf{J}^m \equiv \mathbf{J}^m (A, \mu, \lambda) = I^m(A, 0, \mu, \lambda)$ is defined by

$$\mathbf{J}^0 f(z) = I^0 f(z), \ \mathbf{J}^1 f(z) = R^\lambda f(z), \text{ and for } m \geq 2, \tag{1.6}$$
$$\mathbf{J}^m f(z) = \begin{cases} \mathbf{D}_{A,\mu} \mathbf{J}^{m-1} f(z), & A \neq 0, \\ R^\lambda f(z), & A = 0. \end{cases}$$

Let for some $k \in \mathbb{N}$, $\epsilon_k = \exp\left(\frac{2\pi i}{k}\right)$, and for $f \in T$ of the form (1.2),

$$f_k(z) = \frac{1}{k} \sum_{j=0}^{k-1} \epsilon_k^{-j} f(\epsilon_k^j z) = z - \sum_{n=2}^{\infty} \phi_n^k \ |a_n| \ z^n \in T, \tag{1.7}$$

where $\phi_n^1 = 1$, and

$$\phi_n^k = \frac{1}{k} \sum_{j=0}^{k-1} \epsilon_k^{j(n-1)} = \begin{cases} 1, & n-1 = lk, l \in \mathbb{Z}, \\ 0, & n-1 = lk+p, \ p = 1, 2, 3, \dots, k-1. \end{cases}$$

Points $\epsilon_k^j \, z$ for $j = 1, 2, \dots, k-1$ ($k \in \mathbb{N} - \{1\}$) are called k-symmetric points. Clearly $f_1(z) = f(z)$ and $f_2(z) = \frac{f(z) - f(-z)}{2}$.

Thus, for $f \in T$ of the form (1.2), we write for convenience, the series expansion of (1.3) as

$$I^m f(z) = z - \sum_{n=2}^{\infty} P_n \ |a_n| \ z^n, \tag{1.8}$$

where

$$P_n = \frac{\left(1 + \frac{A}{1+\mu}(n-1)\right)^{m-1}}{(1 + B(n-1))^m} \frac{(\lambda+1)_{n-1}}{(1)_{n-1}} \tag{1.9}$$

and for $f_k \in T$ of the form (1.7),

$$I^m f_k(z) = z - \sum_{n=2}^{\infty} P_n \ \phi_n^k \ |a_n| \ z^n.$$

We also note that for $m \in \mathbb{N}$, the operator $\mathbf{J}^m$ defined by (1.6) includes various earlier defined operators, some of them based on the multiplier P_n

are as follows:

$$\mathbf{J}^m \equiv R^\lambda \text{ if } P_n = \frac{(\lambda+1)_{n-1}}{(1)_{n-1}}, \tag{1.10}$$
$$\mathbf{J}^m \equiv R^\lambda_{m,A} \text{ if } P_n = (1+A(n-1))^{m-1}\frac{(\lambda+1)_{n-1}}{(1)_{n-1}},$$
$$\mathbf{J}^m \equiv R^{\lambda,\mu}_m \text{ if } P_n = \left(\frac{n+\mu}{1+\mu}\right)^{m-1}\frac{(\lambda+1)_{n-1}}{(1)_{n-1}},$$
$$\mathbf{J}^m \equiv J^{m-1}_{A,\mu} \text{ if } P_n = \left(1+\frac{A}{1+\mu}(n-1)\right)^{m-1}.$$

Operator $R^\lambda_{m,A}$ is called the generalized Ruscheweyh derivative operator whereas $R^\lambda_{1,A} \equiv R^\lambda_{m,0} \equiv R^\lambda$ is the Ruscheweyh derivative operator [10]. Operators $R^\lambda_{m,A}, R^\lambda_{m,1}$ and $R^{\lambda,\mu}_m$ are earlier studied in [15] ([12]), [13] and [14] respectively by Shaqsi and Darus. Further, $R^{0,\mu}_m \equiv R^\mu_m$ is studied by Flett [8].

Operator $J^{m-1}_{A,\mu}$ is defined in [2] by Lupaş and is introduced by Cătuş [5] for p-valent functions. Also $J^{m-1}_{A,0} \equiv S^{m-1}_A$ is the generalized Sălăgean operator defined by Al-Oboudi [3] ([1]) whereas $S^{m-1}_0 \equiv S^{m-1}$ is the Sălăgean operator [11] of order $m-1$. The operator $J^{m-1}_{1,\mu}$ is studied by Cho and Srivastava [6] ([7]) and $J^{m-1}_{1,1}$ is studied by Urelgaddi and Somanatha [17].

From the operators I^m and $\mathbf{J}^m$ defined by (1.3) and (1.6), respectively, we find an identity operator: $I = I^0(0,B,\mu,0) \equiv I^1(A,0,\mu,0) \equiv \mathbf{J}^1(A,\mu,0) \equiv \mathbf{J}^m(0,\mu,0) : T \to T$ with $If(z) = f(z)$.

A function $f \in T$ is said to be in $TS_k(I^m;\alpha,\beta)$, a class of starlike functions of order α $(0 \le \alpha < 1)$ and type β $(0 < \beta \le 1)$ with respect to k-symmetric points involving the operator I^m, if and only if for $z \in U$, $I^m f_k(z) \neq 0$,

$$\left|\frac{\frac{z(I^m f(z))'}{I^m f_k(z)} - 1}{\frac{z(I^m f(z))'}{I^m f_k(z)} + (1-2\alpha)}\right| < \beta, z \in U. \tag{1.11}$$

Further, $f \in T$ is said to be in $TC_k(I^m;\alpha,\beta)$, a class of convex functions of order α $(0 \le \alpha < 1)$ and type β $(0 < \beta \le 1)$ with respect to k-symmetric points involving the operator I^m, if and only if $zf' \in TS_k(I^m;\alpha,\beta)$. Denote $TS_1(I^m;\alpha,\beta) = TS(I^m;\alpha,\beta)$; $TC_1(I^m;\alpha,\beta) = TC(I^m;\alpha,\beta)$ and $TS_2(I^m;\alpha,\beta) = TS_s(I^m;\alpha,\beta)$; $TC_2(I^m;\alpha,\beta) = TC_s(I^m;\alpha,\beta)$.

We note that the classes $TS(I;\alpha,\beta) \equiv S^*(\alpha,\beta)$ and $TC(I;\alpha,\beta) \equiv C^*(\alpha,\beta)$ were studied by Gupta and Jain in [9].

In this study, involving a generalized linear operator I^m which is a *differentio-integral* operator, a class $TS_k(I^m;\alpha,\beta)$ $(\subseteq TS(I^m;\alpha,\beta))$ with respect to k-symmetric points of univalent functions belonging to the class T, is defined. Results based on coefficient inequalities and bounds for this class are obtained. Various integral representations and some consequent results are determined for $TS(I^m;\alpha,\beta)$ class. Further, results on partial sums are discussed for the class $TS(\mathbf{J}^m;\alpha,\beta)$.

2. Coefficient inequalities

We now give results for the class $TS_k(I^m;\alpha,\beta)$.

Theorem 1. *Let a linear operator I^m, under its parametric conditions, be defined by (1.3). A function $f(z)=z-\sum\limits_{n=2}^{\infty}|a_n|\ z^n\in TS_k(I^m;\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}\Big[n(1+\beta)-\{1-\beta\,(1-2\alpha)\}\,\phi_n^k\Big]\ P_n\,|a_n|\ \le 2\beta\,(1-\alpha)\,, \tag{2.1}$$

where P_n is given by (1.9). The result is sharp, for the extremal function given for some $n\ge 2$ by

$$f(z)=z-\frac{2\beta\,(1-\alpha)}{[n(1+\beta)-\{1-\beta\,(1-2\alpha)\}\,\phi_n^k]\,P_n}z^n. \tag{2.2}$$

Proof. Consider for $z\in U$,

$$\begin{aligned}
&\Big|z\,(I^m f(z))'-I^m f_k(z)\Big|-\beta\Big|z\,(I^m f(z))'+(1-2\alpha)\,I^m f_k(z)\Big|\\
&\quad<\left|-\sum_{n=2}^{\infty}\left(n-\phi_n^k\right)P_n\ |a_n|\ z^{n-1}\right|\\
&\qquad-\beta\left|2\,(1-\alpha)-\sum_{n=2}^{\infty}\left\{n+(1-2\alpha)\,\phi_n^k\right\}P_n\ |a_n|\ z^{n-1}\right|\\
&\quad\le\sum_{n=2}^{\infty}\left(n-\phi_n^k\right)P_n\ |a_n|\ -2\beta\,(1-\alpha)+\sum_{n=2}^{\infty}\beta\left\{n+(1-2\alpha)\,\phi_n^k\right\}P_n\ |a_n|\\
&\quad\le 0,
\end{aligned}$$

if (2.1) holds. This proves that the condition (1.11) is true. Hence, $f\in TS_k(I^m;\alpha,\beta)$. Conversely, let for $z\in U$, $0\ne I^m f_k(z)\in T$,

$$\left|\frac{\frac{z(I^m f(z))'}{I^m f_k(z)}-1}{\frac{z(I^m f(z))'}{I^m f_k(z)}+(1-2\alpha)}\right|<\beta, z\in U.$$

Since $|\mathrm{Re}(z)| \le |z|, z \in U$, we have

$$\left|\mathrm{Re}\left\{\frac{\frac{z(I^m f(z))'}{I^m f_k(z)} - 1}{\frac{z(I^m f(z))'}{I^m f_k(z)} + (1-2\alpha)}\right\}\right| < \beta.$$

On choosing such real value of $z \in U$ for which $\frac{z(I^m f(z))'}{I^m f_k(z)}$ is real and then taking $z \to 1^-$, we get

$$\frac{\sum\limits_{n=2}^{\infty} \left(n - \phi_n^k\right) P_n \ |a_n|}{2(1-\alpha) - \sum\limits_{n=2}^{\infty} \{n + (1-2\alpha)\phi_n^k\} P_n \ |a_n|} \le \beta,$$

which proves (2.1). Equality in (2.1) holds for the function given by (2.2). This proves Theorem 1. ∎

Corollary 1. *A function* $f(z) = z - \sum\limits_{n=2}^{\infty} |a_n| \ z^n \in TS(I^m;\alpha,\beta)$ *if and only if*

$$\sum_{n=2}^{\infty} [n(1+\beta) - \{1 - \beta(1-2\alpha)\}] \ P_n |a_n| \ \le 2\beta(1-\alpha), \tag{2.3}$$

where P_n *is given by* (1.9)*. The result is sharp, the extremal function is given for some* $n \ge 2$ *by*

$$f(z) = z - \frac{2\beta(1-\alpha)}{[n(1+\beta) - \{1-\beta(1-2\alpha)\}] P_n} z^n.$$

Corollary 2. *A function* $f(z) = z - \sum\limits_{n=2}^{\infty} |a_n| \ z^n \in TS_s(I^m;\alpha,\beta)$ *if and only if*

$$\sum_{n=2}^{\infty} \left[n(1+\beta) - \{1-\beta(1-2\alpha)\} \frac{\{1-(-1)^n\}}{2}\right] \ P_n |a_n| \ \le 2\beta(1-\alpha), \tag{2.4}$$

where P_n *is given by* (1.9)*. The result is sharp, the extremal function is given for some* $n \ge 2$ *by*

$$f(z) = z - \frac{2\beta(1-\alpha)}{\left[n(1+\beta) - \{1-\beta(1-2\alpha)\} \frac{\{1-(-1)^n\}}{2}\right] P_n} z^n.$$

Corollary 3. *A function* $f(z) = z - \sum\limits_{n=2}^{\infty} |a_n| \ z^n \in TS_k(\mathbf{J}^m;\alpha,\beta)$ *if and only if*

$$\sum_{n=2}^{\infty}\left[n(1+\beta)-\{1-\beta\,(1-2\alpha)\}\,\phi_n^k\right]Q_n\,|a_n|\ \le 2\beta(1-\alpha), \tag{2.5}$$

where

$$Q_n=\left(1+\frac{A}{(1+\mu)}(n-1)\right)^{m-1}\frac{(\lambda+1)_{n-1}}{(1)_{n-1}}. \tag{2.6}$$

The result is sharp, for the extremal function given for some $n\ge 2$ by

$$f(z)=z-\frac{2\beta\,(1-\alpha)}{[n(1+\beta)-\{1-\beta\,(1-2\alpha)\}\,\phi_n^k]\,Q_n}z^n.$$

Remark 1. Results, similar to Corollaries 1 and 2 can also be obtained for $TS(\mathbf{J}^m;\alpha,\beta)$ and $TS_s(\mathbf{J}^m;\alpha,\beta)$ classes.

Corollary 4. *If a function $f(z)=z-\sum\limits_{n=2}^{\infty}|a_n|\ z^n\in TS_k(I^m;\alpha,\beta)$, then*

$$\sum_{n=2}^{\infty}P_n\,|a_n|\le\frac{2\beta\,(1-\alpha)}{[1+\beta\,(3-2\alpha)]},$$

where P_n is given by (1.9).

Proof. From Theorem 1, we have

$$[2(1+\beta)-\{1-\beta\,(1-2\alpha)\}]\sum_{n=2}^{\infty}P_n\,|a_n| \tag{2.7}$$
$$\le\sum_{n=2}^{\infty}\left[n(1+\beta)-\{1-\beta\,(1-2\alpha)\}\,\phi_n^k\right]\ P_n\,|a_n|\le 2\beta(1-\alpha),$$

which proves the result. ∎

Theorem 2. *Let the linear operator I^m under its parametric conditions along with $\frac{A}{1+\mu}\le B$, be defined by* (1.3). *If a function $f\in T$ of the form* (1.2) *satisfies*

$$\sum_{n=2}^{\infty}[n(1+\beta)-\{1-\beta\,(1-2\alpha)\}]\frac{(\lambda+1)_{n-1}}{(1)_{n-1}}|a_n| \tag{2.8}$$
$$\le\frac{2\beta\,(1-\alpha)\,(1+B)^m}{\left(1+\frac{A}{1+\mu}\right)^{m-1}},$$

then $f\in TS(I^m;\alpha,\beta)$.

Proof. Since,

$$\sum_{n=2}^{\infty}[n(1+\beta)-\{1-\beta\,(1-2\alpha)\}]\ \ P_n\,|a_n|$$

$$\leq \frac{1}{(1+B)^m}\left(1+\frac{A}{1+\mu}\right)^{m-1}$$

$$\sum_{n=2}^{\infty}[n(1+\beta)-\{1-\beta\,(1-2\alpha)\}]\ \ \frac{(\lambda+1)_{n-1}}{(1)_{n-1}}\,|a_n|$$

$$\leq 2\beta\,(1-\alpha)\,,$$

if (2.3) holds. Hence, by Corollary 1, $f \in TS(I^m;\alpha,\beta)$. ∎

3. Bounds

Theorem 3. *If a function* $f(z) = z - \sum\limits_{n=2}^{\infty} |a_n|\ \ z^n \in TS_k(I^m;\alpha,\beta)$*, then*

$$r-\frac{2\beta\,(1-\alpha)}{[1+\beta\,(3-2\alpha)]}r^2 \leq |I^m f(z)| \leq r+\frac{2\beta\,(1-\alpha)}{[1+\beta\,(3-2\alpha)]}r^2,\ \ |z|=r<1. \tag{3.1}$$

The bounds are sharp for the function:

$$I^m f(z) = z-\frac{2\beta\,(1-\alpha)}{[1+\beta\,(3-2\alpha)]}\ z^2,\ \ z=\pm r\ \ (r<1)\,. \tag{3.2}$$

Proof. From (1.8) and the Corollary 4, we get

$$|I^m f(z)| = \left|z-\sum_{n=2}^{\infty}P_n\ \ |a_n|\ \ z^n\right| \leq |z|+|z|^2\sum_{n=2}^{\infty}P_n\ \ |a_n|$$

$$\leq r+\frac{2\beta\,(1-\alpha)}{[1+\beta\,(3-2\alpha)]}r^2.$$

Similarly, we get the other side of inequality (3.1). Since, equality attains for the function given by (3.2), the sharpness is verified. ∎

Corollary 5. *If a function* $f(z) = z - \sum\limits_{n=2}^{\infty} |a_n|\ \ z^n \in TS_k(I^m;\alpha,\beta)$*, then* $I^m f$ *maps the disk* U *onto a domain that contains the disk* $\left\{w : |w| < \frac{1+\beta}{[1+\beta(3-2\alpha)]}\right\}$.

Corollary 6. *If a function* $f(z) = z - \sum\limits_{n=2}^{\infty} |a_n|\ \ z^n \in TS_k(\mathbf{J}^m;\alpha,\beta)$*, then for* $|z| = r < 1$*,*

$$|f(z)| \leq r + \frac{2\beta(1-\alpha)}{\left(1+\frac{A}{(1+\mu)}\right)^{m-1}(\lambda+1)[1+\beta(3-2\alpha)]}r^2, \tag{3.3}$$

$$|f(z)| \geq r - \frac{2\beta(1-\alpha)}{\left(1+\frac{A}{(1+\mu)}\right)^{m-1}(\lambda+1)[1+\beta(3-2\alpha)]}r^2. \tag{3.4}$$

The bounds are sharp for the function:

$$f(z) = z - \frac{2\beta(1-\alpha)}{\left(1+\frac{A}{(1+\mu)}\right)^{m-1}(\lambda+1)[1+\beta(3-2\alpha)]}\ z^2, \tag{3.5}$$

$$z = \pm r\ \ (r<1),$$

and the disk U is mapped by f onto a domain that contains the disk:

$$\left\{w : |w| < \left(1 - \frac{2\beta(1-\alpha)}{\left(1+\frac{A}{(1+\mu)}\right)^{m-1}(\lambda+1)[1+\beta(3-2\alpha)]}\right)\right\}.$$

From (2.1) and (2.3), we have $TS_k(I^m;\alpha,\beta) \subseteq TS(I^m;\alpha,\beta)$. We prove next results for $TS(I^m;\alpha,\beta)$ class.

4. Integral representations

Theorem 4. *A function $f(z) = z - \sum_{n=2}^{\infty} |a_n|\ z^n \in TS(I^m;\alpha,\beta)$ if and only if*

$$I^m f(z) = z\exp\left\{2(1-\alpha)\int_0^z \frac{\varphi(t)}{(1-t\varphi(t))}dt\right\}, \tag{4.1}$$

where $\varphi(z)$ is analytic and $|\varphi(z)| < \beta$, $z \in U$.

Proof. From (4.1), we get

$$\left[\log\frac{I^m f(z)}{z}\right]' = 2(1-\alpha)\frac{\varphi(z)}{(1-z\varphi(z))}$$

or,

$$\frac{z\left(I^m f(z)\right)'}{I^m f(z)} - 1 = 2(1-\alpha)\frac{z\varphi(z)}{(1-z\varphi(z))}$$

or,

$$\frac{\frac{z(I^m f(z))'}{I^m f(z)} - 1}{\frac{z(I^m f(z))'}{I^m f(z)} + (1-2\alpha)} = z\varphi(z),$$

which proves that $f \in TS(I^m;\alpha,\beta)$. Conversely, let

$$\frac{\frac{z(I^m f(z))'}{I^m f(z)} - 1}{\frac{z(I^m f(z))'}{I^m f(z)} + (1-2\alpha)} = w(z), \tag{4.2}$$

where $w(z)$ is analytic in U with $w(0) = 0$ and $|w(z)| < \beta$. So, letting $w(z) = z\varphi(z)$ with $|\varphi(z)| < \beta$, $z \in U$, we get the representation (4.1). ■

Remark 2. Results of Gupta and Jain [9] for the classes $S^*(\alpha,\beta)$ and $C^*(\alpha,\beta)$ follow from our results obtained in Theorems 1, 3 and 4.

Theorem 5. *A function* $f(z) = z - \sum\limits_{n=2}^{\infty} |a_n|\ z^n \in TS(I^m;\alpha,\beta)$ *if and only if*

$$I^m f(z) = \exp \int_0^z \frac{1+(1-2\alpha)\,w(t)}{t\,(1-w(t))} dt, \tag{4.3}$$

where $w(z)$ *is analytic and* $|w(z)| < \beta$, $z \in U$.

Proof. From (4.3), we get for $z \in U$, $0 \neq I^m f(z)$,

$$(\log I^m f(z))' = \frac{1+(1-2\alpha)\,w(z)}{z\,(1-w(z))}, z \in U,$$

which proves

$$\frac{z\,(I^m f(z))'}{I^m f(z)} = \frac{1+(1-2\alpha)\,w(z)}{1-w(z)}, \tag{4.4}$$

and hence,

$$|w(z)| = \left| \frac{\frac{z(I^m f(z))'}{I^m f(z)} - 1}{\frac{z(I^m f(z))'}{I^m f(z)} + (1-2\alpha)} \right| < \beta,$$

which proves that $f \in TS(I^m;\alpha,\beta)$. Similarly, converse part can be proved. ■

Theorem 6. *A function* $f(z) = z - \sum\limits_{n=2}^{\infty} |a_n|\ z^n \in TS(I^m;\alpha,\beta)$ *if*

$$\frac{z\,(I^m f(z))'}{I^m f(z)} \prec \frac{1+(1-2\alpha)\,\beta z}{1-\beta z}, z \in U. \tag{4.5}$$

Proof. Let (4.5) holds, we get by subordination property

$$\frac{z\left(I^m f(z)\right)'}{I^m f(z)} = \frac{1+(1-2\alpha)\, w(z)}{1-w(z)},$$

where $w(z)$ is analytic and $|w(z)| < \beta \leq 1$, $z \in U$. Hence,

$$|w(z)| = \left| \frac{\frac{z(I^m f(z))'}{I^m f(z)} - 1}{\frac{z(I^m f(z))'}{I^m f(z)} + (1-2\alpha)} \right| < \beta,$$

which proves that $f \in TS(I^m; \alpha, \beta)$. ■

5. Some consequent results

Theorem 7. *If a function* $f(z) = z - \sum\limits_{n=2}^{\infty} |a_n| \;\; z^n \in TS(I^m; \alpha, \beta)$, *then for* $\zeta, |\zeta| = 1$,

(5.1) $$\frac{1}{z}\left[\frac{z+Kz^2}{(1-z)^2} * I^m f(z)\right] \neq 0,$$

where

(5.2) $$K = \frac{\{1-(1-2\alpha)\beta\zeta\}}{2(1-\alpha)\beta\zeta}.$$

Or, equivalently

(5.3) $$1 - \sum_{n=2}^{\infty} \left[n + (n-1)\, K\right] \;\; P_n \left|a_n\right| z^{n-1} \neq 0,$$

where P_n *is given by* (1.9).

Proof. If $f \in TS(I^m; \alpha, \beta)$, then we have

$$\frac{z\left(I^m f(z)\right)' - I^m f(z)}{z\left(I^m f(z)\right)' + (1-2\alpha)\, I^m f(z)} \neq \beta\zeta$$

and hence,

$$\frac{1}{2(1-\alpha)\beta\zeta z}\left\{z\left(I^m f(z)\right)' (1+\beta\zeta) - I^m f(z)\left\{1-(1-2\alpha)\beta\zeta\right\}\right\} \neq 0.$$

On writing

$$I^m f(z) = \frac{z}{(1-z)} * I^m f(z), \;\; z\left(I^m f(z)\right)' = \frac{z}{(1-z)^2} * I^m f(z),$$

we get

$$\frac{1}{z}\left[\frac{z}{(1-z)^2}\left\{1+\frac{\{1-(1-2\alpha)\beta\zeta\}\, z}{2(1-\alpha)\beta\zeta}\right\} * I^m f(z)\right] \neq 0,$$

which proves (5.1). Further, using the expansions, we get (5.3). ∎

Remark 3. For K given by (5.2) and $|z| = r < 1$, we see that

$$\begin{aligned}\left|1-\sum_{n=2}^{\infty}[n+(n-1)\,K]\ \ P_n\,|a_n|\, z^{n-1}\right| &\geq 1-\sum_{n=2}^{\infty}[n+(n-1)\,K]\ \ P_n\,|a_n|\, r^{n-1}\\ &> 1-\sum_{n=2}^{\infty}[n+(n-1)\,K]\ \ P_n\,|a_n| \geq 0,\end{aligned}$$

if (2.3) is true and hence, negation (5.3) is verified for the class $TS(I^m;\alpha,\beta)$.

Theorem 8. *If a function* $f(z) = z - \sum\limits_{n=2}^{\infty} |a_n|\ \ z^n \in TS(I^m;\alpha,\beta)$, *then*

$$\mathrm{Re}\left\{\frac{z\,(I^m f(z))'}{I^m f(z)}\right\} > \frac{1-(1-2\alpha)\,\beta}{1+\beta},\ \ z \in U.$$

Proof. Since,

$$\begin{aligned}\left|\frac{z\,(I^m f(z))'}{I^m f(z)}-1\right| = \left|\frac{-\sum\limits_{n=2}^{\infty}(n-1)\,P_n\ \ |a_n|\, z^{n-1}}{1-\sum\limits_{n=2}^{\infty}P_n\ \ |a_n|\, z^{n-1}}\right| &\leq \frac{\sum\limits_{n=2}^{\infty}(n-1)\,P_n\,|a_n|}{1-\sum\limits_{n=2}^{\infty}P_n\,|a_n|}\\ &\leq \frac{2(1-\alpha)\beta}{1+\beta},\end{aligned}$$

if inequality (2.3) holds and hence, we get the result. ∎

6. Partial sums

Let

$$f_1(z) = z,\ \ f_p(z) = z - \sum_{n=2}^{p} |a_n|\ \ z^n, p \geq 2$$

be the partial sums of functions $f \in T$ of the form (1.2).

Theorem 9. *If a function* $f(z) = z - \sum\limits_{n=2}^{\infty} |a_n|\ \ z^n \in TS(\mathbf{J}^m;\alpha,\beta)$, *then*

$$\mathrm{Re}\left(\frac{f(z)}{f_p(z)}\right) > 1-\frac{1}{c_{p+1}}, \tag{6.1}$$

$$\mathrm{Re}\left(\frac{f_p(z)}{f(z)}\right) > \frac{c_{p+1}}{1+c_{p+1}}. \tag{6.2}$$

Proof. In view of the Remark 1, from Corollary 3, we get

$$\sum_{n=2}^{\infty} c_n \ |a_n| \ \le 1, \tag{6.3}$$

where

$$c_n = \frac{[n(1+\beta) - \{1 - \beta\,(1-2\alpha)\}] \ Q_n}{2\beta\,(1-\alpha)},$$

and Q_n is given by (2.6). It is easy to verify that

$$c_{n+1} > c_n > 1, n \ge 2.$$

Hence,

$$\sum_{n=2}^{p} |a_n| + c_{p+1} \sum_{n=p+1}^{\infty} |a_n| \le \sum_{n=2}^{\infty} c_n \ |a_n| \le 1. \tag{6.4}$$

Set

$$w_1(z) = c_{p+1} \left\{ \frac{f(z)}{f_p(z)} - \left(1 - \frac{1}{c_{p+1}}\right) \right\} = 1 - \frac{c_{p+1} \sum\limits_{n=p+1}^{\infty} |a_n| \ z^{n-1}}{1 - \sum\limits_{n=2}^{p} |a_n| \ z^{n-1}}.$$

With the use of (6.4), we get

$$\left| \frac{w_1(z) - 1}{w_1(z) + 1} \right| = \left| \frac{-c_{p+1} \sum\limits_{n=p+1}^{\infty} |a_n| \ z^{n-1}}{2 - 2\sum\limits_{n=2}^{p} |a_n| \ z^{n-1} - c_{p+1} \sum\limits_{n=p+1}^{\infty} |a_n| \ z^{n-1}} \right|$$

$$\le \frac{c_{p+1} \sum\limits_{n=p+1}^{\infty} |a_n|}{2 - 2\sum\limits_{n=2}^{p} |a_n| \ - c_{p+1} \sum\limits_{n=p+1}^{\infty} |a_n|} \le 1,$$

which yields the result (6.1). Similarly, if

$$w_2(z) = (1 + c_{p+1}) \left\{ \frac{f_p(z)}{f(z)} - \left(\frac{c_{p+1}}{1 + c_{p+1}} \right) \right\}$$

$$= 1 + \frac{(1 + c_{p+1}) \sum\limits_{n=p+1}^{\infty} |a_n| \ z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} |a_n| \ z^{n-1}}.$$

Again, with the use of (6.4), we get

$$\left|\frac{w_2(z)-1}{w_2(z)+1}\right| = \left|\frac{(1+c_{p+1})\sum\limits_{n=p+1}^{\infty}|a_n|\ z^{n-1}}{2-2\sum\limits_{n=2}^{\infty}|a_n|\ z^{n-1}+(1+c_{p+1})\sum\limits_{n=p+1}^{\infty}|a_n|\ z^{n-1}}\right|$$

$$\leq \frac{(1+c_{p+1})\sum\limits_{n=p+1}^{\infty}|a_n|}{2-2\sum\limits_{n=2}^{p}|a_n|\ -(c_{p+1}-1)\sum\limits_{n=p+1}^{\infty}|a_n|} \leq 1,$$

which yields (6.2). This proves Theorem 9. ∎

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