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Gopal Datt, Deepak Kumar Porwal

PRODUCT OF WEIGHTED HANKEL AND WEIGHTED TOEPLITZ OPERATORS

Abstract. In this paper, we discuss some properties of the weighted Hankel operator H_ψ^β and describe the conditions on which the weighted Hankel operator H_ψ^β and weighted Toeplitz operator T_ϕ^β, with $\phi, \psi \in L^\infty(\beta)$ on the space $H^2(\beta)$, $\beta = \{\beta_n\}_{n\in\mathcal{Z}}$ being a sequence of positive numbers with $\beta_0 = 1$, commute. It is also proved that if a non-zero weighted Hankel operator H_ψ^β commutes with T_ϕ^β, which is not a multiple of the identity, then $H_\psi^\beta = \mu T_\phi^\beta$, for some $\mu \in \mathbb{C}$.

1. Preliminaries

Let $\beta = \{\beta_n\}_{n\in\mathcal{Z}}$ be a sequence of positive numbers with $\beta_0 = 1$. Let $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, be the formal Laurent series (whether or not the series converges for any values of z). Define $\|f\|_\beta$ as

$$\|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 {\beta_n}^2.$$

The space $L^2(\beta)$ consists of all $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ for which $\|f\|_\beta < \infty$. The space $L^2(\beta)$ is a Hilbert space with the norm $\|\cdot\|_\beta$ induced by the inner product

$$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} a_n\ \bar{b}_n {\beta_n}^2,$$

for $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $g(z) = \sum_{n=-\infty}^{\infty} b_n z^n$. The collection $\{e_n(z) = z^n/\beta_n\}_{n\in\mathcal{Z}}$ form an orthonormal basis for $L^2(\beta)$.

The collection of all $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (formal power series) for which $\|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 {\beta_n}^2 < \infty$, is denoted by $H^2(\beta)$. $H^2(\beta)$ is a subspace of $L^2(\beta)$.

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Let $L^\infty(\beta)$ denotes the set of formal Laurent series $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ such that $\phi L^2(\beta) \subseteq L^2(\beta)$ and there exists some $c > 0$ satisfying $\|\phi f\|_\beta \leq c\|f\|_\beta$, for each $f \in L^2(\beta)$. For $\phi \in L^\infty(\beta)$, define the norm $\|\phi\|_\infty$ as

$$\|\phi\|_\infty = \inf\{c > 0 : \|\phi f\|_\beta \leq c\|f\|_\beta, \text{ for all } f \in L^2(\beta)\}.$$

$L^\infty(\beta)$ is a Banach space with respect to $\|\cdot\|_\infty$. $H^\infty(\beta)$ denotes the set of formal Power series ϕ such that $\phi H^2(\beta) \subseteq H^2(\beta)$.

The study over these spaces is more interesting as well as demandable because of the tendency of these spaces to cover Bergman spaces, Hardy spaces and Dirichlet spaces (see [11]). Reference [11] provides a nice survey over the historical growth, details and applications of these spaces. If $\beta_n = 1$, for each $n \in \mathcal{Z}$ and the functions under considerations are complex-valued measurable functions defined over the unit cicle $\mathbb{T}$ then these spaces coincide with classical spaces $L^2(\mathbb{T})$, $H^2(\mathbb{T})$, $L^\infty(\mathbb{T})$ and $H^\infty(\mathbb{T})$. In this literature, we consider the spaces $L^2(\beta)$, $H^2(\beta)$, $L^\infty(\beta)$ and $H^\infty(\beta)$ under the assumption that $\beta = \{\beta_n\}_{n\in\mathcal{Z}}$ is a sequence of positive numbers with $\beta_0 = 1$, $r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$, for $n \geq 0$ and $r \leq \frac{\beta_n}{\beta_{n-1}} \leq 1$, for $n \leq 0$, for some $r > 0$.

2. Motivations and aims

In 1964, Brown and Halmos [1] studied algebraic properties of a class of operators on the space $H^2(\mathbb{T})$ known as Toeplitz operators $T_\phi = PM_\phi$, where P is an orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. In 1980, Power [8] studied the Hankel operators $S_\phi = PJM_\phi$ defined on the space $H^2(\mathbb{T})$, where J is the reflection operator. We would prefer the references [8] and [10] for the readers to get in touch of the study made about these operators in the past for quite some times.

In [11], Shield made a comprehensive study of the operator $M_\phi^\beta(f \mapsto \phi f)$ on $L^2(\beta)$ with the symbol $\phi \in L^\infty(\beta)$.

Let $P^\beta : L^2(\beta) \to H^2(\beta)$ be the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$ and $J^\beta : L^2(\beta) \to L^2(\beta)$ denote the reflection operator defined as $J^\beta f = \sum_{n=-\infty}^{\infty} a_n \beta_n e_{-n}$, for each $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ in $L^2(\beta)$, where $\{e_n(z) = z^n/\beta_n\}_{n\in\mathcal{Z}}$ is the orthonormal basis of $L^2(\beta)$. In the year 2005, Lauric [5] discussed the notion of weighted Toeplitz operator $T_\phi^\beta = P^\beta M_\phi^\beta$ on $H^2(\beta)$ and made a comprehensive study towards the commutant of these operators. Motivated by the work of Power [8], authors in [2] made a study on the weighted Hankel operator $H_\phi^\beta = P^\beta J^\beta M_\phi^\beta$ on $H^2(\beta)$ and proved that if $\beta = \{\beta_n\}_{n\in\mathcal{Z}}$ is a semi-dual sequence (i.e. $\beta_{-n} = \beta_n$ for each natural number n) then the weighted Hankel and weighted Toeplitz operators suggests the following connections between them

(2.1) $$H^{\beta}_{z\tilde{\phi}}H^{\beta}_{z\psi} = T^{\beta}_{\phi\psi} - T^{\beta}_{\phi}T^{\beta}_{\psi} \text{ and } H^{\beta}_{z\tilde{\phi}}T^{\beta}_{z^{-1}\psi} = H^{\beta}_{\tilde{\phi}\psi} - T^{\beta}_{\phi}H^{\beta}_{\psi}.$$

These relations were known among the Hankel and Toeplitz operator from long back (see [8]). In [9], D. Sarason made use of these relations to study the semicommutators of the Toeplitz operators. The work presented in this note is due to a motivation from the work of Sarason [9] and Power [8]. We are focussed to identify the weighted Toeplitz operators and weighted Hankel operators that commute. During the course of study various compact Hankel operators are obtained. It is also shown that a weighted Hankel operator on $H^2(\beta)$ cannot be an isometry and then a characterization for the symbol $\psi \in L^\infty(\beta)$ is obtained so that the product of induced weighted Toeplitz operator T^{β}_{ψ} with any weighted Hankel operator becomes a weighted Hankel operator (see Theorem 3.6).

3. Product of H^{β}_{ϕ} and T^{β}_{ψ}

Let H^{β}_{ϕ} be the weighted Hankel operator on $H^2(\beta)$ with the symbol $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ in $L^\infty(\beta)$, then for $j \geq 0$,

$$H^{\beta}_{\phi} e_j = \frac{1}{\beta_j} \sum_{n=0}^{\infty} a_{-n-j}\beta_{-n} e_n.$$

The adjoint $H^{\beta *}_{\phi}$ of the weighted Hankel operator H^{β}_{ϕ} is given by

$$H^{\beta *}_{\phi} e_j = \beta_{-j} \sum_{n=0}^{\infty} \overline{a}_{-n-j} \frac{e_n}{\beta_n}, \quad \text{for } j \geq 0.$$

We begin with the following observation.

Theorem 3.1. *A weighted Hankel operator on $H^2(\beta)$ cannot be an isometry.*

Proof. Let $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$. If possible, a non-zero weighted Hankel operator H^{β}_{ϕ} on $H^2(\beta)$ is an isometry. Then for each $j \geq 0$, $\|H^{\beta}_{\phi} e_j\| = 1$ or equivalently

(3.1.1) $$\frac{1}{\beta_j^2} \sum_{n=0}^{\infty} |a_{-n-j}|^2 \beta_{-n}^2 = 1.$$

Now, we use equation (3.1.1) and apply the strong induction method to achieve the goal. Indeed, equation (3.1.1) with $j = 0, 1$ gives $\sum_{n=0}^{\infty} |a_{-n}|^2 \beta_{-n}^2 = 1$ and

$$1 = \frac{1}{\beta_1^2} \sum_{n=0}^{\infty} |a_{-n-1}|^2 \beta_{-n}^2 \leq \frac{1}{\beta_1^2} \sum_{n=0}^{\infty} |a_{-n-1}|^2 \beta_{-n-1}^2$$

$$= \frac{1}{\beta_1^2} \sum_{n=0}^{\infty} |a_{-n}|^2 \beta_{-n}^2 - \frac{|a_0|^2 \beta_0^2}{\beta_1^2} \leq 1 - \frac{|a_0|^2 \beta_0^2}{\beta_1^2} \leq 1.$$

As a consequence, $\frac{|a_0|^2}{\beta_1^2} = 0$ so that $a_0 = 0$. Now assume that for a natural number n_0, $a_0 = a_{-1} = a_{-2} = \cdots = a_{-n_0+1} = 0$. Then equation (3.1.1), for $j = n_0$ and $n_0 + 1$ gives

$$\frac{1}{\beta_{n_0}^2} \sum_{n=0}^{\infty} |a_{-n-n_0}|^2 \beta_{-n}^2 = 1$$

and

$$\begin{aligned} 1 = \frac{1}{\beta_{n_0+1}^2} \sum_{n=0}^{\infty} |a_{-n-n_0-1}|^2 \beta_{-n}^2 &\leq \frac{1}{\beta_{n_0+1}^2} \sum_{n=0}^{\infty} |a_{-n-n_0-1}|^2 \beta_{-n-1}^2 \\ &= \frac{1}{\beta_{n_0+1}^2} \Big(\sum_{n=0}^{\infty} |a_{-n-n_0}|^2 \beta_{-n}^2 - |a_{-n_0}|^2 \Big) \\ &= \frac{1}{\beta_{n_0+1}^2} (\beta_{n_0}^2 - |a_{-n_0}|^2) \\ &\leq 1 - \frac{|a_{-n_0}|^2}{\beta_{n_0+1}^2} \leq 1. \end{aligned}$$

This implies that, $\frac{|a_{-n_0}|^2}{\beta_{n_0+1}^2} = 0$ and hence $a_{-n_0} = 0$. Therefore, by the principle of mathematical induction, $a_n = 0$, for each $n \leq 0$. Consequently, $\phi \in zH^\infty(\beta)$ and hence $H_\phi^\beta = 0$. This is a contradiction. Hence the result. ∎

It is interesting to note that each symbol ϕ in $\{z^{-j} : \ j \geq 0\}$ induces a non-zero weighted Hankel operator on $H^2(\beta)$. In fact, each such H_ϕ^β is a finite rank operator, as for $j \geq 0$,

$$H_{z^{-j}}^\beta e_m = \begin{cases} \frac{\beta_{m-j}}{\beta_m} e_{-m+j}, & \text{if } 0 \leq m \leq j, \\ 0, & \text{otherwise.} \end{cases}$$

With this observation, it is easy to extend the result if ϕ is a Laurent polynomial i.e. $\phi(z) = a_{-n}z^{-n} + a_{-n+1}z^{-n+1} + \cdots + a_{-1}z^{-1} + a_0 + a_1 z^1 + \cdots + a_m z^m$, for some $m, n \geq 0$.

Theorem 3.2. *If $\phi \in L^\infty(\beta)$ is a Laurent polynomial then the weighted Hankel operator H_ϕ^β on $H^2(\beta)$ is a compact operator.*

Proof. In this situation, $H_\phi^\beta = a_{-n} H_{z^{-n}} + a_{-n+1} H_{z^{-n+1}} + \cdots + a_{-1} H_{z^{-1}} + a_0 H_1$, which is a compact operator being a finite sum of compact operators. ∎

It immediately yields the following.

Corollary 3.3. *If $\phi \in L^\infty(\beta)$ is of the form $\phi(z) = \sum_{n=-m}^{\infty} a_n z^n$, $m \geq 0$ then the weighted Hankel operator H_ϕ^β on $H^2(\beta)$ is a compact operator.*

Now we show an example insuring that these are not the only ϕ inducing the compact weighted Hankel operators.

Example. Consider the sequence $\beta = \{\beta_n\}_{n\in\mathcal{Z}}$, where $\beta_n = 1$, for each n. Let $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, where

$$a_n = \begin{cases} 2^n, & \text{if } n \leq 0, \\ 0, & \text{if } n > 0. \end{cases}$$

Then $\phi \in L^\infty(\beta)$ as the series $\sum_{n=-\infty}^{\infty} a_{-n} z^n \in L^\infty(\beta)$ being bounded and analytic in $|z| \leq 1$ (see [11, Theorem 10'(vii)]) and $L^\infty(\beta) \equiv L^\infty(\mathbb{T})$.

$$\begin{aligned} \sum_{j=0}^{\infty} \|H_\phi^\beta e_j\|^2 = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} |a_{-n-j}|^2 &= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{2^{n+j}} \\ &= \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{j=0}^{\infty} \frac{1}{2^j} < \infty. \end{aligned}$$

Hence, H_ϕ^β is a Hilbert–Schmidt operator and so is compact.

Corollary 3.4. *If $\phi \in L^\infty(\beta)$ is of the form $\phi(z) = \sum_{n=-m}^{\infty} a_n z^n$, $m \in \mathbb{Z}$ then the weighted Hankel operator H_ϕ^β on $H^2(\beta)$ is hyponormal if and only if it is normal.*

Proof. If $m < 0$, then $H_\phi^\beta = 0$ and hence, there is nothing to prove. For $m \geq 0$, the "only if" part follows as hyponormal compact operator is always normal and "if" part is trivial. ∎

It is interesting to find some spaces $H^2(\beta)$ where hyponormal weighted Hankel operators are always normal, even if the inducing symbols are not of the form as in Corollary 3.4.

Theorem 3.5. *If $\phi \in L^\infty(\beta)$ is not of the form $\phi(z) = \sum_{n=-m}^{\infty} a_n z^n$, $m \in \mathbb{Z}$ then the hyponormal weighted Hankel operator H_ϕ^β on $H^2(\beta)$ is normal if and only if $\beta_n = 1$ for each n.*

Proof. Let the hyponormal weighted Hankel operator H_ϕ^β on $H^2(\beta)$ is normal. Then, for each $j \geq 0$, $\|H_\phi^{\beta*} e_j\| = \|H_\phi^\beta e_j\|$ which implies that

$$\beta_{-j}^2 \sum_{n=0}^{\infty} \frac{|a_{-n-j}|^2}{{\beta_n}^2} = \frac{1}{{\beta_j}^2} \sum_{n=0}^{\infty} |a_{-n-j}|^2 \beta_{-n}^2.$$

For $j = 0$ this means that $\sum_{n=0}^{\infty}(|a_{-n}|^2\beta_{-n}^2 - \frac{|a_{-n}|^2}{{\beta_n}^2}) = 0$ and since each term in the bracket is positive, we get

$$\frac{|a_{-n}|^2}{{\beta_n}^2} = |a_{-n}|^2\beta_{-n}^2,$$

for each $n \geq 0$. Let $k \geq 0$ be an arbitrary number, then we can find $k_0 > k$ such that $a_{-k_0} \neq 0$. Now, $\frac{|a_{-k_0}|^2}{{\beta_{k_0}}^2} = |a_{-k_0}|^2\beta_{-k_0}^2$ provides that $\beta_{k_0} = \beta_{-k_0} = 1$. Hence, $\beta_m = 1$ for $|m| \leq k_0$. In particular, $\beta_k = 1$. As $k \geq 0$ is arbitrary so $\beta_n = 1$ for each n.
Converse is obvious as every hyponormal Hankel operator is normal (see Power [8]). ∎

If T_ψ^β denote the weighted Toeplitz operator on $H^2(\beta)$ with the symbol $\psi(z) = \sum_{n=-\infty}^{\infty} b_n z^n$, for $j \geq 0$ then

$$T_\psi^\beta e_j = \frac{1}{\beta_j}\sum_{n=0}^{\infty} b_{n-j}\beta_n e_n$$

and

$$T_\psi^{\beta *} e_j = \beta_j \sum_{n=0}^{\infty} \bar{b}_{j-n}\frac{e_n}{\beta_n}.$$

Analogously, for $\phi \in L^\infty(\beta)$ and $\psi \in H^\infty(\beta)$, the product $H_\phi^\beta T_\psi^\beta$ of the weighted Hankel operator H_ϕ^β and the weighted Toeplitz operator T_ψ^β is a weighted Hankel operator. In fact, it is $H_{\phi\psi}^\beta$, being $T_\psi^\beta = M_\psi^\beta|_{H^2(\beta)}$. However, in the main theorem of this section we prove the following.

Theorem 3.6. *Let H_ϕ^β and T_ψ^β be non-zero operators for $\phi, \psi \in L^\infty(\beta)$. Then the product $H_\phi^\beta T_\psi^\beta$ is a weighted Hankel operator if and only if $\psi \in H^\infty(\beta)$.*

Proof. We only prove that if $\phi, \psi \in L^\infty(\beta)$ are such that $H_\phi^\beta \neq 0, T_\psi^\beta \neq 0$ and $H_\phi^\beta T_\psi^\beta = H_\xi^\beta$, for some $\xi \in L^\infty(\beta)$ then $\psi(z) \in H^\infty(\beta)$. Suppose $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $\psi(z) = \sum_{n=-\infty}^{\infty} b_n z^n$ and $\xi(z) = \sum_{n=-\infty}^{\infty} c_n z^n$. Now, for each $j \geq 0$,

$$\begin{aligned} H_\phi^\beta T_\psi^\beta e_j &= H_\phi^\beta\left(\frac{1}{\beta_j}\sum_{k=0}^{\infty} b_{k-j}\beta_k e_k\right) = \frac{1}{\beta_j}\sum_{k=0}^{\infty} b_{k-j}\beta_k\left(\frac{1}{\beta_k}\sum_{n=0}^{\infty} a_{-n-k}\beta_{-n}e_n\right) \\ &= \frac{1}{\beta_j}\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{\infty} b_{k-j}a_{-n-k}\Big)\beta_{-n}e_n \end{aligned}$$

and

$$H_\xi^\beta e_j = \frac{1}{\beta_j} \sum_{n=0}^{\infty} c_{-n-j} \beta_{-n} e_n.$$

Therefore, for $n, j \geq 0$, we have

$$c_{-n-j} = \sum_{k=0}^{\infty} b_{k-j} a_{-n-k}. \tag{3.6.1}$$

Since H_ϕ^β is a non-zero operator, we assume that $a_{-l_0} \neq 0$, for some $l_0 \geq 0$. We prove the result using the strong induction by considering the two cases.

CASE (i). Let $a_0 \neq 0$. Then equation (3.6.1) gives, for each $l \geq 0$, $c_{-l} = c_{-l-0} = \sum_{k=0}^{\infty} b_k a_{-l-k}$. Also, $c_{-l} = c_{-0-l} = \sum_{k=0}^{\infty} b_{k-l} a_{-k}$. Consequently, for $l \geq 1$

$$\sum_{k=0}^{\infty} b_k a_{-l-k} = \sum_{k=0}^{\infty} b_{k-l} a_{-k} = \sum_{k=0}^{l-1} b_{k-l} a_{-k} + \sum_{k=l}^{\infty} b_{k-l} a_{-k}.$$

A change of variables shows that both the infinite summands in the above equation are equal and hence, for $l \geq 1$

$$\sum_{k=0}^{l-1} b_{k-l} a_{-k} = 0. \tag{3.6.2}$$

Putting $l = 1$, we get $b_{-1} a_0 = 0$ so that $b_{-1} = 0$. Now, assume that $b_{-1} = b_{-2} = b_{-3} = \cdots = b_{-n} = 0$. Then equation (3.6.2) with $l = n + 1$ becomes $b_{-n-1} a_0 = 0$, which implies $b_{-n-1} = 0$. Thus, $b_{-p} = 0$, for each $p \geq 1$ and hence $\psi(z) = \sum_{n=0}^{\infty} b_n z^n \in H^\infty(\beta)$.

CASE (ii). Let $a_0 = a_{-1} = a_{-2} = a_{-3} = \cdots = a_{-(k_0-1)} = 0$ and $a_{-k_0} \neq 0$. Again, by equation (3.6.1), for $l \geq 0$,

$$\begin{aligned}
\sum_{k=0}^{\infty} b_k a_{-k-k_0-l} &= c_{-(k_0+l)-0} = c_{-k_0-l} \\
&= c_{-0-(k_0+l)} = \sum_{k=0}^{\infty} b_{k-k_0-l} a_{-k} \\
&= \sum_{k=0}^{k_0+l-1} b_{k-k_0-l} a_{-k} + \sum_{k=k_0+l}^{\infty} b_{k-k_0-l} a_{-k} \\
&= \sum_{k=0}^{k_0+l-1} b_{k-k_0-l} a_{-k} + \sum_{k=0}^{\infty} b_k a_{-k-k_0-l}.
\end{aligned}$$

This yields that for $l \geq 1$,

$$\sum_{k=0}^{k_0+l-1} b_{k-k_0-l} a_{-k} = 0. \tag{3.6.3}$$

Now, on proceeding as in case (i) and using equation (3.6.3) in place of (3.6.2), we can prove that $b_{-p} = 0$, for each $p \geq 1$. It provides that $\psi \in H^\infty(\beta)$. Hence the theorem is proved. ∎

4. When do the weighted Hankel and weighted Toeplitz operators commute?

The study made in the paper so far was in a more general setting, however, we prefer to impose some restrictions to progress further. In what follows, we will consider $\beta = \{\beta_n\}_{n\in\mathcal{Z}}$ as a semi-dual sequence. Now onwards, the spaces $L^2(\beta), H^2(\beta)$ or $L^\infty(\beta)$ all are considered with $\beta = \{\beta_n\}_{n\in\mathcal{Z}}$ as a semi-dual sequence of positive numbers with $\beta_0 = 1$, $r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$, for $n \geq 0$ and $r \leq \frac{\beta_n}{\beta_{n-1}} \leq 1$, for $n \leq 0$, for some $r > 0$. For $\phi \in L^\infty(\beta)$, with formal Laurent series expression $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, we use the symbol $\widetilde{\phi}$ to represent the expression $\widetilde{\phi}(z) = \sum_{n=-\infty}^{\infty} a_{-n} z^n$. Now, if ϕ is in $L^\infty(\beta)$, then all the functions $\widetilde{\phi}, \phi + \widetilde{\phi}$ and $\phi\widetilde{\phi}$ belong to $L^\infty(\beta)$. In the next result, we identify the weighted Toeplitz operators induced by symbol ϕ with $\phi = \widetilde{\phi}$ that can commute with any weighted Hankel operator.

Theorem 4.1. *Let $\phi \in L^\infty(\beta)$ be such that $\phi = \widetilde{\phi}$. If any non-zero weighted Hankel operator H_ψ^β on $H^2(\beta)$, $\psi \in L^\infty(\beta)$ commutes with the weighted Toeplitz operator T_ϕ^β on $H^2(\beta)$ then ϕ is a constant function.*

Proof. Let $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ and $\psi(z) = \sum_{n=-\infty}^{\infty} b_n z^n$ be in $L^\infty(\beta)$ satisfying $0 \neq H_\psi^\beta$, $H_\psi^\beta T_\phi^\beta = T_\phi^\beta H_\psi^\beta$. This gives that, for each $i, j \geq 0$,

$$\langle H_\psi^\beta T_\phi^\beta e_j, e_i \rangle = \langle T_\phi^\beta H_\psi^\beta e_j, e_i \rangle.$$

However, one can see that

$$\langle H_\psi^\beta T_\phi^\beta e_j, e_i \rangle = \frac{\beta_i}{\beta_j} \sum_{k=0}^{\infty} a_{k-j} b_{-i-k} \text{ and } \langle T_\phi^\beta H_\psi^\beta e_j, e_i \rangle = \frac{\beta_i}{\beta_j} \sum_{k=0}^{\infty} a_{i-k} b_{-k-j}.$$

This yields that

$$\sum_{k=0}^{\infty} a_{k-j} b_{-i-k} = \sum_{k=0}^{\infty} a_{i-k} b_{-k-j}, \tag{4.1.1}$$

for each $i, j \geq 0$.

If we put $i = 0$, then equation (4.1.1) becomes

$$\sum_{k=0}^{j-1} a_{k-j} b_{-k} + \sum_{k=j}^{\infty} a_{k-j} b_{-k} = \sum_{k=0}^{\infty} a_{-k} b_{-k-j},$$

for each $j > 0$. This implies that for each $j > 0$,

$$\sum_{k=0}^{j-1} a_{k-j} b_{-k} = 0. \tag{4.1.2}$$

Combining (4.1.1) with the fact that $\phi = \widetilde{\phi}$ i.e. $a_n = a_{-n}$, we find that for $j > i > 0$,

$$\sum_{k=0}^{j-i-1} a_{k-j} b_{-i-k} = 0. \tag{4.1.3}$$

Since H_ψ^β is a non-zero weighted Hankel operator, we can find a non-negative integer n_0 such that $b_{-n_0} \neq 0$.

In case $n_0 = 0$, the repetitive use of (4.1.2), for $j = 1, 2, 3, \cdots$ gives $a_j = a_{-j} = 0$. Hence, ϕ is a constant function.

In case $n_0 > 0$, then using (4.1.3) for $i = n_0$, we have

$$\sum_{k=0}^{j-n_0-1} a_{k-j} b_{-n_0-k} = 0,$$

for $j > n_0$. Now, on taking the values $j = n_0+1, n_0+2, n_0+3, \cdots$ successively, we get $a_{n_0+s} = a_{-(n_0+s)} = 0$, for each $s \geq 1$.

If we apply (4.1.3) for $j = n_0 + 1$, we get

$$\sum_{k=1}^{n_0-i} a_{k-n_0-1} b_{-i-k} = 0,$$

for $n_0 + 1 > i$ and then on setting $i = n_0 - 1, n_0 - 2, \ldots, 2, 1$, it gives $a_s = a_{-s} = 0$, for each $2 \leq s \leq n_0$. Now on putting $j = n_0 + 1$ in equation (4.1.2), we get $a_{-1} = 0$. Hence in this case also, ϕ becomes a constant function. Hence the result. ∎

Almost along the same arguments as in Theorem 4.1, we can prove the following.

Theorem 4.2. *Let $\phi, \psi \in L^\infty(\beta)$ and H_ψ^β be a non-zero weighted Hankel operator on $H^2(\beta)$. Then $T_{\widetilde{\phi}}^\beta H_\psi^\beta = H_\psi^\beta T_\phi^\beta$ if and only if $\phi \in H^\infty(\beta)$.*

Corollary 4.3. *If $\phi, \psi \in L^\infty(\beta)$ are such that H_ψ^β is a non-zero weighted Hankel operator and $H_\psi^\beta T_\phi^\beta$ is a weighted Hankel operator on $H^2(\beta)$ then $T_{\widetilde{\phi}}^\beta H_\psi^\beta$ is a weighted Hankel operator on $H^2(\beta)$.*

Proof. Theorem 3.6 gives $\phi \in H^{\infty}(\beta)$ and then result follows by using Theorem 4.2. ∎

Now, we show that the commutativity of a weighted Hankel operator with weighted Toeplitz operators T^{β}_{ϕ} or $T^{\beta}_{\widetilde{\phi}}$ is the one and same.

Theorem 4.4. *Let $\phi, \psi \in L^{\infty}(\beta)$. Then $T^{\beta}_{\phi}H^{\beta}_{\psi} = H^{\beta}_{\psi}T^{\beta}_{\phi}$ if and only if $T^{\beta}_{\widetilde{\phi}}H^{\beta}_{\psi} = H^{\beta}_{\psi}T^{\beta}_{\widetilde{\phi}}$.*

Proof. Suppose $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ and $\psi(z) = \sum_{n=-\infty}^{\infty} b_n z^n$ in $L^{\infty}(\beta)$ are such that $T^{\beta}_{\phi}H^{\beta}_{\psi} = H^{\beta}_{\psi}T^{\beta}_{\phi}$. An usual computation show that for $i, j \geq 0$

$$\begin{aligned}\left\langle T^{\beta}_{\phi}H^{\beta}_{\psi}e_i, e_j\right\rangle = \left\langle H^{\beta}_{\psi}e_i, T^{\beta *}_{\phi}e_j\right\rangle &= \left\langle \frac{1}{\beta_i}\sum_{k=0}^{\infty} b_{-k-i}\beta_{-k}e_k, \beta_j\sum_{l=0}^{\infty}\frac{\overline{a}_{j-l}}{\beta_l}e_l\right\rangle \\ &= \frac{\beta_j}{\beta_i}\sum_{k=0}^{\infty} a_{j-k}b_{-k-i}\end{aligned}$$

and

$$\begin{aligned}\left\langle H^{\beta}_{\psi}T^{\beta}_{\phi}e_i, e_j\right\rangle = \left\langle T^{\beta}_{\phi}e_i, H^{\beta *}_{\psi}e_j\right\rangle &= \left\langle \frac{1}{\beta_i}\sum_{k=0}^{\infty} a_{k-i}\beta_k e_k, \beta_{-j}\sum_{l=0}^{\infty}\frac{\overline{b}_{-j-l}}{\beta_l}e_l\right\rangle \\ &= \frac{\beta_j}{\beta_i}\sum_{k=0}^{\infty} a_{k-i}b_{-j-k}.\end{aligned}$$

Thus,

$$\begin{aligned}T^{\beta}_{\phi}H^{\beta}_{\psi} = H^{\beta}_{\psi}T^{\beta}_{\phi} &\iff \sum_{k=0}^{\infty} a_{j-k}b_{-k-i} = \sum_{k=0}^{\infty} a_{k-i}b_{-j-k} \\ &\iff \left\langle T^{\beta}_{\widetilde{\phi}}H^{\beta}_{\psi}e_j, e_i\right\rangle = \left\langle H^{\beta}_{\psi}T^{\beta}_{\widetilde{\phi}}e_j, e_i\right\rangle \\ &\iff T^{\beta}_{\widetilde{\phi}}H^{\beta}_{\psi} = H^{\beta}_{\psi}T^{\beta}_{\widetilde{\phi}}. \ \blacksquare\end{aligned}$$

Corollary 4.5. *If $\phi \in L^{\infty}(\beta)$ is such that $T^{\beta}_{\phi}H^{\beta}_{\psi} = H^{\beta}_{\psi}T^{\beta}_{\phi}$, for some $\psi \in L^{\infty}(\beta)$ then $\phi + \widetilde{\phi}$ is a constant function.*

Proof. With the given conditions, $T^{\beta}_{\phi+\widetilde{\phi}}$ commutes with H^{β}_{ψ} and then applying Theorem 4.1, we get the result. ∎

Theorem 4.6. *Let $\phi, \psi \in L^{\infty}(\beta)$ be such that T^{β}_{ϕ} on $H^2(\beta)$ is not a multiple of identity operator and H^{β}_{ψ} on $H^2(\beta)$ is a non-zero operator. Then $T^{\beta}_{\phi}H^{\beta}_{\psi} = H^{\beta}_{\psi}T^{\beta}_{\phi}$ implies that $H^{\beta}_{\psi} = H^{\beta}_{\mu z\phi}$, for some complex number μ.*

Proof. Let $\phi, \psi \in L^\infty(\beta)$ be such that $T_\phi^\beta H_\psi^\beta = H_\psi^\beta T_\phi^\beta$. For $m, n \geq 0$, let $C_{n,m} = \langle H_\psi^\beta T_\phi^\beta e_m, e_n \rangle$ and $D_{n,m} = \langle T_\phi^\beta H_\psi^\beta e_m, e_n \rangle$. Then $C_{n,m} = D_{n,m}$, for each $m, n \geq 0$. One can see that, for $m > 0$ and $n \geq 0$,

$$\begin{aligned} D_{n+1,m-1} &= \frac{\beta_{n+1}}{\beta_{m-1}} \sum_{k=0}^{\infty} b_{-(k+m-1)} a_{n+1-k} \\ &= \frac{\beta_{n+1}}{\beta_{m-1}} b_{-(m-1)} a_{n+1} + \frac{\beta_{n+1}}{\beta_{m-1}} \frac{\beta_m}{\beta_n} D_{n,m} \\ &= \frac{\beta_{n+1}}{\beta_{m-1}} b_{-(m-1)} a_{n+1} + \frac{\beta_{n+1}}{\beta_{m-1}} \frac{\beta_m}{\beta_n} C_{n,m} \end{aligned}$$

and

$$\begin{aligned} C_{n,m} &= \frac{\beta_n}{\beta_m} \sum_{k=0}^{\infty} a_{k-m} b_{-n-k} \\ &= \frac{\beta_n}{\beta_m} a_{-m} b_{-n} + \frac{\beta_n}{\beta_m} \frac{\beta_{m-1}}{\beta_{n+1}} C_{n+1,m-1} \\ &= \frac{\beta_n}{\beta_m} a_{-m} b_{-n} + \frac{\beta_n}{\beta_m} \frac{\beta_{m-1}}{\beta_{n+1}} D_{n+1,m-1} \\ &= \frac{\beta_n}{\beta_m} a_{-m} b_{-n} + \frac{\beta_n}{\beta_m} \frac{\beta_{m-1}}{\beta_{n+1}} \left(\frac{\beta_{n+1}}{\beta_{m-1}} b_{-(m-1)} a_{n+1} + \frac{\beta_{n+1}}{\beta_{m-1}} \frac{\beta_m}{\beta_n} C_{n,m} \right) \\ &= \frac{\beta_n}{\beta_m} a_{-m} b_{-n} + \frac{\beta_n}{\beta_m} b_{-(m-1)} a_{n+1} + C_{n,m}. \end{aligned}$$

This gives $a_{n+1} b_{-(m-1)} = -a_{-m} b_{-n}$, for each $m > 0$ and $n \geq 0$. Using Corollary 4.5, $\phi + \widetilde{\phi}$ is a constant function so that $a_m + a_{-m} = 0$, for each $m > 0$. Therefore, we have

$$a_{n+1} b_{-(m-1)} = a_m b_{-n}, \tag{4.6.1}$$

for each $m > 0$ and $n \geq 0$.

We claim that $a_{n+1} = 0$ if and only if $b_{-n} = 0$, for $n \geq 0$.

If there exists $n_0 \geq 0$ such that $b_{-n_0} = 0$, then (4.6.1) implies that either $a_{n_0+1} = 0$ or $b_{-m+1} = 0$, for all $m > 0$. But the latter would imply that $H_\psi^\beta = 0$, hence $a_{n_0+1} = 0$. Conversely, if there exists $n_0 \geq 0$ such that $a_{n_0+1} = 0$, then (4.6.1) yields that either $a_m = 0$, for all $m > 0$ or $b_{-n_0} = 0$. The former implies that ϕ is a constant function so that $T_\psi^\beta = 0$ is a multiple of the identity, which is a contradiction. Hence, $b_{-n_0} = 0$. Therefore the claim.

H_ψ^β being a non-zero operator, take a non-negative integer n_0 such that $b_{-n_0} \neq 0$. Define $\lambda = \frac{b_{-n_0}}{a_{n_0+1}}$. Then (4.6.1) gives $b_{-m+1} = \mu a_{-m}$, for each $m > 0$, where $\mu = -\lambda$. Moreover, $H_\psi^\beta = H_{\mu z \phi}^\beta$. ∎

Theorem 4.7. *Let $\phi \in L^\infty(\beta)$. If the weighted Hankel operator $H^\beta_{z\phi}$ commutes with the weighted Toeplitz operator T^β_ϕ then $\phi\widetilde{\phi}$ is a constant function.*

Proof. Evidentally, $H^\beta_{z\phi}$ commutes with $H^\beta_{z\phi}H^\beta_{z\phi}$. By using the equation (2.1), we have $H^\beta_{z\phi}H^\beta_{z\phi} = T^\beta_{\phi\widetilde{\phi}} - T^\beta_{\widetilde{\phi}}T^\beta_\phi$. Now Theorem 4.4 helps to conclude that $H^\beta_{z\phi}$ commutes with $T^\beta_{\phi\widetilde{\phi}}$. Moreover, $\widetilde{\phi\widetilde{\phi}} = \phi\widetilde{\phi}$ and now the result follows using Theorem 4.1. ■

An additional use of Theorem 4.6 gives the following.

Corollary 4.8. *If H^β_ψ is a non-zero weighted Hankel operator such that $T^\beta_\phi H^\beta_\psi = H^\beta_\psi T^\beta_\phi$, where T^β_ϕ is not a multiple of the identity operator on $H^2(\beta)$ then $H^\beta_\psi = H^\beta_{\mu z\phi}$, for some complex number μ. In this case, $\phi + \widetilde{\phi}$ and $\phi\widetilde{\phi}$ are constant functions.*

We conclude the study with the result which identifies the weighted Toeplitz operators commuting with a weighted Hankel operators.

Theorem 4.9. *Let $\phi \in L^\infty(\beta)$ be such that $\phi\widetilde{\phi}$ and $\phi + \widetilde{\phi}$ are constant, then the weighted Hankel operator $H^\beta_{z\phi}$ commutes with the weighted Toeplitz operator T^β_ϕ.*

Proof. Put $\phi + \widetilde{\phi} = c$ and $\phi\widetilde{\phi} = d$. Then $\phi\widetilde{\phi}$ and $\phi + \widetilde{\phi}$ are in $H^\infty(\beta)$. Hence $H^\beta_{zc} = H^\beta_{zd} = 0$. Now, by applying the equation (2.1), we have

$$\begin{aligned} 0 = H^\beta_{zd} = H^\beta_{z\phi\widetilde{\phi}} &= T^\beta_\phi H^\beta_{z\phi} + H^\beta_{z\widetilde{\phi}} T^\beta_\phi \\ &= T^\beta_\phi H^\beta_{z\phi} + H^\beta_{z(c-\phi)} T^\beta_\phi \\ &= T^\beta_\phi H^\beta_{z\phi} - H^\beta_{z\phi} T^\beta_\phi . \end{aligned}$$

Hence the theorem. ■

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DEPARTMENT OF MATHEMATICS
PGDAV COLLEGE
UNIVERSITY OF DELHI
DELHI - 110065, INDIA
E-mails: gopal.d.sati@gmail.com
porwal1987@gmail.com

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